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Structure of locally conformally flat manifolds satisfying some weakly-Einstein conditions

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ABSTRACT

It is given a complete study of locally conformally flat metrics satisfying some weakly-Einstein conditions. It is shown that they are either a product $M^n(c) \times M^n(-c)$ or a warped product $\mathbb{R} \times_f \mathbb{R}^{n-1}$ for some specific warping function. Moreover, some conditions on locally conformally flat fixed points for the RG2 flow are pointed out.

1. Introduction

Let (M, g) a n-dimensional Riemannian manifold and R its curvature tensor given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. A manifold is called locally conformally flat if for every point in M, it exists a neighborhood of the point such that a flat space can be assigned to it via a conformal change. Moreover, it is a known fact that a manifold is locally conformally flat if and only if its Weyl tensor is vanishing. In that case, the curvature is determined by the Ricci tensor, given by $\rho(X, Y) := \operatorname{tr}\{Z \mapsto R(Z, X)Y\}$. Thus, the curvature tensor can be written as

$$\begin{aligned} R(X,Y)Z = &-\frac{\tau}{(n-2)(n-1)}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{1}{(n-2)}\{\rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}, \end{aligned} \tag{1.1}$$

where Q denote the Ricci operator, $\rho(X, Y) = g(QX, Y)$ and $\tau = \operatorname{tr}\rho$ is the scalar curvature. Therefore, the study of the different terms coming from the curvature is a much simpler task.

Besides, Berger, in [2], showed the following universal identity in dimension four.

$$\left(\check{R} - \frac{\|R\|^2}{4}g\right) + \tau\left(\rho - \frac{\tau}{4}g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4}g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right) = 0, \tag{1.2}$$

where $\check{R}_{ij} = R_{iabc}R_j^{abc}$, $\check{\rho}_{ij} = \rho_{ia}\rho_j^a$ and $R[\rho]_{ij} = R_{iabj}\rho^{ab}$. Now, in the light of this identity, if we assume that the metric is Einstein (i.e., $\rho = \frac{\tau}{n}g$), then, all the brackets in (1.2) vanish, so the Einstein condition for a metric automatically satisfies that the other three tensors are a multiple of the metric. So now a natural question that arises is the converse: If any of these

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three tensors is a multiple of the metric, does a metric fulfill the Einstein condition? There are some counterexamples. In [6] it is shown a product manifold of two surfaces with opposite curvature $M^2(c) \times M^2(-c)$, which satisfies that everyone of the three presented tensors are a multiple of the metric whereas it is not Einstein. So it is a legitimate task looking for examples that satisfies these conditions but the Einstein one. We define the following classes.

Definition 1. A non–Einstein Riemannian manifold is called:

- $\check{R}$-Einstein if $\check{R} = \frac{||R||^2}{n} g$.
- $\check{\rho}$-Einstein if $\check{\rho} = \frac{||\rho||^2}{n} g$.
- $R[\rho]$-Einstein if $R[\rho] = \frac{||\rho||^2}{n} g$.

Moreover, if a Riemannian manifold (M, g) (respectively, a metric) satisfies any of these three conditions, then we will say that the manifold (the metric) satisfies a weakly-Einstein condition.

Notice that we use a slightly different definition for weakly-Einstein metrics. In [1] and [6], for example, the authors define these conditions as what we call $\check{R}$-Einstein.

Einstein metrics are a main topic in differential geometry and they appear as critical metrics for the Hilbert functional, $g \mapsto \int \tau dvol_g$, restricted to volume one metrics. Weakly-Einstein metrics also appears naturally in the study of critical metrics for some specific functional, for instance, if we take the functional

$$\mathcal{F}_{t,s} : g \longmapsto \mathcal{F}_{t,s}(g) = \int_M \{||\rho||^2 + t\tau^2 + s||R||^2\} dv_g$$

and compute its gradient [4],

$$\begin{aligned} \nabla \mathcal{F}_{t,s} = &- (1+4s)\Delta\rho + (1+2t+2s)\,\mathrm{Hess}(\tau) - \frac{1+4t}{2}\Delta\tau g \\ &- 2t\tau\left(\rho - \frac{\tau}{4}g\right) - 2s\left(\check{R} - \frac{\|R\|^2}{4}g\right) + 4s\left(\check{\rho} - \frac{\|\rho\|^2}{4}g\right) \\ &- 2(1+2s)\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right), \end{aligned}$$

it involves all the tensors mentioned in the definition of the weakly-Einstein classes. Moreover, $\check{R}$-Einstein condition has seem to receive attention in other fields

