POSITIVE PERIODIC SOLUTIONS FOR LOTKA-VOLTERRA SYSTEMS WITH A GENERAL ATTACK RATE

CRISTINA LOIS-PRADOS AND RADU PRECUP

Abstract. The paper deals with a non-autonomous Lotka-Volterra type system, which in particular may include logistic growth of the prey population and hunting cooperation between predators. We focus on the existence of positive periodic solutions by using an operator approach based on the Krasnosel'skii homotopy expansion theorem. We give sufficient conditions in order that the localized periodic solution does not reduce to a steady state. Particularly, two typical expression for the functional response of predators are discussed.

Key words: non-autonomous Lotka-Volterra system, hunting cooperation, logistic growth, periodic solution, existence, localization

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1. Introduction

In this paper we consider non-autonomous Lotka-Volterra type systems with a general attack rate

$$\begin{cases} x' = a(t)xg(x) - \varphi(t,x,y)xy \\ y'(t) = -b(t)y + c(t)\varphi(t,x,y)xy, \end{cases} \tag{1.1}$$

where $a, b, c \in \mathcal{C}(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic for the same period $\omega > 0$, $a, b \not\equiv 0$, $\min_{s\in[0,\omega]} c(s) > 0$; $\varphi \in \mathcal{C}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ is such that $\varphi(\cdot, x, y)$ is ω-periodic for every $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$; and $g \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ is decreasing, with $g(0) \leq 1$.

In particular, we consider

$$\begin{array}{ll} g(x) \equiv 1 & \text{(linear growth of the prey)}, \quad \text{or} \\ g(x) \equiv 1 - \frac{x}{K} & \text{(logistic growth of the prey)} \end{array}$$

and one of the following expressions for the attack rate φ,

$$\varphi_I(t,x,y) \equiv \lambda(t) + \alpha(t)\,y,$$

$$\varphi_{II}(t,x,y) \equiv \frac{\lambda(t) + \alpha(t)\,y}{1 + \beta(t)(\lambda(t) + \alpha(t)\,y)\,x},$$

both used in the literature to simulate cooperation between predators [12]. Here, we assume that α, β, $\lambda \in \mathcal{C}(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic functions and λ, $\beta \not\equiv 0$.

Lotka-Volterra type systems are commonly use to describe interactions between two species, prey and predator. In the autonomous case, these models have a Kolmogorov structure, being of the form

$$\begin{cases} x' = x\,F(x,y) \\ y' = y\,G(x,y), \end{cases}$$

and most of them satisfy the following conditions:

$$F_y(x,y) > 0, \quad G_x(x,y) > 0 \text{ and } \quad G_y(x,y) \leq 0$$

(see, e.g., [3, Section 5.4]). For non-autonomous Kolmogorov type systems, we refer the reader to the paper by Zanolin [15].

The original model given by Lotka [8] and Volterra [14] is the following one

$$\begin{cases} x' = ax \, - \, \lambda xy \\ y' = -by \, + \, c\lambda xy. \end{cases} \tag{1.2}$$

For a historical note on this classical model, see [1]. As suggested by Volterra himself, a more realistic prey growth is the logistic one, as is the case of Rosenzweig-MacArthur model [11], namely

$$\begin{cases} x' = ax\left(1 - \frac{x}{K}\right) - \phi(x)y \\ y' = -by + c\phi(x)y. \end{cases}$$

Some generalizations of the Rosenzweig-MacArthur model are given in [6], where in particular, it is consider the logistic growth for both prey and predator populations (see also [4]).

As regards the function φ involved in the functional response of predators to the change of densities in (1.1), we can mention the paper by Berec [2], where φ has the form

$$\varphi(x,y) = \frac{\lambda + \alpha y}{1 + \beta(y)(\lambda + \alpha y)x}, \tag{1.3}$$

which expresses the effects of hunting cooperation between predators. Also, in a recent paper by Alves & Hilker [12], there are used the particular expressions of φ_I, φ_{II} involving constant coefficients, namely

$$\varphi(x,y) = \lambda + \alpha y \quad (\lambda > 0, \;\; \alpha \geq 0)$$

and

$$\varphi(x,y) = \frac{\lambda + \alpha y}{1 + \beta(\lambda + \alpha y)x} \quad (\beta > 0).$$

In this paper, we consider the more general system (1.1), which is non-autonomous, involves a general prey growth g, and a general functional response of predators. We proceed as follows:

In Section 2, we give the integral version of the system, we state the Krasnosel'skii type homotopy fixed point theorem, which is our main tool, and we give some useful notations.

