Competitive Equilibrium and Indivisibles

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Abstract

This paper studies the existence of competitive equilibrium in markets with indivisibles. The basis of the paper is a theorem that relates the existence of equilibrium to the core of certain associated cooperative games. From this result, we obtain a significant improvement on a classical result in this literature. We then characterize the existence of equilibrium in a two-agent three-good economy, showing that, in this particular context, the existence of equilibrium is more likely when goods are complements than when they are substitutes.

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1 Introduction

In our opinion, the main motivation of an economic model should be to reflect the reality it wants to describe as closely as possible. In that sense, considering money to be an infinitely divisible good seems a reasonable assumption but, in many contexts, the classical hypothesis of the infinite divisibility of goods is undoubtedly unrealistic. It seems, then, that considering indivisible goods is a good wTheoremay for models to come a little closer to the realities they want to explain. Along these lines, in this paper we therefore consider a model of goods as indivisible and the classical notion of competitive equilibrium in economies. Since many aspects of such models have already been analysed in the literature, we focus here on the existence of competitive equilibrium. Studies of the existence of equilibrium in models with indivisible goods admit several classifications, one of the most important of which is obtained by observing where attention is focused.

Some studies focus on demand functions, where gross substitutability, a property introduced by Kelso and Crawford (1982), is perhaps the most studied. If the demands of our model are verified, this property is sufficient to guarantee the existence of competitive equilibrium. This property, used in several works and models, such as matching, housing, and labour market models, has proved to be a condition that guarantees the existence of competitive equilibrium. Generalizing Kelso and Crawford's result, Sun and Yang (2006) assume the existence of complementary and substitute goods and prove that a competitive equilibrium exists if the demand functions satisfy the gross substitutability and complementarity conditions (for a review of relevant classical contributions on the subject, see Sönmez and Ünver (2011)). A relatively recent work in this field is by Baldwin and Klemperer (2019), who formally classify demand types according to the point where small variations occur in the price of demand bundles. That paper, which uses hard theoretical geometry, obtains existence results according to certain properties that must be satisfied by the demands.

Other studies focus on the utility functions of the agents. In the context of our research, we draw most on a seminal study by Bikhchandani and Mamer (1997), sharing the model and much of the notation. This work characterizes the existence of equilibrium prices in terms of the efficiency of indivisible allocations in the associated divisibles market. Specifically, Bikhchandani and Mamer (1997) show that an economy has competitive equilibrium if and only if every efficient allocation in the indivisibles market is also efficient in the associated divisibles market. However, this result does not solve our problem: although the calculation of an efficient indivisible allocation may be simple, this is not the case for a divisible allocation. In a context where all agents share the same utility function, Méndez-Naya and Méndez-Naya (2025) give interesting conditions for the existence of equilibrium. Some of those conditions are generalized in Méndez-Naya and Méndez-Naya (2024), which also puts forward manageable equivalences in the context of convexity. As shown by Beviá, Quinzii, and Silva (1999), if functions are concave and cardinal, then equilibrium exists.

Another author addressing the same problem is Ma (1998), who follows the ideas of Kelso and Crawford (1982). In that paper, a cooperative game of transferable utility, "played" by agents and goods, is associated with the economy. In this associated game, the utility of a coalition S of goods and agents is given by the maximum utility that the members of S can achieve among themselves. It is then shown that the economy has equilibrium if and only if this associated game has a non-empty core. In Theorem 2 we improve this result by simplifying the associated cooperative game, finding that the characterization of the existence of equilibrium still holds for our game. Going a little further in simplifying the conditions for equilibrium, with each economy we associate a cooperative game involving only the goods of the economy. We thus obtain the principal result of our work: characterization of the existence of equilibria in terms of the associated cooperative games (of fewer agents) having a non-empty core (Theorem 1). This result is merely a generalization of Theorem 1 in Méndez-Naya and Méndez-Naya (2025) for the context of multiple utility functions.

Even in the simplest case, i.e., two agents and three goods as dealt with here, the subject is not trivial. Our main motivation is to search for conditions in the utility functions of the two agents that are equivalent to the existence of equilibrium with three goods. The essential point is Theorem 3, which guarantees that an economy in this class has equilibrium if and only if one of two easily valid alternatives is given. The interest of our result lies in reducing an infinite search to verification of a few inequalities. If the economy has equilibrium, then calculation of the associated equilibrium prices is immediate. Using Theorem 3, many of the results already known in more general contexts are easily deduced. Others specific results are also deduced in the context of superadditivity and the same function for both agents. We again show that, in this context, equilibrium is much more likely to exist with complementary goods than with substitute goods.

The paper is organized as follows. Section 2 describes the model and the basic notation. Section 3 is devoted to the central result of the paper (Theorem 1). Section 4 introduces the *VL* game and proves Theorem 2, which enables us to improve on the results of Ma (1998). Section 5 is devoted to the class of an economy with two agents and three goods and to proving our central result (Theorem 3) characterizing the existence of equilibrium in this class; also demonstrated are other interesting results, in particular that the existence of equilibrium is more likely when goods are complementary than when they are substitutes.

2 The model

Consider an exchange economy $\mathbf{E} = \{M, N, \{V_j, w_j^0\}_{j\in M}\}$ that consists of indivisible goods $i \in N = \{1, ..., n\}$ and agents $j \in M = \{1, ..., m\},\ m > 1$, who possess an initial endowment (w_j^0) of perfectly divisible money and whose preferences are expressed by reservation value functions $V_j : P(N) \to R$ (where $P(Z)$ is the set of all subsets of any set Z), where $V_j(T)$ is the greatest sum of money that agent j is willing to pay for the bundle T. We assume $\forall j \in M$, that $V_j(\emptyset) = 0$, V_j is monotone (i.e., $T \subset U$, and thus, $V_j(T) \leq V_j(U)$), that $w_j^0 \geq V_j(N)$, and that the utility functions U_j are quasi-linear with respect to money. If agent j has a bundle T and money w_j, then

$$U_j = U_j(T, w_j) = V_j(T) + w_j \ \ \forall T \subseteq N, \forall j \in M.$$

Note that we use $A \subset B$ when $A \subseteq B$ and $A \neq B$. A feasible allocation is a set of m bundles of goods $S = (S_1, ..., S_m)$ such that $S_j \cap S_k = \emptyset\ \forall j, k \in M$ and $\cup_{j\in M} S_j = N$. Note that S_j may be empty (agent j receives no goods). The social value of a feasible allocation S is $V(S) = \Sigma_{j \in M} V_j(S_j)$, and a feasible allocation $\overline{S}$ is efficient if it maximizes V, i.e., if

$$MI = V(\overline{S}) \geq V(S)$$

for every feasible allocation S. We can think of MI as the maximum indivisible utility that can be achieved in the economy. Since the set of allocations is finite, at least one efficient allocation exists.

Goods are sold in the market at prices $p_i \geq 0$, expressed in units of money. The demand correspondence of agent j is

$$D_j(p) = \{T^* \subseteq N \mid V_j(T^*) - p(T^*) \geq V_j(T) - p(T) \ \forall T \subseteq N\},$$

where $p = (p_1, ..., p_n)$ and $p(T) = \Sigma_{i \in T} p_i$ (in general, if $x \in I\!R^z$ and $T \subseteq \{1, ..., z\}$, we denote $x(T) = \sum_{i \in T} x_i$). Given a price vector $p \in I\!R^n_+$, $D_j(p)$ is the set of all bundles that maximizes consumer agent j 's surplus, $e_j = V_j(T_j) - p(T_j)$. Note that since

$$V_j(T^*) - p(T^*) \geq V_j(\emptyset) - p(\emptyset) = 0$$

$V_j(T^*) \geq p(T^*)$. Hence, the assumption $w^0_j \geq V_j(N)$ and the monotonicity of V_j together guarantee that agent j has enough money to buy any bundle in their demand set.

Definition 1 *A competitive equilibrium for* $\mathbf{E}$ *is a pair* (p, S) *such that* $p \in I\!R^n_+$, $S = (S_1, ..., S_m)$ *is a feasible allocation and* $S_j \in D_j(p)$ $\forall j \in M$. *The vector* p *is therefore an equilibrium price and* S *is an equilibrium allocation supported by* p.