In [1], Arias-Marco and Kowalski classified $\check{R}$-Einstein four dimensional Lie groups, and following this work, a classification on the same field was given in [9], completing all the casuistic. In [5], Chen study $\check{R}$-Einstein almost contact manifolds and in [3], the authors study this tensor in the context of compact manifolds with boundary.

On the other hand, $\check{R}$ tensor appears in other fields, such as in the study of the two-loop renormalization flow. The two-loop renormalization flow (or $RG2$ flow) appears as a perturbation of the Ricci flow (see [11–13]) and it is given by

$$\frac{\partial}{\partial_t} g_t = RG[g], \tag{1.3}$$

where $RG[g] = -2\rho - \frac{\alpha}{2}\check{R}$ and α is a positive coupling constant.

One can study genuine fixed points of (1.3), i.e., metrics satisfying $\rho + \frac{\alpha}{4}\check{R} = 0$.

In dimension two the condition reduces to constant negative curvature. In dimension three, they were studied by Gimre, Guenther and Isenberg in [11], where they showed solutions with Ricci curvatures $Q_\rho = -2\,\mathrm{diag}[\frac{1}{\alpha}, \alpha, 0]$ or $Q_\rho = -2\,\mathrm{diag}[\frac{2}{\alpha}, \alpha, \frac{1}{\alpha}]$. Einstein metrics are genuine fixed points of this flow in dimension four since if the Ricci tensor is a multiple of the metric, the $\check{R}$ tensor is as well. In [10], it is given a classification of genuine fixed points in four dimensional Lie groups. This flow has also been applied in the study of black holes metrics, analyzing how they evolved along it and for the study of entropy, which has been stated as monotonic along this same flow. The reason to use this in such cases is that the singularities appearing in the study of other flows disappear in RG2, being this a better approximation to higher curvature effects [14,15].

The main aim of this work is classifying these conditions, both weakly-Einstein and fixed points, in the field of locally conformally flat manifolds. The $\check{R}$-Einstein condition has been already studied in [8], where the metrics satisfying it were classified as a product $M^n(c) \times M^n(-c)$ or a warped product $\mathcal{I} \times_f N(c)$ of a real interval and a manifold of constant sectional curvature c with some specific real function solving the differential equation $f'(t)^2 + f(t)f''(t) - c = 0$. Therefore, we will focus on the other two left, the $\check{\rho}$-Einstein and the $R[\rho]$-Einstein conditions along sections 2 and 3, completing the study and giving the whole classification of weakly-Einstein locally conformally flat Riemannian manifolds and finding new examples of this sort of manifolds, of which there is a lack of them along all the literature. During section 4, we study fixed points for the RG2 flow, obtaining an algebraic condition. Finally, in section 5, we are going to study the condition $R[\rho]$-Einstein in a particular casuistic in order to try to give some light on it as it seems to be the one that remains with no new examples.

2. $R[\rho]$-Einstein condition

$R[\rho]$-Einstein conditions seems to be too much rigid and these fields provides none new examples for it. The result of its calculation is briefed in the following statement.