In Section 3, we first study the steady states of the system giving a necessary and sufficient condition for the existence of positive equilibriums. Then, in Subsection 3.2, we state and prove the main result about the existence and localization of periodic solutions, whose sufficient conditions are particularized for the cases of linear and logistic prey growth. In both situations, the possibility of localization is discussed separately for $\varphi = \varphi_I$ and $\varphi = \varphi_{II}$. The method that we use is based on a completely different topological argument as compared to the one in [13], where only a particular case of our system is studied. Our proof appears to be simpler and more natural. In Subsection 3.3, we give sufficient conditions in order that the localized periodic solution does not reduce to a steady state, while in Subsection 3.4, a positivity result about nonconstant periodic solutions is included. Next, in Subsection 3.5, the existence results are improved for the case that φ does not depend on time. Finally, the autonomous system is discussed as a very particular case.

2. Preliminaries

We are interested in ω-periodic solutions of system (1.1) and for this purpose we use an operator approach as in [10] (see also [13]). This approach is based on the fact that for every ω-periodic functions f_1, $f_2 \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $a, b \in \mathcal{C}(\mathbb{R},\mathbb{R}_+)$, $a, b \not\equiv 0$, there is a unique ω-periodic solution (x, y) of the system

$$\begin{cases} x' = a(t)x - f_1(t) \\ y' = -b(t)y + f_2(t), \end{cases} \tag{2.1}$$

given by

$$\begin{cases} x(t) = \int_t^{t+\omega} H_1(t,s)\, f_1(s)\, ds \\ y(t) = \int_t^{t+\omega} H_2(t,s)\, f_2(s)\, ds, \end{cases}$$

where

$$H_1(t,s) = \frac{e^{-\int_t^s a(\tau)d\tau}}{1 - e^{-\int_0^\omega a(\tau)d\tau}}, \quad H_2(t,s) = \frac{e^{\int_t^s b(\tau)d\tau}}{e^{\int_0^\omega b(\tau)d\tau} - 1} \quad ((t,s) \in \mathbb{R} \times \mathbb{R})\,.$$

Now, if instead of linear system (2.1), we consider the nonlinear system

$$\begin{cases} x' = a(t)x - f_1(t,x,y) \\ y' = -b(t)y + f_2(t,x,y), \end{cases}$$

then its ω-periodic solutions are exactly the ω-periodic solutions of the nonlinear integral system

$$\begin{cases} x(t) = \int_t^{t+\omega} H_1(t,s)\, f_1(s, x(s), y(s))\, ds \\ y(t) = \int_t^{t+\omega} H_2(t,s)\, f_2(s, x(s), y(s))\, ds, \end{cases}$$

that can be studied as a fixed point equation.

In this paper, the fixed point arguments are based on the following Krasnosel'skii type result in cones (see, for example, [9, Theorem 10.8]). We recall that by a cone C in a Banach space X, we mean a closed convex set such that $\lambda C \subset C$, for every $\lambda \in \mathbb{R}_+$, and $C \cap (-C) = \{0\}$.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space, $C \subset X$ a cone, $0 < r < R$ and $N : C_R \longrightarrow C$ a compact operator, where $C_R = \{u \in C : \ \|u\| \leq R\}$. Assume that*

- (E$_1$) *$N(u) \neq \lambda u$, for every $u \in C$, $\|u\| = r$ and all $\lambda > 1$,*
- (E$_2$) *there exists $v \in C \setminus \{0\}$ such that $u - N(u) \neq \lambda v$, for every $u \in C$, $\|u\| = R$ and all $\lambda > 0$.*

Then, N has a fixed point u in C with $r \leq \|u\| \leq R$.