The precise quantities of the initial endowments of money are irrelevant to the existence of equilibrium because Definition 1 neither explicitly nor implicitly mentions endowments. This is a consequence of the assumption that each agent initially has sufficient money to buy any basket of goods, and this, in turn, allows us to define the demand sets without reference to budgetary limitations. Note that the economy is specified for our purposes by its valuation functions $\{V_j\}_{j \in M}$, functions which (abusing the language somewhat) we will sometimes refer to as utility functions. For an economy $\mathbf{E} = \{M, N, \{V_j\}_{j \in M}\}$, Bikhchandani and Mamer (1997) show that every equilibrium allocation is efficient and also that, if a price vector supports one efficient allocation, then it supports all efficient allocations.

3 The main theorem

In this part we first state and prove the central result of this work, namely, Theorem 1. The novelty of our approach consists in determining the existence of equilibrium from the point of view of excesses. Our condition requires the existence of a vector of excesses r such that a certain associated cooperative set V_r has a non-empty core. This simple idea will give us sufficient room to work with, as we will see throughout the paper. We begin by setting out specific notations and definitions.

For an economy $\mathbf{E} = \{M, N, \{V_j\}_{j\in M}\}$ and $r = (r_1, ..., r_m) \in I\!R^m_+$, we let $\bar{r} = \sum_{j=1}^m r_j$ and we define

$$V^r(T) = \begin{cases} MI - \bar{r} & if \quad T = N \\ \max_{j\in M}\{V_j(T) - r_j\} & if \quad T \subset N \end{cases}$$

In general, the functions V^r are unconventional games in the sense that the image of some coalitions may be negative and they may not be weakly increasing functions. Nonetheless, we can define their core and work with this concept. The core of a function W from $P(N)$ in $I\!R$ is

$$C(W) = \{(x_1, ..., x_n) \in I\!R^n \mid (x(T) \geq W(T)\, \forall T \subseteq N) \wedge (x(N) = W(N))\}.$$

We are now in a position to state and prove the main result of this work.

Theorem 1 *An economy* $\mathbf{E} = \{M, N, V\}$ *has competitive equilibrium if and only if there exists* $r = (r_1, ..., r_m) \in I\!R^m_+$ *such that*

a) $\bar{r} - r_j \leq MI - V_j(N) \quad \forall j \in M$

b) $C(V^r) \neq \varnothing$

Proof. **(IF)** Let $r = (r_1, ..., r_m)$ be a vector in $I\!R^m_+$ verifying **a)** and **b)**, and let $p = (p_1, ..., p_n)$ be a vector in $C(V^r)$. We now prove that p is an equilibrium price. We know that p verifies

$$p(N) = V^r(N) = MI - \bar{r} \tag{1}$$

and $\forall T \subset N$,

$$p(T) \geq V^r(T) = \max_{j\in M}\{V_j(T) - r_j\} \tag{2}$$

In particular, if $S = (S_1, ..., S_m)$ is an efficient allocation, then $\forall j \in M$,

$$p(S_j) \geq V^r(S_j) \geq V_j(S_j) - r_j. \tag{3}$$

Summing Eq. (3) over j, and considering Eq. (1),

$$\Sigma_{j=1}^m p(S_j) \geq \Sigma_{j=1}^m V_j(S_j) - \bar{r} = MI - \bar{r} = p(N).$$

However, as S is a feasible allocation, $N = \cup_{j\in M} S_j$, and we verify that $\Sigma_{j=1}^m p(S_j) = p(N)$. Thus, $\Sigma_{j=1}^m p(S_j) = \Sigma_{j=1}^m V_j(S_j) - \bar{r}$, which, along with Eq. (3), implies that $\forall j \in M$, $p(S_j) = V_j(S_j) - r_j$, i.e.

$$r_j = V_j(S_j) - p(S_j). \tag{4}$$

Taking Eq. (2), $\forall j \in M, \forall T \subset N$, we verify that

$$r_j \geq V_j(T) - p(T) \tag{5}$$

and for $T = N$, from **a)** we have

$$\bar{r} - r_j \leq MI - V_j\left(N\right) \quad \forall j \in M$$

From Eq. (1) we obtain

$$MI - p\left(N\right) = \bar{r}$$

Summing the two last expressions, we obtain

$$r_j \geq V_j(N) - p(N)$$

Together, Eqs. (4) and (5) imply that $\forall j \in M$, S_j belongs to the demand set of agent j . To prove that (p, S) is a competitive equilibrium we must prove that p is a valid price vector, i.e., $p_i \geq 0$ $\forall i \in N$.

However, we first analyse a very particular case. Suppose that $m = 2$ and $S = (i, N\backslash i)$; in this case $MI = V_1\left(i\right) + V_2\left(N\backslash i\right)$, and, based on **a)**,

$$r_1 + r_2 - r_2 \leq MI - V_2\left(N\right)$$

Hence

$$\begin{aligned} r_1 &\leq V_1\left(i\right) + V_2\left(N\backslash i\right) - V_2\left(N\right) \\ &\Longrightarrow V_1\left(i\right) - r_1 \geq V_2\left(N\right) - V_2\left(N\backslash i\right) \geq 0 \end{aligned}$$

and, as $p \in C\left(V^r\right)$,

$$p_i \geq V^r\left(i\right) \geq V_1\left(i\right) - r_1 \geq 0.$$

Note that in all other cases, since S is an efficient allocation, for each $i \in N$, $\exists\ j \in M$ such that $i \notin S_j, (S_j \cup \{i\} \neq N)$, according to Eq. (5),

$$r_j \geq V_j(S_j \cup \{i\}) - p\left(S_j \cup \{i\}\right) = V_j(S_j \cup \{i\}) - p(S_j) - p_i$$

Therefore, given Eq. (4) and given that V_j is weakly increasing,

$$p_i \geq V_j(S_j \cup \{i\}) - p(S_j) - r_j = V_j(S_j \cup \{i\}) - V\left(S_j\right) \geq 0$$

This proves that $p_i \geq 0\ \ \forall i \in N$, and hence, (p, S) is a competitive equilibrium.

(ONLY IF) Let $S = (S_1, ..., S_m)$ be an equilibrium allocation supported by a price vector $p =$

$(p_1, ..., p_n)$; hence, S is efficient and $MI = \sum_{j=1}^{m} V_j(S_j)$. For each $j \in M$, we take $r_j = V_j(S_j) - p(S_j)$, and obtain $\bar{r} = \sum_{j=1}^{m} r_j = MI - p(N)$. We see that $p \in C(V^r)$, and it is clear that

$$V^r(N) = MI - \bar{r} = p(N)$$

Hence, the first core condition is satisfied. As each $S_j \in D_j(p)$ that we have for each $T \subseteq N$,

$$r_j = V_j(S_j) - p(S_j) \geq V_j(T) - p(T) \quad \forall j \in M, \tag{6}$$

then

$$p(T) \geq V_j(T) - r_j \quad \forall j \in M \quad \forall T \subset N$$
$$p(T) \geq \max_{j \in M} (V_j(T) - r_j) = V^r(T) \quad \forall T \subset N$$

and p also verifies the second core condition, proving **b)**. From Eq. (6) , for each $j \in M$, we have $r_j \geq V_j(N) - p(N)$ (or $-r_j \leq -V_j(N) + p(N)$) and $\bar{r} = MI - p(N)$. Hence, summing up,

$$\bar{r} - r_j \leq MI - V_j(N) \quad \forall j \in M$$

which proves **a)** and so the proof is complete. ■

The following example, already explored in Example 1 in Ma (1998), shows the greater effectiveness and clarity of our Theorem 1 in determining the existence of equilibrium.