Theorem 2. *A locally conformally flat Riemannian manifold is $R[\rho]$-Einstein if and only if $M = M^{n_1}(c) \times M^{n_2}(-c)$ with $n_1 = n_2$.*

Proof. First of all, we are going to compute the $R[\rho]$ $(1,1)$-tensor, which is given by $R[\rho](X,Y) = g(Q_{R[\rho]}(X),Y)$. Using (1.1), a straightforward calculation shows that

$$Q_{R[\rho]} = -\frac{2}{(n-2)}Q^2 + \frac{n\tau}{(n-1)(n-2)}Q + \left\{\frac{1}{(n-2)}||\rho||^2 - \frac{\tau^2}{(n-1)(n-2)}\right\}\mathrm{Id}\,.$$

Since we want to see when this tensor is a multiply of the identity, we need that

$$Q_{R[\rho]} - \frac{||\rho||^2}{n}\,\mathrm{Id} = 0,$$

or equivalently,

$$-\frac{2}{(n-2)}Q^2 + \frac{n\tau}{(n-1)(n-2)}Q + \left\{\frac{2}{n(n-2)}||\rho||^2 - \frac{\tau^2}{(n-1)(n-2)}\right\}\mathrm{Id} = 0. \tag{2.4}$$

This equation needs to be satisfied by every eigenvalue of the tensor, and since it is a quadratic, then we have two at most, but if we have just one, then the manifold would be Einstein, so assume that we have eigenvalues of the Ricci operator λ and μ with multiplicities m and $n-m$, respectively. Moreover, using the Vieta's Formulae [7], we obtain that

$$\lambda + \mu = \frac{n\tau}{2(n-1)}. \tag{2.5}$$

Thus, as $\tau = m\lambda + (n-m)\mu$, both eigenvalues are related by

$$\mu = \frac{2(n-1) - mn}{n(n-m) - 2(n-1)}\lambda. \tag{2.6}$$

Next, on the one hand, let $S = \frac{1}{n-2}(\rho - \frac{\tau}{2(n-1)}g)$ be the Schouten tensor, and on the other hand, let us introduce the following technical result.

Lemma 3. [16] *Let T be a Codazzi tensor. Let γ be an eigenfunction of T with eigenspace V_γ. If $\dim V_\gamma \geq 2$, then $\nabla\gamma$ is orthogonal to V_γ. Moreover, if T has exactly two different eigenfunctions γ and δ with $\dim V_\gamma \leq \dim V_\delta$, then*

(i) *M is locally a product if $\dim V_\gamma \geq 2$.*
(ii) *M is locally a warped product with one-dimensional if and only if*
  (ii.a) $\dim V_\gamma = 1$,
  (ii.b) *the eigenfunction δ is not constant and $\nabla\gamma$ is orthogonal to V_δ.*

It is well known that S is Codazzi if the manifold is Locally conformally flat and from (2.6) one can obtain, through a standard calculation, that the Schouten tensor has two different eigenvalues (call them $\bar{\lambda}$ and $\bar{\mu}$), and thus, we can apply the lemma.

If $\dim V_{\bar{\lambda}} \geq 2$, then M is a locally a product by assertion (i) and due to locally conformally flatness it can be either $\mathbb{R} \times N(c)$ or $M^{n_1}(c) \times M^{n_2}(-c)$. The first case implies that one of the eigenvalues is zero and, as they are a multiple of each other, then both are vanishing, so M is flat. Regarding the second case, one can easily see that a product manifold $M^{n_1}(c_1) \times M^{n_2}(c_2)$ is $R[\rho]$-Einstein if and only if

$$c_1^2(n_1-1)^2 = c_2^2(n_2-1)^2,$$

and since in this case $c_1 = -c_2$, then $n_1 = n_2$.

If $\dim V_{\bar{\lambda}} = 1$, then $\dim V_{\bar{\mu}} = n-1$, and so $\nabla_{\bar{\mu}}$ is orthogonal to $V_{\bar{\mu}}$, but, as $\bar{\lambda}$ is a multiple of $\bar{\mu}$, then $\nabla_{\bar{\lambda}}$ is also orthogonal to $V_{\bar{\mu}}$. Besides, $\bar{\mu}$ cannot be constant. Otherwise, $\bar{\lambda}$ would be constant as well, which would imply that λ and μ would be constant. Hence, we would have a locally conformally flat manifold with constant Ricci curvatures, which is curvature homogeneous, and by [18], it would be locally symmetric. Then M would split as a product of the form $\mathbb{R} \times N(c)$, whose factors correspond to the Ricci curvatures, so λ would be vanishing and so μ, and therefore, M would be flat. Thus, applying the previous lemma, we have a warped product and due to locally conformally flatness, the fiber has to be of constant sectional curvature.