We conclude this preliminary section by the list of notations that are useful to simplify the computations related to the application of Theorem 2.1 to our system (1.1).

$$\underline{a} := \min_{s\in[0,\omega]} a(s), \quad \underline{b} := \min_{s\in[0,\omega]} b(s), \quad \underline{c} := \min_{s\in[0,\omega]} c(s),$$

$$\overline{a} := \max_{s\in[0,\omega]} a(s), \quad \overline{b} := \max_{s\in[0,\omega]} b(s), \quad \overline{c} := \max_{s\in[0,\omega]} c(s),$$

$$
\begin{aligned}
m_1 &:= \min_{(t,s)\in[0,\omega]\times[t,t+\omega]} H_1(t,s) = \frac{1}{e^{\int_0^\omega a(\tau)d\tau}-1},\\
\underline{m}_1 &:= \min_{t\in[0,\omega]} \int_t^{t+\omega} H_1(t,s)\,ds,\\
m_2 &:= \min_{(t,s)\in[0,\omega]\times[t,t+\omega]} H_2(t,s) = \frac{1}{e^{\int_0^\omega b(\tau)d\tau}-1},\\
\underline{m}_2 &:= \min_{t\in[0,\omega]} \int_t^{t+\omega} H_2(t,s)\,ds,\\
M_1 &:= \max_{(t,s)\in[0,\omega]\times[t,t+\omega]} H_1(t,s) = \frac{1}{1-e^{-\int_0^\omega a(\tau)d\tau}},\\
\overline{M}_1 &:= \max_{t\in[0,\omega]} \int_t^{t+\omega} H_1(t,s)\,ds,\\
M_2 &:= \max_{(t,s)\in[0,\omega]\times[t,t+\omega]} H_2(t,s) = \frac{1}{1-e^{-\int_0^\omega b(\tau)d\tau}},\\
\overline{M}_2 &:= \max_{t\in[0,\omega]} \int_t^{t+\omega} H_2(t,s)\,ds,\\
q_1 &:= \frac{m_1}{M_1}, \quad q_2 := \frac{m_2}{M_2},\\
m_3 &:= q_1\, q_2\, \min\{m_1, \underline{c}\, m_2\}, \quad M_3 := \max\{M_1, \overline{c}\, M_2\},\\
\underline{m}_3 &:= q_1\, q_2\, \min\{\underline{m}_1, \underline{c}\, \underline{m}_2\}, \quad \overline{M}_3 := \max\{\overline{M}_1, \overline{c}\, \overline{M}_2\}.
\end{aligned}
$$

We shall also use the notation $\|x\|_\infty$ for the max norm of $x \in \mathcal{C}([0,\omega],\mathbb{R})$, i.e.,

$$\|x\|_\infty = \max_{t\in[0,\omega]} |x(t)|.$$

3. Main results

3.1. **Steady states.** We begin by looking for the steady states of system (1.1), that is for points $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$
\begin{aligned}
x_0\,(a(t)g(x_0) - \varphi(t,x_0,y_0)y_0) &= 0,\\
y_0(c(t)\varphi(t,x_0,y_0)x_0 - b(t)) &= 0,
\end{aligned}
\tag{3.1}
$$

for all $t \in \mathbb{R}$.

It is clear that $(0,0)$ is a solution of (3.1). In the following, we distinguish three cases:

Case I: $x_0 = 0$, $y_0 > 0$. Under this conditions, the first equation in (3.1) is obviously satisfied, while from the second one we have $y_0 b(t) = 0$ for every $t \in [0,\omega]$, which is not possible for $y_0 > 0$ and $b \not\equiv 0$. Therefore, there are no steady states of the type $(0, y_0)$, with $y_0 > 0$.

Case II: $x_0 > 0$, $y_0 = 0$. Now the second equation in (3.1) trivially holds, while the first one gives $a(t)g(x_0) = 0$ for all $t \in [0,\omega]$. As $a \not\equiv 0$, one must have $g(x_0) = 0$. Therefore, a point of the form $(x_0, 0)$, $x_0 > 0$ is a steady state if and only if $g(x_0) = 0$.

Case III: $x_0 > 0$, $y_0 > 0$. Under this situation, system (3.1) is equivalent to

$$\varphi(t,x_0,y_0) = \frac{a(t)g(x_0)}{y_0} = \frac{b(t)}{c(t)x_0} \quad \text{for every} \quad t \in [0,\omega].$$

The conclusions about the steady states of system (1.1) are collected in the following proposition.