Example 1 *For each* $\alpha_1, \alpha_2 \in [0, 3]$, *and* $\alpha_3 \in [0, 1]$, *we consider the economy* $\mathbf{E} = (M, N, (V_1, V_2))$ *where*

	1	2	3	12	13	23	123
V_1	4	4	$4 + \alpha_1$	$7 + \alpha_3$	7	7	9
V_2	$4 + \alpha_2$	4	4	7	7	$7 + \alpha_3$	9

It is clear that $MI = 11 + \max\{\alpha_1, \alpha_2, \alpha_3\}$. *Let us first assume that* $\alpha_1 = \max\{\alpha_1, \alpha_2, \alpha_3\}$ *and apply Theorem 1. We thus have*

	1	2	3	12	13	23	123
$V_1 - r_1$	$4 - r_1$	$4 - r_1$	$4 + \alpha_1 - r_1$	$7 + \alpha_3 - r_1$	$7 - r_1$	$7 - r_1$	$9 - r_1$
$V_2 - r_2$	$4 + \alpha_2 - r_2$	$4 - r_2$	$4 - r_2$	$7 - r_2$	$7 - r_2$	$7 + \alpha_3 - r_2$	$9 - r_2$

The game $V_{(r_1,r_2)}$ *is given by*

	1	2	3
$V_{(r_1,r_2)}$	$4 + \max\{-r_1, \alpha_2 - r_2\}$	$4 + \max\{-r_1, -r_2\}$	$4 + \max\{\alpha_1 - r_1, -r_2\}$

12	13	23	123
$7 + \max\{\alpha_3 - r_1, -r_2\}$	$7 + \max\{-r_1, -r_2\}$	$7 + \max\{-r_1, \alpha_3 - r_2\}$	$11 + \alpha_1 - r_1 - r_2$

Two necessary conditions for $C\left(V_{(r_1,r_2)}\right) \neq \varnothing$ *are*

$$\begin{aligned} V_{(r_1,r_2)}(3) + V_{(r_1,r_2)}(12) &\leq V_{(r_1,r_2)}(123) \\ \Longrightarrow 4 + \alpha_1 - r_1 + 7 + \alpha_3 - r_1 &\leq 11 + \alpha_1 - r_1 - r_2 \\ \Longrightarrow \alpha_3 &\leq r_1 - r_2 \end{aligned} \tag{7}$$

and

$$\begin{aligned} V_{(r_1,r_2)}(1) + V_{(r_1,r_2)}(23) &\leq V_{(r_1,r_2)}(123) \\ \Longrightarrow 4 + \alpha_2 - r_2 + 7 + \alpha_3 - r_2 &\leq 11 + \alpha_1 - r_1 - r_2 \\ \Longrightarrow r_1 - r_2 &\leq \alpha_1 - \alpha_2 - \alpha_3 \end{aligned} \tag{8}$$

Hence,

$$\alpha_3 \leq r_1 - r_2 \leq \alpha_1 - \alpha_2 - \alpha_3$$

Consequently, if $\alpha_1 = \max\{\alpha_1, \alpha_2, \alpha_3\}$*, a necessary condition for equilibrium to exist is*

$$2\alpha_3 \leq \alpha_1 - \alpha_2.$$

If we now consider

$$\begin{aligned} r_1 &= 1 + \alpha_1 - \alpha_3 \\ r_2 &= 1 + \alpha_2 \end{aligned}$$

it is easy to verify that we comply with the conditions of Theorem 1 and, hence, equilibrium exists. Based on symmetry, the result is analogous when $\alpha_2 = \max\{\alpha_1, \alpha_2, \alpha_3\}$*, in which case a necessary condition for equilibrium is*

$$2\alpha_3 \leq \alpha_2 - \alpha_1$$

Furthermore, taking

$$\begin{aligned} r_1 &= 1 + \alpha_1 \\ r_2 &= 1 + \alpha_2 - \alpha_3 \end{aligned}$$

we derive that, in this case, equilibrium also exists when the necessary condition is verified.

The last case we consider is $\alpha_3 = \max\{\alpha_1, \alpha_2, \alpha_3\}$*. Applying the same necessary conditions as in Eqs.* (7) *and* (8)*, we obtain*

$$\alpha_1 \leq r_1 - r_2 \leq -\alpha_2$$

which is only possible if $\alpha_1 = \alpha_2 = 0$*. Again taking* $r_1 = r_2 = 1 - \alpha_3$*, Theorem 1 guarantees equilibrium in that economy. In sum, for* $\alpha_1, \alpha_2 \in [0, 3]$ *and* $\alpha_2 \in [0, 1]$*, there is equilibrium in the* *economy if and only if*

$$2\alpha_3 \leq |\alpha_1 - \alpha_2| \;\; or \; \alpha_1 = \alpha_2 = 0.$$

Remark 1 *Note that, to analyse the existence of equilibrium, application of Theorem 1 is generally simpler than the method proposed in Ma (1988), since in Ma's work it is necessary to study the core of a game of n+m players, while Theorem 1 relies on games of only n players. Obviously the difference becomes more noticeable for large values of m and n. Further comments on this matter are made in the Remark 2.*

4 A theoretical application

Ma (1998) associates each economy with indivisibles $\mathbf{E} = (M, N, V)$ with a cooperative game of $n + m$ players, and then shows that the economy has equilibrium if and only if the associated game has a non-empty core. Here, our aim is to find a cooperative game with properties similar to those in Ma (1998) but considerably simpler. We begin by defining both games: that considered in Ma (1998) is denoted *VM*, and that introduced here is denoted *VL*.

For $T_1 \subseteq N$ and $T_2 \subseteq M$, T_2 of cardinal t_2, we define $P_{T_2}(T_1)$ as the set of partitions of T_1 of t_2 elements

$$P_{T_2}(T_1) = \{K = (K_1, ..., K_{t_2}) / K_j \cap K_k = \varnothing \; \forall j,k \in T_2, \cup_{j \in T_2} K_j = T_1\}$$

For an economy $\mathbf{E} = \{M, N, V\}$ and for each $T = T_1 \cup T_2 \subseteq N \cup M$, $\varnothing \neq T_1 \subseteq N$, and $\varnothing \neq T_2 \subseteq M$, we define

$$VL(T) = \begin{cases} MI & if \quad T = N \cup M \\ \max_{j \in T_2} V_j(T_1) & if \quad T \subset N \cup M \end{cases}$$

$$VM(T) = \max_{K \in P_{T_2}(T_1)} \sum_{j \in T_2} V_j(K_j)$$

Of course, if $T_1 = \varnothing$ or $T_2 = \varnothing$, then the value of both games is 0. Note that, in both games, the utility that can be guaranteed by a grand coalition is MI. The game *VL* represents, for each $T \subset N \cup M$, the best use that one agent in T_2 can make of T_1, and the game *VM* represents the optimal way to distribute the goods in T_1 between the agents in T_2. We are now in a position to state and prove the corresponding central result.

Theorem 2 *An economy* $\mathbf{E} = \{M, N, V\}$ *has competitive equilibrium if and only if* $C(VL) \neq \varnothing$.

Proof. (ONLY IF) Taking into account that $VL(T) \leq VM(T) \; \forall T \in N \cup M$ and $VL(N \cup M) = VM(N \cup M) = MI$, it is obvious that $C(VM) \subset C(VL)$. If competitive equilibrium exists in $\mathbf{E}$, by Theorem 1 in Ma (1998), we know that $C(VM) \neq \varnothing$. As a consequence $C(VL) \neq \varnothing$, which proves the first implication.