Now, we want to determine the warping function. In order to do that, we use the following result.

Lemma 4 *([17]). Let $B \times_f F$ be a warped product with $\dim F = d$. Let $X, Y \in T_pB$ and $V, W \in T_pF$. Then*

- $\rho(X,Y) = \rho^B(X,Y) - \dfrac{d}{f} Hess(f)(X,Y).$
- $\rho(X,V) = 0.$
- $\rho(V,W) = \rho^F(V,W) - \left\{ \dfrac{\Delta f}{f} + (d-1)\dfrac{g(\operatorname{grad} f, \operatorname{grad} f)}{f^2} \right\} g(V,W).$

In our current situation, we are in a warped product $\mathbb{R} \times_f N(c)$, so the Ricci operator is written by

$$Q(\partial_t) = -(n-1)\frac{f''}{f}\partial_t, \tag{2.7}$$

$$Q(X) = \left((n-2)\frac{c}{f^2} - (n-2)\frac{f'^2}{f^2} - \frac{f''}{f} \right) X.$$

Since the Ricci eigenvalues are related by $\lambda = (n-1)\mu$, we obtain the differential equation

$$f'^2 - c = 0,$$

which only have a suitable solution if $c > 0$, and in that case, it is linear, what gives Einstein metrics. Therefore we cannot have $R[\rho]$-Einstein warped products in this field, which completes the proof. □

3. $\check{\rho}$-Einstein condition

In sharp contrast with the previous case, we can get new examples for this metrics. We state the following.

Theorem 5. *A locally conformally flat Riemannian manifold is $\check{\rho}$-Einstein if and only if $M = M^{n_1}(c) \times M^{n_2}(-c)$ with $n_1 = n_2$ or a warped product $\mathbb{R} \times_f \mathbb{R}^{n-1}$ with*

$$f(t) = \left(\frac{2(n-1)(at+b)}{n} \right)^{\frac{n}{2(n-1)}},$$

with $a, b \in \mathbb{R}$ and $t \in \left(\frac{-b}{a}, +\infty \right)$.

Proof. We proceed in the same way. This time, the equation desire equation is

$$Q^2 - \frac{||\rho||^2}{n}\operatorname{Id} = 0. \tag{3.8}$$

Consequently, we have two eigenvalues again and due to Vieta's formulae they are related by $\mu = -\lambda$. We shall use Lemma 3 again. Therefore, if $\dim V_\lambda \geq 2$ then we have a product $M^{n_1}(c) \times M^{n_2}(-c)$ and the condition to a product of this kind to be $\check{\rho}$-Einstein is that

$$c_1^2(n_1-1)^2 = c_2^2(n_2-1)^2,$$

so $n_1 = n_2$.

If $\dim V_\lambda = 1$, then we have a warped product $\mathbb{R} \times_f N(c)$, and as we know that $\mu = -\lambda$, using ((2.7)), we obtain

$$nff'' + (n-2)f'^2 - (n-2)c = 0.$$

Now, taking the derivative of this equation one obtains

$$nff''' + (3n-4)f'f'' = 0.$$

Since f and f'' cannot be zero (otherwise the metric is flat), then one can divide by these factors and thus

$$\frac{(4-3n)}{n}\frac{f'}{f}=\frac{f'''}{f''}.$$

Next, integrate both parts of the equations and get

$$\frac{(4-3n)}{n}\ln f=\ln f''+K.$$

Taking the exponential, the equations become

$$f^{\frac{(4-3n)}{n}}=e^K f'',$$

and now multiply both sides by $2f'$,

$$2e^{-K}f'f^{\frac{(4-3n)}{n}}=2f'f''.$$

Call the constant part $\bar{K}$. We have standard integrals on both parts, so we get