Proposition 3.1. *A point* $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ *is a steady state of system* (1.1), *if and only if one of the following conditions holds:*

- (a) $x_0 = 0$ *and* $y_0 = 0$*;*
- (b) $x_0 > 0$, $g(x_0) = 0$ *and* $y_0 = 0$*;*
- (c) $x_0 > 0$, $y_0 > 0$ *and*

$$\varphi(t, x_0, y_0) = \frac{a(t)g(x_0)}{y_0} = \frac{b(t)}{c(t)x_0} \quad \textit{for every} \ \ t \in [0, \omega]. \tag{3.2}$$

According to this proposition, in case of considering the linear growth of the prey population, case (b) is not possible, and steady states of the form (x_0, y_0) with x_0, $y_0 > 0$ exist if and only if

$$\varphi(t, x_0, y_0) = \frac{a(t)}{y_0} = \frac{b(t)}{c(t)x_0} \quad \text{for every} \ \ t \in [0, \omega]. \tag{3.3}$$

If one considers the logistic growth of the prey population, then from case (b) we have the steady state $(K, 0)$, and steady states of the form (x_0, y_0) with x_0, $y_0 > 0$ exist if and only if

$$\varphi(t, x_0, y_0) = \frac{a(t)\left(1 - \frac{x_0}{K}\right)}{y_0} = \frac{b(t)}{c(t)x_0} \quad \text{for every} \ \ t \in [0, \omega]. \tag{3.4}$$

Coming back to the general system (1.1), let us note that if there is not any constant $k > 0$ such that

$$a(t) = k\,\frac{b(t)}{c(t)} \quad \text{for all} \ \ t \in [0, \omega], \tag{3.5}$$

then the system has no steady states (x_0, y_0) with $x_0, y_0 > 0$. Therefore, under condition (3.5), the orbits of all ω-periodic solutions (x, y), with $x(t)$, $y(t) > 0$ for every $t \in [0, \omega]$, do not reduce to points.

3.2. **Existence of periodic solutions.** In this section, we prove the existence of ω-periodic solutions of system (1.1). To this aim, we introduce the following conditions on φ:

- (i) there exists $\eta \in \mathcal{C}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ such that
  $\eta(\cdot, x, y)$ is ω-periodic for every $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$,
  $\eta(t, \cdot, y)$, $\eta(t, x, \cdot)$ are increasing functions for every (t, y), $(t, x) \in \mathbb{R} \times \mathbb{R}_+$,
  $\varphi(t, x, y) \le \eta(t, x, y)$ for every $(t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$,
- (ii) there exists $\psi \in \mathcal{C}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+)$ such that
  $\psi(\cdot, z)$ is ω-periodic for every $z \in \mathbb{R}_+$,
  $\psi(t, \cdot)$ is decreasing for every $t \in \mathbb{R}$,
  $\varphi(t, x, y) \ge \psi(t, x + y)$ for every $(t, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$.

The main result of this paper is the following

Theorem 3.1. *Let conditions* (i) *and* (ii) *hold. If there exist* r, $R \in \mathbb{R}$, $0 < r < R$ *such that*

$$1 \ge \overline{a}\,\overline{M}_1\,(1 - g(r))\,, \tag{3.6}$$

$$\overline{M}_1\,\overline{a}\,(1 - g(r)) + M_3\,r \int_0^\omega \eta(s, r, r)ds \le 2, \tag{3.7}$$

$$R\int_0^\omega \psi(s,R)ds \geq \frac{2}{m_3}, \tag{3.8}$$

then system (1.1) *has an ω-periodic solution $(x,y)\in C$ such that*

$$r \leq \|(x,y)\| = \|x\|_\infty + \|y\|_\infty \leq R.$$

Proof. We apply Theorem 2.1 in the Banach space

$$X_\omega := \left\{(x,y)\in \mathcal{C}(\mathbb{R},\mathbb{R})^2 : \ x(t)=x(t+\omega), \ \ y(t)=y(t+\omega) \ \text{ for every } t\in\mathbb{R}\right\},$$

endowed with the norm

$$\|(x,y)\| := \|x\|_\infty + \|y\|_\infty$$

and with the cone

$$C := \{(x,y)\in X_\omega : \ x(t)\geq q_1\|x\|_\infty, \ \ y(t)\geq q_2\|y\|_\infty \ \text{ for every } t\in\mathbb{R}\},$$

to the operator $N=(N_1,N_2)$, where

$$\begin{aligned} N_1(x,y)(t) &:= \int_t^{t+\omega} H_1(t,s)\left[a(s)\,x(s)(1-g(x(s))) + \varphi(s,x(s),y(s))\,x(s)\,y(s)\right]ds, \\ N_2(x,y)(t) &:= \int_t^{t+\omega} H_2(t,s)c(s)\varphi(s,x(s),y(s))\,x(s)\,y(s)ds. \end{aligned} \tag{3.9}$$

As shown in Preliminaries, the ω-periodic solutions of system (1.1) are the fixed points in X_ω of the operator N.