(IF) Letting $((p_1, ..., p_n), (r_1, ..., r_m)) \in C(VL)$, we set $p = (p_1, ..., p_n)$, $r = (r_1, ..., r_m)$. Taking into account that $VL(M \cup N) = MI = \sum_{i=1}^{n} p_i + \sum_{j=1}^{m} r_j = p(N) + \bar{r}$, we have

$$\begin{aligned} VL(N \cup \{j\}) = V_j(N) &\leq p(N) + r_j \ \forall j \in M \\ p(N) + \bar{r} &= MI \end{aligned}$$

Summing we have

$$\bar{r} - r_j \leq MI - V_j(N) \ \ \forall j \in M$$

Hence, r verifies part **a)** of Theorem 1. We also can prove that r verifies part **b)** since

$$\begin{aligned} &((p_1, ..., p_n), (r_1, ..., r_m)) \in C(VL) \Longrightarrow \\ &VL(T_1 \cup T_2) = \max_{j \in T_2} V_j(T_1) \leq p(T_1) + r(T_2) \ \ \forall T_1 \cup T_2 \subseteq N \cup M \Longrightarrow \\ &VL(T_1 \cup \{j\}) = V_j(T_1) \leq p(T_1) + r_j \ \forall T_1 \subseteq N, \ \forall j \in M \Longrightarrow \\ &V_j(T_1) - r_j \leq p(T_1) \ \ \forall T_1 \subseteq N, \ \forall j \in M \Longrightarrow \\ &\max_{j \in M} (V_j(T_1) - r_j) \leq p(T_1) \ \ \forall T_1 \subseteq N \Longrightarrow \\ &p \in C(V^r) \end{aligned}$$

Hence, r verifies Theorem 1 **a)** and **b)** both, and, as a consequence, the economy $\mathbf{E} = \{M, N, V\}$ has competitive equilibrium. ■

Remark 2 *Note that VL is a much simpler game than VM. Looking again at Example 1, to calculate VM, it is necessary, for each $S \subseteq N$, to calculate $VM(S, j, k)$, which is the optimal use that $\{j, k\}$ can jointly make of the goods in S; for VL this calculation is not necessary. Of course, the difference in efficiency between the games becomes more noticeable as we increase the number of goods and agents. Actually, Theorem 1 can be seen as a simplification of the study of the VL core, and VL as a simplification of VM.*

5 A practical application

From now on we focus on games with two players, $m = 2$, $M = \{1, 2\}$, and three goods, $n = 3$, $N = \{1, 2, 3\}$. Although the context is very particular and the extension to more goods and players does not seem easy, we consider that the characterization given in Theorem 3 and the conclusions that follow from it are of interest in themselves.

5.1 The theorem

To prove the corresponding main result, we start by providing a number of definitions. The set of all partitions of $N = \{1, 2, 3\}$ of t elements, with $t = 1, 2, 3$, is defined as

$$P_t = \{(S_1, ..., S_t) / S_j \subseteq N,\ S_j \cap S_k = \varnothing\ \forall j, k \in M,\ \cup_{j \in M} S_j = N\}$$

The utility functions of the agents are defined as

$$V_j : P(N) \to I\!R_+ \quad j = 1, 2$$

The values of a series of constants associated with the economy are defined as

$$\begin{aligned}
MI &= \max_{(S_1,S_2)\in P_2} \{V_1(S_1) + V_2(S_2)\} \\
V^1_{112} &= \max_{(i,j,k)\in P_3} \{V_1(i) + V_1(j) + V_2(k)\} \\
V^1_{122} &= \max_{(i,j,k)\in P_3} \{V_1(i) + V_2(j) + V_2(k)\} \\
V^2_{112} &= \max_{(i,j,k)\in P_3} \{V_1(ij) + V_1(jk) + V_2(ik)\} \\
V^2_{122} &= \max_{(i,j,k)\in P_3} \{V_1(ij) + V_2(jk) + V_2(ik)\} \\
V^2_{111} &= V_1(12) + V_1(13) + V_1(23) \\
V^2_{222} &= V_2(12) + V_2(13) + V_2(23)
\end{aligned}$$

$$\begin{aligned}
V_{11} &= \max_{(i,jk)\in P_2} \{V_1(i) + V_1(jk)\} \\
V_{22} &= \max_{(i,jk)\in P_2} \{V_2(i) + V_2(jk)\}
\end{aligned}$$

Also defined are

$$\begin{aligned}
x_1 &= \max\left\{0, V^1_{112} - MI\right\} \\
x_2 &= \min\left\{2MI - V^2_{122},\ MI - V_2(N)\right\} \\
y_1 &= \max\left\{0, V^1_{122} - MI\right\} \\
y_2 &= \min\left\{2MI - V^2_{112},\ MI - V_1(N)\right\}
\end{aligned}$$

Finally, for $j = 1, 2$ we define

$$\begin{aligned} a_j =& MI - V_{jj} \\ b_j =& MI - V_j(1) - V_j(2) - V_j(3) \\ c_j =& 2MI - V_j(12) - V_j(23) - V_j(13) \end{aligned}$$

The following lemma expresses the existence condition given in Theorem 1 in a more manageable form for our purposes.

Lemma 1 *Let* **E** *be a two-agent three-good economy.* **E** *has equilibrium if and only if a vector* $(x_e, y_e) \in I\!R^2$ *exists, verifying that*

$$\begin{aligned} &a)\, x_1 \leq x_e \leq x_2 \\ &b)\, y_1 \leq y_e \leq y_2 \\ &c)\, x_e - a_2 \leq y_e \leq x_e + a_1 \\ &d)\, 2x_e - c_2 \leq y_e \leq 2x_e + b_1 \\ &e)\, \tfrac{x_e - b_2}{2} \leq y_e \leq \tfrac{x_e + c_1}{2} \end{aligned}$$

Proof. As is well known (see Shapley, 1967), for a three-agent cooperative game

$$W : P(N) \rightarrow I\!R_+$$

the properties

$$\begin{cases} 1)\, W(1) + W(2) + W(3) \leq W(123) \\ 2)\, W(i) + W(r,s) \leq W(123) \text{ for any } (\{i\}, \{r,s\}) \text{ partition of } N \\ 3)\, W(12) + W(13) + W(23) \leq 2W(123). \end{cases} \tag{9}$$

are equivalent to a non-empty core of W. The proof of the lemma is reduced to a rewriting of the properties of Eq. (9) for the cooperative game $V^r = V_{(x_e,y_e)}$ of Theorem 1, taking into account the different possibilities for maxima to be reached in the definition of this game. For example

$$\begin{aligned} V_{(x_e,y_e)}(1) + V_{(x_e,y_e)}(23) &\leq V_{(x_e,y_e)}(123) \\ \max\{V_1(1) - x_e, V_2(1) - y_e\} + \max\{V_1(23) - x_e, V_2(23) - y_e\} &\leq MI - x_e - y_e \end{aligned} \tag{10}$$

Taking into account that

$$\begin{aligned} V_1(1) + V_2(23) \leq MI \\ V_2(1) + V_1(23) \leq MI \end{aligned}$$

Eq. (10) is equivalent to

$$\begin{cases} V_1(1) + V_1(23) \leq MI + x_e - y_e \Leftrightarrow y_e \leq x_e + MI - V_1(1) - V_1(23) \\ V_2(1) + V_2(23) \leq MI + y_e - x_e \Leftrightarrow y_e \geq x_e + V_2(1) - V_2(23) - MI \end{cases}$$

Repeating the process for partitions $(\{2\}, \{13\})$ and $(\{3\}, \{12\})$, we obtain the six conditions equivalent to

$$c)\; x_e - a_2 \leq y_e \leq x_e + a_1$$

The remainder of the proof is implemented in the same way. ■

Note that Lemma 1 yields a characterization of the existence of equilibrium that is not very applicable in practice. The logical question is: when does such $(x_e, y_e) \in \mathbb{R}^2$ exist? In fact, to answer this question we would have to represent in $\mathbb{R}^2$ all the inequalities given in Lemma 1 . We would thus obtain the points of a rectangle (as given by the limits of variation of x and y) intersecting with the (possibly empty) surface between two lines of slope 2, between two lines of slope 1, and between two lines of slope $\frac{1}{2}$.

The following result is an efficient and rapid method to determine whether or not the intersection is empty. In fact, what Theorem 3 guarantees is that, if that intersection in $\mathbb{R}^2$ is non-empty, then there is an element of either the form (x_1, y) or the form (x, y_2) in that intersection. Note that both conditions are immediately verified.

This theorem is an essential result, since in transforming an "infinite search" into the verification of a few inequalities, it represents a simple means of checking whether or not an economy of the corresponding characteristics has equilibrium. Obviously, knowing an "equilibrium excess"(x_e, y_e), calculating equilibrium prices is reduced to calculating the elements of the corresponding game core.