$$\bar{K}\frac{n}{4-2n}f^{\frac{4-2n}{n}}=f'^2.$$

Finally, isolating f', we get

$$f'=\tilde{K}f^{\frac{2-n}{n}},$$

which solution is

$$f(t)=\left(\frac{2(n-1)\left(\tilde{K}t+a\right)}{n}\right)^{\frac{n}{2(n-1)}},$$

where $a\in\mathbb{R}$. Therefore, we obtain a solution for the second equation, which was the derivative of the one we got in first place. Now, if some function is a solution for the first equations, it is a solution for its derivative, and as we know the solutions for this last one, the solution of the original equations need to be of this form. So if we put this f in the original equations, we get that it is a solution for it if and only if $c=0$. Therefore, we are in a warped product of the form $\mathbb{R}\times_f\mathbb{R}^{n-1}$ and we have no other possibility here. □

Remark 6. Notice that using these techniques on the warped products, we shall give a simpler proof for the classification of the $\check{R}$-Einstein case given in [8]. From there, we have that the relation between both eigenvalues was given by

$$\mu=-\frac{2m+(n-1)(n-4)}{2(n-m)+(n-1)(n-4)}\lambda,$$

and if $m=1$, then

$$\mu=-\frac{2+(n-1)(n-4)}{2(n-1)+(n-1)(n-4)}\lambda.$$

Using now (2.7), we obtain the differential equation

$$f'^2+ff''-c=0,$$

which is the one that gives $\check{R}$-Einstein metrics.

4. Locally conformally flat fixed points of the RG2-flow

In this section we classify fixed points in the context of locally conformally flat manifolds.

Theorem 7. *Let (M,g) be a n-dimensional locally conformally flat fixed point for the two-loop renormalization group flow with coupling constant α. Then*

(1) *If $n\neq 4$, then (M,g) is homothetic to a product $M_1^{n_1}(c)\times M_2^{n_2}(-c)$ with $n_1=n_2$ or to a warped product $\mathbb{R}\times_f N(c)$ with non trivial warping function satisfying*

$$\alpha(n-2)((n-6)n+6)\left(f'^2-c\right)+\alpha((n-4)(n-2)n-4)ff''+2(n-2)^2f^2=0.$$

(2) *If* $n=4$, *then* $\|R\|^2 = \|\rho\|^2 = \frac{\tau^2}{3}$.

Proof. Let us recall, on the one hand, that fixed points for the $RG2$ flow is given by a metric fulfilling $\rho + \frac{\alpha}{4}\check{R} = 0$. On the other hand, one can see from [8] that $Q_{\check{R}}$ operator is given by

$$Q_{\check{R}} = \frac{2}{(n-2)^2}\left\{(n-4)Q^2 + \frac{2\tau}{(n-1)}Q + \frac{(n-1)\|\rho\|^2 - \tau^2}{(n-1)}\,\mathrm{Id}\right\}.$$

Combining these two identities, one can get that a metric in this field is a fixed point if $Q + \frac{\alpha}{4}Q_{\check{R}} = 0$, which is

$$Q + \frac{\alpha}{2(n-2)^2}\left\{(n-4)Q^2 + \frac{2\tau}{(n-1)}Q + \frac{(n-1)\|\rho\|^2 - \tau^2}{(n-1)}\,\mathrm{Id}\right\} = 0,$$

and then,

$$\frac{\alpha(n-4)}{2(n-2)}Q^2 + \frac{\alpha\tau + (n-1)(n-2)^2}{(n-1)(n-2)^2}Q + \frac{\alpha((n-1)\|\rho\|^2 - \tau^2)}{2(n-2)^2(n-1)}\,\mathrm{Id} = 0. \tag{4.9}$$