Theorem 3 *Let* **E** *be a two-agent three-good economy. The economy has equilibrium if and only if one of the following conditions holds:*

$$a)\; x_1 \leq x_2,\; \max\left\{y_1, x_1 - a_2, 2x_1 - c_2, \frac{x_1 - b_2}{2}\right\} \leq \min\left\{y_2, x_1 + a_1, 2x_1 + b_1, \frac{x_1 + c_1}{2}\right\}$$

$$b)\; y_1 \leq y_2,\; \max\left\{x_1, y_1 - a_1, \frac{y_1 - b_1}{2}, 2y_1 - c_1\right\} \leq \min\left\{x_2, y_1 + a_2, \frac{y_1 + c_2}{2}, 2y_1 + b_2\right\}$$

Proof. See Appendix. ■

5.2 Applications

Of course, results for the existence of equilibrium in situations more general than the two-agent three-good scenario are also valid here, and all of them are immediate consequences of our Theorem 3. Below we explore in more detail what our result yields in specific contexts.

5.2.1 Convexity, concavity, and gross substitutes

In Méndez-Naya and Méndez-Naya (2025) it is proved that, if all agents have the same quasi-symmetric and concave utility function, then equilibrium exists. This result, in our particular case, is an immediate consequence of Theorem 3. Furthermore, in Méndez-Naya and Méndez-Naya (2024) it is proved that a two-agent economy in which the utility functions are convex always has equilibrium, and again, in the case $n = 3$, this result is an immediate consequence of Theorem 3.

Even in our particular context, it is known (Bikhchandani and Mamer, 1997) that the utility concavity condition is not sufficient to guarantee the existence of equilibrium. By means of a simple numerical simulation and making use of our Theorem 3, we can verify that, if the utility functions of our economy V_1 and V_2 are concave, then the probability of equilibrium is very high – approximately 0.993. This amounts to saying that examples of markets with concave utility functions that do not have equilibrium are very rare. If we want to guarantee the existence of equilibrium we have to resort to gross substitutability. Using the characterization of this property as given in Fujishige and Yang (2003), it is easy to see that for $n = 3$, V verifies gross substitutability if and only if V is concave and there exists a re-enumeration $\{i, j, k\} = N$ such that

$$V(ij) + V(k) = V(ik) + V(j) \geq V(jk) + V(i) \tag{11}$$

Obviously, the probability of equality in Eq. (11) is zero, i.e., within the concave functions, those verifying gross substitutability are practically negligible.

Remark 3 *Even though it is true, in our case at least, that gross substitutability is a sufficient condition for the existence of equilibrium, contrary to what is usually believed, gross substitutability is very far from being a necessary condition; indeed, not even the much weaker condition of concavity is a necessary condition.*

5.2.2 Superadditivity

Let us look now at superadditive functions, which are those that verify that

$$V(A) + V(B) \leq V(A \cup B) \quad \forall A, B \subseteq N,\ A \cap B = \varnothing$$

When the inequality is in the opposite direction, the function is called subadditive. Supposing that in our economy V_1 and V_2 are superadditive, then it is immediate to verify that

$$x_1 = 0, y_1 = 0, a_1 \geq 0, a_2 \geq 0, b_1 \geq 0, b_2 \geq 0$$

and the two conditions of Theorem 3 we are left with are

$$a)\, 0 \leq x_2,\ \max\{0, -c_2\} \leq \min\left\{y_2, a_1, b_1, \frac{c_1}{2}\right\}$$
$$b)\, 0 \leq y_2,\ \max\{0, -c_1\} \leq \min\left\{x_2, a_2, b_2, \frac{c_2}{2}\right\}$$

Taking into account that $MI \geq V_2(N)$, we always have

$$0 \leq x_2 \Leftrightarrow 2MI \geq V_{122}^2$$

and analogously

$$0 \leq y_2 \Leftrightarrow 2MI \geq V_{112}^2$$

On the other hand,

$$-c_2 \leq y_2 \Leftrightarrow \begin{cases} 4MI \geq V_2(12) + V_2(13) + V_2(23) + V_{112}^2 \\ 3MI \geq V_2(12) + V_2(13) + V_2(23) + V_1(N) \end{cases}$$

but for some i, j, k

$$\begin{aligned} V_2(12) + V_2(13) + V_2(23) + V_{112}^2 &= V_2(12) + V_2(13) + V_2(23) + V_1(ij) + V_1(ik) + V_2(jk) \\ &\leq V_{122}^2 + V_{122}^2 \leq 2MI + 2MI = 4MI \end{aligned}$$

As a consequence

$$-c_2 \leq y_2 \Leftrightarrow 3MI \geq V_2(12) + V_2(13) + V_2(23) + V_1(N)$$

In addition $-c_2 \leq a_1$, because for some i, j, k

$$\begin{aligned} V_2(12) + V_2(13) + V_2(23) - 2MI \leq MI - V_{11} \Leftrightarrow & \\ & 3MI \geq V_2(12) + V_2(13) + V_2(23) + V_1(ij) + V_1(k) \Leftrightarrow \\ & 3MI \geq V_1(k) + V_2(ij) + V_2(ik) + V_2(jk) + V_1(ij) \end{aligned}$$

That is always true because

$$\begin{gathered} V_1(k) + V_2(ij) \leq MI \\ V_2(ik) + V_2(jk) + V_1(ij) \leq V_{122}^2 \leq 2MI \end{gathered}$$

In the same way it can be proved that $-c_2 \leq b_1$ and $-c_2 \leq \frac{c_1}{2}$. Hence, the only condition that we must include to verify

$$\max\{0, -c_2\} \leq \min\left\{y_2, a_1, b_1, \frac{c_1}{2}\right\}$$

is that $c_1 \geq 0$. Making the same arguments for part **b)** of Theorem 3, equilibrium is verified to exist if and only if

$$\begin{cases} 2MI \geq V_{122}^2 \\ 2MI \geq V_{112}^2 \\ \begin{cases} \begin{cases} 3MI \geq V_2(12) + V_2(13) + V_2(23) + V_1(N) \\ 2MI \geq V_1(12) + V_1(13) + V_1(23) \end{cases} \\ \text{or} \\ \begin{cases} 3MI \geq V_1(12) + V_1(13) + V_1(23) + V_2(N) \\ 2MI \geq V_2(12) + V_2(13) + V_2(23) \end{cases} \end{cases} \end{cases}$$

That is to say

$$\begin{cases} 2MI \geq \max\left\{V_{112}^2, V_{122}^2\right\} \\ \begin{cases} \begin{cases} 3MI \geq V_{222}^2 + V_1(N) \\ 2MI \geq V_{111}^2 \end{cases} \\ \text{or} \\ \begin{cases} 3MI \geq V_{111}^2 + V_2(N) \\ 2MI \geq V_{222}^2 \end{cases} \end{cases} \end{cases}$$

As a consequence we prove the following interesting proposition.

Proposition 1 *Let* **E** *be a two-agent three-good economy such that V_1 and V_2 are superadditive. The economy has equilibrium if and only if*

$$\begin{cases} 2MI \geq \max\left\{V_{112}^2, V_{122}^2\right\} \\ \begin{cases} \begin{cases} 3MI \geq V_{222}^2 + V_1(N) \\ 2MI \geq V_{111}^2 \end{cases} \\ or \\ \begin{cases} 3MI \geq V_{111}^2 + V_2(N) \\ 2MI \geq V_{222}^2 \end{cases} \end{cases} \end{cases}$$

Note that, if the market is superadditive, it is normal that the equilibrium conditions occur in "large" coalitions. In Proposition 1 the values of single component baskets do not appear explicitly (only implicitly when monotonic and superadditive). This is natural, given superadditivity, and also Proposition 2 of Bikhchandani and Mamer (1997) , which seeks for MI to be as large as possible.

On the other hand, in view of Proposition 1, and given that $MI \geq V_j(N)$, $j = 1, 2$, a sufficient condition for a superadditive market to have equilibrium is

$$2MI \geq \max\left\{V_{122}^2, V_{112}^2, V_{111}^2, V_{222}^2\right\} \tag{12}$$

This condition is almost necessary, in the sense that the only possibility with equilibrium that does not verify the condition is when, for some $j \neq k$,

$$V_{jjj}^2 > 2MI$$

and

$$3MI \leq V_{jjj}^2 + V_k(N)$$

Using Proposition 1 and implementing a simple numerical simulation, the probability that Eq. (12) is not a necessary condition for the existence of equilibrium is very small, i.e., 0.024 . The sufficient equilibrium condition in Eq. (12) simply tells us that, if we duplicate the goods and distribute the three baskets (jk) between the two agents, then the joint utility divided by two does not increase the initial joint utility. Denoting as $U(N) = MI$ the initial joint utility that can be achieved with N goods, and denoting as $U(2N)$ the joint utility provided by the duplicated N goods, then the condition for sufficient equilibrium is

$$2U(N) \geq U(2N) \Leftrightarrow U(N) \geq \frac{1}{2}U(2N)$$

which evokes diminishing returns to scale for the total market utility function.

Remark 4 *When individual utilities show complementarity between goods, equilibrium exists when the joint utility function for the market shows substitutability in the sense described above.*

5.2.3 Complementarity versus substitutability

In a given economy, goods are complementary when utility functions are superadditive, and are substitutes when utility functions are subadditive. For the case of two agents and convex utility functions (hence, superadditive), there is always equilibrium, but this is not the case for concave utility functions (hence, subadditive). This fact may lead one to think that equilibrium is more likely with complementary goods than with substitute goods. However, here we not only prove that this is true, but also quantify the corresponding probabilities. The problem is delicate because there are "more" superadditive functions than subadditive functions. In view of Proposition 1 it is clear that, if the utility functions of a market with equilibrium are superadditive, then we can indefinitely increase the value of the basket of all goods, N (at least for one agent), with the utility functions remaining superadditive and the market continuing to have equilibrium. This fact notoriously distorts the probability of equilibrium existing in this case, which obviously does not happen for the subadditive case. Our way of implementing the numerical simulation takes this issue into account and, to some extent, resolves it. Our two-agent three-good economy is represented as

	1	2	3	12	13	23	123
V_1	α_1	α_2	α_3	$\alpha_1 + \alpha_2 + \alpha_{12}$	$\alpha_1 + \alpha_3 + \alpha_{13}$	$\alpha_2 + \alpha_3 + \alpha_{23}$	$\alpha_1 + a_2 + \alpha_3 + \alpha_{123}$
V_2	β_1	β_2	β_3	$\beta_1 + \beta_2 + \beta_{12}$	$\beta_1 + \beta_3 + \beta_{13}$	$\beta_2 + \beta_3 + \beta_{23}$	$\beta_1 + \beta_2 + \beta_3 + \beta_{123}$

It is clear that, if the constants α_i, β_i are positive, then V_1 and V_2 are superadditive if and only if

$$0 \leq \alpha_{ij} \leq \alpha_{123}$$
$$0 \leq \beta_{ij} \leq \beta_{123}$$

and are subadditive if and only if

$$0 \geq \alpha_{ij} \geq \alpha_{123}$$
$$0 \geq \beta_{ij} \geq \beta_{123}$$

We now use numerical simulation to determine whether equilibrium is more or less likely to exist in the case of complementarity or substitutability. We first randomly simulate the constants $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ in an interval $(0, T)$, where the value of T is irrelevant. We then randomly simulate the values of α_{ij}, β_{ij} such that all are negative and the utility functions are monotonic. With these values for α_{ij}, β_{ij} , in the same way we simulate a_{123} and β_{123}, i.e., $\alpha_{123} \leq \min \{\alpha_{ij}/i, j \in N\}, \beta_{123} \leq \min \{\beta_{ij}/i, j \in N\}$ such that V_1, V_2 are monotonic. We thus have a pair of monotone and subadditive functions. To this pair of functions we associate the obtained superadditive functions by simply changing the signs of $\alpha_{ij}, \beta_{ij}, \alpha_{123}, \beta_{123}$. We now examine whether or not the pair of subadditive functions and the pair of superadditive functions have equilibrium. Repeating this process 20000 times, we obtain equilibrium probabilities for a subadditive economy and a superadditive economy of 0.962 and of 0.982, respectively. These data confirm our suspicion.

Remark 5 *While equilibrium is more likely with complementary goods than with substitute goods, the difference is very small.*

5.2.4 A particular case

We now consider a rather general situation in which the conditions of Theorem 3 are greatly simplified, both in the substitute good and complementary good cases. Note that for the complementary good, Proposition 1 already gives a simple expression for the equilibrium condition. However, we have been less successful in obtaining an equivalent result for the substitute good, and we therefore have to lose some generality in our model. For this purpose, we consider a two-agent three-good market in which both agents equally value baskets of a single good. The utility functions are

	1	2	3	12	13	23	123
V_1	δ_1	δ_2	δ_3	$\delta_1 + \delta_2 + \alpha_{12}$	$\delta_1 + \delta_3 + \alpha_{13}$	$\delta_2 + \delta_3 + \alpha_{23}$	$\delta_1 + \delta_2 + \delta_3 + \alpha_{123}$
V_2	δ_1	δ_2	δ_3	$\delta_1 + \delta_2 + \beta_{12}$	$\delta_1 + \delta_3 + \beta_{13}$	$\delta_2 + \delta_3 + \beta_{23}$	$\delta_1 + \delta_2 + \delta_3 + \beta_{123}$

We assume that the unit valuations are positive, $\delta_i \geq 0 \ \forall i \in N$, and that both V_1 and V_2 are monotonic. Hence, for all $i, j, k \in N$, distinct from each other (we assume that the considered subscripts are always distinct), we need to verify

$$\begin{gathered}
\delta_i \leq \delta_i + \delta_j + \alpha_{ij} \Leftrightarrow 0 \leq \delta_j + \alpha_{ij} \\
\delta_i \leq \delta_i + \delta_j + \beta_{ij} \Leftrightarrow 0 \leq \delta_j + \beta_{ij} \\
\delta_i + \delta_j + \alpha_{ij} \leq \delta_i + \delta_j + \delta_k + \alpha_{123} \Leftrightarrow \alpha_{ij} \leq \delta_k + \alpha_{123} \\
\delta_i + \delta_j + \beta_{ij} \leq \delta_i + \delta_j + \delta_k + \beta_{123} \Leftrightarrow \beta_{ij} \leq \delta_k + \beta_{123}
\end{gathered}$$

Clearly, to say that V_1 and V_2 are superadditive is equivalent to saying that

$$\begin{gathered}
\alpha_{123} \geq \alpha_{ij} \geq 0 \ \forall i, j \in N \\
\beta_{123} \geq \beta_{ij} \geq 0 \ \forall i, j \in N
\end{gathered}$$

Denoting $\delta = \delta_1 + \delta_2 + \delta_3$, it is clear that

$$MI = \delta + \max\{\alpha_{123}, \beta_{123}\}$$

Easily obtained in this case is the equilibrium condition of Proposition 1

$$\begin{cases}
2\max\{\alpha_{123}, \beta_{123}\} \geq \max\{\alpha_{ij} + \alpha_{ik} + \beta_{jk}, \alpha_{ij} + \beta_{ik} + \beta_{jk} \ /i, j, k \in N\} \\
\begin{cases}
\begin{cases}
3\max\{\alpha_{123}, \beta_{123}\} \geq \beta_{12} + \beta_{13} + \beta_{23} + \alpha_{123} \\
2\max\{\alpha_{123}, \beta_{123}\} \geq \alpha_{12} + \alpha_{13} + \alpha_{23}
\end{cases} \\
\text{or} \\
\begin{cases}
3\max\{\alpha_{123}, \beta_{123}\} \geq \alpha_{12} + \alpha_{13} + \alpha_{23} + \beta_{123} \\
2\max\{\alpha_{123}, \beta_{123}\} \geq \beta_{12} + \beta_{13} + \beta_{23}
\end{cases}
\end{cases}
\end{cases}$$