Now we have two different possibilities depending on the dimension. If $n \neq 4$, then we have a quadratic equation on the Ricci operator, so we have two Ricci eigenvalues related by

$$\lambda + \mu = -\frac{2(\alpha\tau + (n-1)(n-2)^2)}{\alpha(n-1)(n-2)(n-4)}.$$

Thus, we have two eigenvalues, one a multiple of the other, and as the Schouten tensor is Codazzi and it has also two eigenvalues, one a multiple of the other, then we have either a warped product $\mathbb{R} \times_f N(c)$, with f a non trivial real warping function and $N(c)$ an $(n-1)$-dimensional Riemannian manifold of constant curvature or a Riemannian product $M_1^{n_1}(c) \times M_2^{n_2}(-c)$, such that $n_1 = n_2$. In order to determine the function f, assuming that λ has multiplicity one, then both are related by

$$\lambda + \mu = \frac{2\left(\alpha(\lambda + \mu(n-1)) + (n-1)(n-2)^2\right)}{\alpha(n-4)(n-2)(n-1)},$$

and using the formulas from (2.7) for the Ricci operator, we get that f must satisfy the differential equation

$$\alpha(n-2)((n-6)n+6)\left(f'^2 - c\right) + \alpha((n-4)(n-2)n-4)ff'' + 2(n-2)^2f^2 = 0$$

If $n=4$, then equation (4.9) becomes

$$12(\alpha\tau + 12)Q + \alpha(3\|\rho\|^2 - \tau^2)\,\mathrm{Id} = 0.$$

Since this is a lineal equation, this only can have one solution, and then, the Ricci operator has only one eigenvalue, so the metric is Einstein as long as the equation is not identically zero. In order to have that, we need that $\alpha = -\frac{12}{\tau}$ and $\|\rho\| = \frac{\tau^2}{3}$. Moreover, taking traces in $\rho + \frac{\alpha}{4}\check{R} = 0$, one can obtain that $\alpha = -4\tau\|R\|^{-2}$, then $\|R\|^2 = \frac{\tau^2}{3}$, and hence $\|R\|^2 = \|\rho\|^2$. Notice that α cannot be vanishing since, in that case, the Ricci tensor is as well. □

5. A note on $R[\rho]$-Einstein condition

During these sections we are not going to assume that the manifold is locally conformally flat. Since this condition seems to be the most rigid one, we may think other ways to try to obtain examples. We may think of an easier casuistic in order to get some suitable algebraic condition. For that, assume that the Ricci operator has two eigenvalues, one simple, i.e., $Q_\rho = \mathrm{diag}[\lambda, \mu, \dots, \mu]$. In this situation, as the ρ_{ij} are all vanishing whenever $i \neq j$, then the tensor reduces to $R[\rho]_{ij} = \sum_a R_{aija}\rho_{aa}$, having a much simpler casuistic.

Now we are computing $R[\rho]_{11}$.

$$\begin{aligned} R[\rho]_{11} =& R_{2112}\rho_{22} + R_{3113}\rho_{33} + \cdots + R_{n11n}\rho_{nn} \\ =& \mu(R_{2112} + R_{3113} + \cdots + R_{n11n}) = \mu\rho_{11} = \mu\lambda \end{aligned}$$

Now, recall that $\|\rho\|^2 = \lambda^2 + (n-1)\mu^2$, so one of the equations that needs to be satisfies, in order to have a multiply of the identity, is

$$\lambda^2 + (n-1)\mu^2 - n\lambda\mu = 0,$$

which, after doing some simplifications, is equivalent to

$$(\lambda-\mu)(\lambda-(n-1)\mu)=0.$$

Therefore, since if $\lambda\neq\mu$, we obtain that $\lambda=(n-1)\mu$, which makes the component $R[\rho]_{11}=(n-1)\mu^2$.

The condition of $R[\rho]$ being a multiple of the metric must fulfill that $R[\rho]_{ii}=R[\rho]_{jj}=(n-1)\mu^2$ and $R[\rho]_{ij}=0$, for all $i,j\in\{1,\dots,n\}$, $i\neq j$. Because of that, we are studying the rest of the components in three steps.