Let us now see what happens when V_1 and V_2 are subadditive. The conditions for V_1 and V_2 are

$$\begin{gathered}
\alpha_{123} \leq \alpha_{ij} \leq 0 \ \forall i, j \in N \\
\beta_{123} \leq \beta_{ij} \leq 0 \ \forall i, j \in N
\end{gathered}$$

and it is verified that

$$MI = \delta + \max\{\alpha_{ij}, \beta_{ij} / i, j \in N\}$$

Applying Theorem 3, simple algebraic calculations lead us to

$$\begin{gathered}
x_1 \leq x_2 \Leftrightarrow \beta_{123} \leq 2\max\{\alpha_{ij}, \beta_{ij} / i, j \in N\} \\
y_1 \leq y_2 \Leftrightarrow \alpha_{123} \leq 2\max\{\alpha_{ij}, \beta_{ij} / i, j \in N\}
\end{gathered}$$

and the remaining conditions of the theorem are verified. We conclude that if V_1 and V_2 are subadditive, then equilibrium exists if and only if

$$\max\{\alpha_{123}, \beta_{123}\} \leq 2\max\{\alpha_{ij}, \beta_{ij} / i, j \in N\}$$

which gives us a very simple version of Proposition 1 for the substitute good case.

5.2.5 A unique utility function

Let us now study what Theorem 3 tells us when the two agents have the same utility function.

Proposition 2 *Let* $\mathbf{E}$ *be a two-agent three-good economy such that* $V_1 = V_2 = V$*. The economy has equilibrium if and only if we verify*

$$a)\, 2MI \geq V(12) + V(13) + V(23)$$
$$b)\, 2MI \geq V(1) + V(2) + V(3) + V(N)$$

Proof. See Appendix. ■

Interestingly, the existence condition given in Bikhchandani and Mamer (1997) concerning divisible allocations in the context $m = 2$, $n = 3$, $V_1 = V_2$ can be drastically simplified. In fact, what Proposition 2 assures us is that, to guarantee the existence of equilibrium, all we need to do is to verify Bikhchandani and Mamer (1997)'s condition for two very concrete divisible allocations

$$\left(\frac{1}{2}V(12), \frac{1}{2}V(13) + \frac{1}{2}V(23)\right)$$
$$\left(\frac{1}{2}V(1) + \frac{1}{2}V(2), \frac{1}{2}V(3) + \frac{1}{2}V(N)\right)$$

Using our Proposition 2 we now prove, for a simple two-agent three-good economy, that if the agents have the same concave utility function, then the economy, of necessity, has equilibrium.

Corollary 1 *Let* $\mathbf{E}$ *be a two-agent three-good economy such that* $V_1 = V_2 = V$*. If* V *is concave, then the economy has equilibrium.*

Proof. If V is indeed concave, then parts a) and b) of Proposition 2 are both verified. Clearly if V is concave

$$V(12) \leq V(1) + V(2)$$

then

$$V(12) + V(13) + V(23) \leq V(1) + V(23) + V(2) + V(13) \leq 2MI$$

which proves a). On the other hand, taking into account that

$$V(3) + V(N) \leq V(13) + V(23)$$

we can conclude that

$$V(1) + V(2) + V(3) + V(N) \leq V(1) + V(2) + V(13) + V(23) \leq 2MI$$

which proves b). This proves the corollary. ■

6 Concluding remarks

In this paper several significant contributions are made to the literature on economies with indivisibles. Our central result, Theorem 1, can be considered an alternative to the Bikhchandani and Mamer (1997) theorem widely cited in the literature. While an alternative approach is in itself meritorious, Theorem 1, is also essential to proving the remaining interesting results of our paper. As to whether Theorem 1 improves on the result of Bikhchandani and Mamer, it is true that to apply our theorem one must search for the vector r, and it is no less true that determining whether a cooperative set has an empty core requires proving that no balanced collection has a value greater than $V(N)$. To apply the result of Bikhchandani and Mamer, however, an analogous proof is necessary for a very much larger set of all divisible allocations.

Another contribution, in Section 4 on the theoretical application of Theorem 1, is that we prove Theorem 2. This proof significantly improves on the result given in Ma (1998), as we characterize the existence of equilibrium by a much more tractable cooperative game.

Finally, in characterizing, from Theorem 1, the existence of equilibrium in two-agent three-good economies, we obtain a number of interesting results. While it is true that the context is very particular, it is no less true that a characterization such as that of Theorem 3 is tremendously useful, especially for the purpose of contrasting properties that can then be extended to more general contexts. This will be the goal of future research.

One could consider extending Theorem 3 to four or more goods, but such an extension would not be simple. Just as the step from $n = 3$ to $n = 4$ in terms of the number of conditions to verify is frankly large, we believe that, if we tried to extend Theorem 3, something analogous would result from calculating the core of the cooperative game (Shapley, 1967).

Numerical simulations based on Theorem 3 suggest an interesting way of working with many mathematical properties. When we write A implies B but B does not imply A, to what extent can we numerically quantify this non-implication? In other words, how likely is it that B occurs and not A? How far is B from being a sufficient condition for A? Gross substitution, for instance, is a sufficient property for equilibrium to exist but, as we see here, in our particular context it is certainly far from being a necessary condition.

7 Appendix

Proof of Theorem 3:

(IF) Let us suppose that **a)** is verified. It is clear, taking

$$\bar{y} = \max\left\{y_1, x_1 - a_2, 2x_1 - c_2, \frac{x_1 - b_2}{2}\right\}$$

that $(x_1, \bar{y})$ would be excesses verifying all the conditions of Lemma 1 , and consequently, there would be equilibrium. The proof of part **b)** is analogous, and this proves the (**IF**) part.

(ONLY IF) Suppose now that the economy has equilibrium. Lemma 1 guarantees that there are (x_e, y_e) verifying the five conditions of the statement. It is therefore verified that

$$x_1 \leq x_e \leq x_2; \; y_1 \leq y_e \leq y_2 \tag{13}$$

$$\max\left\{y_1, x_e - a_2, 2x_e - c_2, \frac{x_e - b_2}{2}\right\} \leq y_e \leq \min\left\{y_2, x_e + a_1, 2x_e + b_1, \frac{x_e + c_1}{2}\right\} \tag{14}$$

$$\max\left\{x_1, y_e - a_1, \frac{y_e - b_1}{2}, 2y_e - c_1\right\} \leq x_e \leq \min\left\{x_2, y_e + a_2, \frac{y_e + c_2}{2}, 2y_e + b_2\right\} \tag{15}$$

We demonstrate by reductio ad absurdum. Let us suppose that neither **a)** nor **b)** in the statement is verified. In that case, Eq. (13) implies, of necessity, that

$$\max\left\{y_1, x_1 - a_2, 2x_1 - c_2, \frac{x_1 - b_2}{2}\right\} > \min\left\{y_2, x_1 + a_1, 2x_1 + b_1, \frac{x_1 + c_1}{2}\right\} \tag{16}$$

$$\max\left\{x_1, y_1 - a_1, \frac{y_1 - b_1}{2}, 2y_1 - c_1\right\} > \min\left\{x_2, y_1 + a_2, \frac{y_1 + c_2}{2}, 2y_1 + b_2\right\} \tag{17}$$

It is simple to prove that, since the four functions (thought of as functions of x) for which the maximum is taken are linear, then the maximum function of the four is always convex (as x increases, the maximum must be reached in function of the greater slope). For analogous reasons, the minimum function is always concave. Hence, the only possibility for Eqs. (14) and (16) to be verified simultaneously is that, in $x = x_1$, the maximum is reached in function of a smaller slope than that in which the minimum is reached. In other words, the only possibilities to consider are

	max	min
i)	y_1	$\frac{x_1+c_1}{2}$
ii)	y_1	$x_1 + a_1$
iii)	y_1	$2x_1 + b_1$
iv)	$\frac{x_1-b_2}{2}$	$x_1 + a_1$
v)	$\frac{x_1-b_2}{2}$	$2x_1 + b_1$
vi)	$x_1 - a_2$	$2x_1 + b_1$
vii)	$2x_1 - c_2$	$-$
viii)	$-$	y_2

The option *viii)* cannot be given, since, as already said, if

$$\max\left\{y_1, x_1 - a_2, 2x_1 - c_2, \frac{x_1 - b_2}{2}\right\} = 2x_1 - c_2 > \min\left\{y_2, x_1 + a_1, 2x_1 + b_1, \frac{x_1 + c_1}{2}\right\}$$

then $\forall x \geq x_1$, and we would have

$$\max\left\{y_1, x-a_2, 2x-c_2, \frac{x-b_2}{2}\right\} = 2x-c_2 > \min\left\{y_2, x+a_1, 2x+b_1, \frac{x+c_1}{2}\right\}$$

since the slope of the minimum function is always less than or equal to 2, and since Eq. (14) could never be verified. In an analogous way, *vii*), *vi*), and *v*) cannot be given either. Furthermore, reasoning in a similar way for condition **b)** of the statement, we obtain that the maxima and minima for Eqs. (16) and (17) can only be attained at

(16)

	max	min
i)	y_1	$\frac{x_1+c_1}{2}$
ii)	y_1	x_1+a_1
iii)	y_1	$2x_1+b_1$
iv)	$\frac{x_1-b_2}{2}$	x_1+a_1

(17)

	max	min
i)	x_1	$\frac{y_1+c_2}{2}$
ii)	x_1	y_1+a_2
iii)	x_1	$2y_1+b_2$
iv)	$\frac{y_1-b_1}{2}$	y_1+a_2

The rest of the proof is reduced to demonstrating that none of the 16 possible combinations from the two tables above can occur.

1. Case $i), i)$. In this case we have

$$y_1 > \frac{x_1+c_1}{2}$$
$$x_1 > \frac{y_1+c_2}{2}$$

Hence

$$y_1 > \frac{x_1+c_1}{2} > \frac{\frac{y_1+c_2}{2}+c_1}{2}$$

In other words

$$3y_1 > 2c_1 + c_2$$

from where

$$3y_1 > 4MI - 2V_1(12) - 2V_1(13) - 2V_1(23) + 2MI - V_2(12) - V_2(13) - V_2(23)$$

If $y_1 = 0$ we obtain

$$MI < \frac{1}{3}V_1(12) + \frac{1}{3}V_1(13) + \frac{1}{3}V_1(23) + \frac{1}{6}V_2(12) + \frac{1}{6}V_2(12) + \frac{1}{6}V_2(12)$$

which would give us a valid divisible allocation with a value greater than the optimal indivisible allocation (MI). However, according to Proposition 2 of Bikhchandani and Mamer (1997) this cannot be, since we know that the economy has equilibrium. If

$$y_1 = V_{122}^1 - MI$$

we obtain

$$3V_{122}^{1} - 3MI > 4MI - 2V_1(12) - 2V_1(13) - 2V_1(23) + 2MI - V_2(12) - V_2(13) - V_2(23)$$

$$9MI < 2V_1(12) + 2V_1(13) + 2V_1(23) + V_2(12) + V_2(13) + V_2(23) + 3V_1(i) + 3V_2(j) + 3V_2(k)$$

$$MI < \tfrac{2}{9}V_1(12) + \tfrac{2}{9}V_1(13) + \tfrac{2}{9}V_1(23) + \tfrac{1}{9}V_2(12) + \tfrac{1}{9}V_2(13) + \tfrac{1}{9}V_2(23) + \tfrac{1}{3}V_1(i) + \tfrac{1}{3}V_2(j) + \tfrac{1}{3}V_2(k)$$

Therefore, as before, we would obtain a valid divisible allocation with a value greater than MI, which we already know cannot be. Similar arguments prove that none of the remaining 15 cases can be given either. This completes the proof. ∎

Proof of Proposition 2:

(ONLY IF) Suppose that our economy has equilibrium. Applying Theorem 3 and taking into account that $V_1 = V_2$ we have

$$a_1 = a_2 = a,\ b_1 = b_2 = b,\ c_1 = c_2 = c,\ x_1 = y_1,\ x_2 = y_2$$

Hence, both equilibrium conditions coincide and we are left with

$$a)\, x_1 \leq x_2,\ \max\left\{x_1, x_1 - a, 2x_1 - c, \frac{x_1 - b}{2}\right\} \leq \min\left\{x_2, x_1 + a, 2x_1 + b, \frac{x_1 + c}{2}\right\}. \qquad (18)$$

If $x_1 \leq x_2$, then it must be verified that

$$x_1 = \max\{0, V(1) + V(2) + V(3) - MI\} \leq \min\{2MI - V(12) - V(13) - V(23), MI - V(N)\} = x_2$$

It also has to be verified that

$$0 \leq 2MI - V(12) - V(13) - V(23)$$

which is equivalent to part a) of Proposition 2. Also to be verified is

$$V(1) + V(2) + V(3) - MI \leq MI - V(N)$$

which is equivalent to part b). Hence, the **(ONLY IF)** part is proved.

(IF) Assuming now that a) and b) of Proposition 2 are verified, then it will suffice to prove that the two inequalities are verified in Eq.(18). Let us first verify that $x_1 \leq x_2$, or what amounts to the same, that

$$x_1 = \max\{0, V(1) + V(2) + V(3) - MI\} \leq$$

$$\min\{2MI - V(12) - V(13) - V(23), MI - V(N)\} = x_2$$

In fact, proving that the above inequality is verified is equivalent to proving four inequalities. As we see in (**ONLY IF**), two of these are equivalent to a) and b), and we will now see that the remaining two are always verified. The first one

$$0 \leq MI - V(N)$$

is obviously always true because $S = (N, \varnothing)$ is a feasible allocation. The second one

$$V(1) + V(2) + V(3) - MI \leq 2MI - V(12) - V(13) - V(23) \Leftrightarrow$$

$$V(1) + V(23) + V(2) + V(13) + V(3) + V(12) \leq 3MI$$

again is trivially verified, and this proves that $x_1 \leq x_2$. Let us now see that

$$\max\left\{x_1, x_1 - a, 2x_1 - c, \frac{x_1 - b}{2}\right\} \leq \min\left\{x_2, x_1 + a, 2x_1 + b, \frac{x_1 + c}{2}\right\} \tag{19}$$

We now test, depending on whether

$$x_1 = \max\{0, V(1) + V(2) + V(3) - MI\} = 0$$

or

$$x_1 = V(1) + V(2) + V(3) - MI$$

If $x_1 = 0$

$$V(1) + V(2) + V(3) - MI \leq 0$$

and then $b \geq 0$. From *a)* we verify $c \geq 0$, and since $S = (i, kl)$ is a feasible allocation for every partition (i, kl) of N, then

$$V(i) + V(kl) \leq MI \Rightarrow a \geq 0$$

Hence, Eq. (19) is reduced to

$$0 \leq x_2$$

which is trivially verified because, always, $0 = x_1 \leq x_2$. Let us now see what happens when

$$x_1 = V(1) + V(2) + V(3) - MI$$

Obviously, in such a case

$$x_1 = -b \geq 0,\ c \geq 0,\ a \geq 0,\ x_2 \geq 0$$

and Eq. (19) is reduced to

$$\max\left\{x_1, x_1 - a, x_1 - c, \frac{x_1 - b}{2}\right\} = x_1 \leq \min\left\{x_2, \frac{x_1 + c}{2}\right\}$$

However

$$
\begin{aligned}
x_1 \leq & \frac{x_1 + c}{2} \Leftrightarrow x_1 \leq c \\
\Leftrightarrow & V(1) + V(2) + V(3) - MI \leq 2MI - V(12) - V(13) - V(23) \\
\Leftrightarrow & V(1) + V(23) + V(2) + V(13) + V(3) + V(12) \leq 3MI
\end{aligned}
$$

which is always true. As we already know that $x_1 \leq x_2$, Eq. (19) is verified. This concludes the proof. ■

Compliance with Ethical Standards:

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