Firstly we are analyzing $R[\rho]_{1\alpha}$, with $\alpha>1$.

$$\begin{aligned}R[\rho]_{1\alpha}=&R_{21\alpha2}\rho_{22}+R_{31\alpha3}\rho_{33}+\cdots+R_{n1\alpha n}\rho_{nn}\\ =&\mu(R_{21\alpha2}+R_{31\alpha3}+\cdots+R_{n1\alpha n})=\mu\rho_{1\alpha}=0.\end{aligned}$$

Using the same computations, we obtain that $R[\rho]_{\alpha\alpha}=\mu((n-2)R_{1\alpha\alpha1}+\mu)$, with $\alpha>1$, and $R[\rho]_{\alpha\beta}=\mu(n-2)R_{1\alpha\beta1}$, with $\alpha\neq\beta$, $1<\alpha<\beta$. Therefore, from these two components, we obtain the equations

$$\begin{aligned}\mu((n-2)R_{1\alpha\alpha1}+\mu)-\mu^2(n-1)=0\\ \mu(n-2)R_{1\alpha\beta1}=0\end{aligned}$$

On the one hand, as μ cannot be zero, we get that $R_{1\alpha\alpha1}=\mu$, and since

$$\rho_{\alpha\alpha}=\mu=R_{1\alpha\alpha1}+R_{2\alpha\alpha2}+\cdots+R_{n\alpha\alpha n},$$

we get the condition

$$\sum_{i=2}^{n}R_{i\alpha\alpha i}=0.$$

On the other hand, we automatically get that $R_{1\alpha\beta1}=0$. Thus, we have the following result.

Lemma 8. *Let (M,g) be an n-dimensional Riemannian manifold with two different eigenvalues for the Ricci operator, one of them simple. Then, (M,g) is $R[\rho]$-Einstein if and only if*

(i) $Q_\rho=\operatorname{diag}[(n-1)\mu,\mu,\dots,\mu]$.
(ii) $\sum_{i>1}R_{i\alpha\alpha i}=0$, *with* $1<\alpha\leq n$.
(iii) $R_{1\alpha\beta1}=0$ *for all* α,β *such that* $1<\alpha<\beta\leq n$.

Remark 9. The suitable curvature tensor could have different special conditions depending on the dimension we are working on. For instance, if $n=4$, then

$$\begin{aligned}\rho_{23}=0=&R_{1231}+R_{4234}\\ \rho_{24}=0=&R_{1241}+R_{3243}\\ \rho_{34}=0=&R_{1341}+R_{2342}.\end{aligned}$$

Using (iii) from the Lemma, shows that $R_{4234}=R_{3243}=R_{2342}=0$. Moreover, (ii) give us the system of equations

$$\begin{aligned}&R_{3223}+R_{4224}=0\\ &R_{2332}+R_{4334}=0\\ &R_{2442}+R_{3443}=0,\end{aligned}$$

which only possible solution, due to the symmetries of the curvature tensor, is that every $R_{\alpha\beta\beta\alpha}$, with $1<\alpha<\beta\leq4$ is vanishing.

However, if $n=5$, this last system becomes

$$\begin{aligned}&R_{3223}+R_{4224}+R_{5225}=0\\ &R_{2332}+R_{4334}+R_{5335}=0\\ &R_{2442}+R_{3443}+R_{5445}=0\\ &R_{2552}+R_{3553}+R_{4554}=0,\end{aligned}$$

which does not imply that every term is vanishing.

Remark 10. In the context of locally conformally flat metrics, (i) is satisfied, but this does not imply that M is locally conformally flat. For example, if $n = 4$, the term $W(e_1, e_2)$ of the Weyl tensor is given by

$$W(e_1, e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{1223} & -R_{1224} \\ 0 & R_{1223} & 0 & -R_{1234} \\ 0 & R_{1224} & R_{1234} & 0 \end{pmatrix}.$$

In locally conformally flat metrics with this kind of Ricci operator, the curvature components where we have three different indices are zero, whereas in this case we do not need any special condition for these components.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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