First note that since a, c, φ, $1-g$ and H_1, H_2 are nonnegative functions, one has $N(C)\subset C$. In addition, the compactness of N immediately follows from the Arzelà–Ascoli theorem. It remains to prove that conditions (E_1) and (E_2) hold, where the element $v\in C\setminus\{0\}$ is chosen to be any (x_0,y_0) with x_0, $y_0>0$.

We start by proving condition (E_1), which in our case reads as follows

$$(N_1(x,y),N_2(x,y)) \neq \lambda(x,y) \quad \text{for every } \ (x,y)\in C, \ \|(x,y)\|=r \ \text{ and all } \ \lambda>1. \tag{3.10}$$

To this aim, we consider three cases:

(*a*) Assume $x\equiv 0$, $y\not\equiv 0$. Then condition (3.10) trivially holds since $N_1(x,y), N_2(x,y)\equiv 0$.

(*b*) If $x\not\equiv 0$ and $y\equiv 0$, then $N_2(x,y)\equiv 0$ and (3.10) reduces to

$$N_1(x,0)\neq \lambda x \quad \text{for all } \ (x,0)\in C, \ \|x\|_\infty = r \ \text{ and all } \ \lambda>1. \tag{3.11}$$

To proof this, assume the contrary, namely that there exist $(x,0)\in C$, $\|x\|_\infty=r$ and $\lambda>1$ such that

$$N_1(x,0)(t)=\lambda x(t) \quad \text{for every } \ t\in[0,\omega].$$

Let $t_0\in[0,\omega]$ be such that $x(t_0)=\|x\|_\infty = r>0$, then also using the property that $-g$ is increasing, one has

$$\begin{aligned} r = x(t_0) < \lambda x(t_0) &= \int_{t_0}^{t_0+\omega} H_1(t_0,s)a(s)x(s)\left(1-g(x(s))\right)ds \\ &\leq \overline{a}\,r\,(1-g(r))\int_{t_0}^{t_0+\omega} H_1(t_0,s)\,ds \leq \overline{a}\,r\,(1-g(r))\,\overline{M}_1. \end{aligned}$$

Dividing by $r>0$ yields $1<\overline{a}\,(1-g(r))\,\overline{M}_1$, which contradicts our hypothesis (3.6). Thus, condition (3.11) holds.

(*c*) Finally, we prove that condition (3.10) holds for $x,\, y \not\equiv 0$. If it does not hold, then there exists such a pair $(x,y) \in C$, $\|(x,y)\| = r$ and $\lambda > 1$ with

$$N_1(x,y)(t) = \lambda x(t), \quad N_2(x,y)(t) = \lambda y(t) \ \text{ for every } \ t \in [0,\omega].$$

Let $t_0 \in [0,\omega]$ be such that $\|x\|_\infty = x(t_0)$. Then also using condition (i) over φ, one has

$$\begin{aligned} \|x\|_\infty &= x(t_0) < \lambda x(t_0) = N_1(x,y)(t_0) \\ &= \int_{t_0}^{t_0+\omega} H_1(t_0,s)[a(s)x(s)(1-g(x(s))) + \varphi(s,x(s),y(s))\, x(s)y(s)]ds \\ &\le \overline{a}\|x\|_\infty(1-g(\|x\|_\infty))\overline{M}_1 + \|x\|_\infty\, \|y\|_\infty\, M_1 \int_{t_0}^{t_0+\omega} \eta(s,\|x\|_\infty,\|y\|_\infty)ds. \end{aligned} \tag{3.12}$$

After dividing by $\|x\|_\infty$ and using the fact that $\|x\|_\infty,\, \|y\|_\infty < \|(x,y)\| = r$, it gives

$$1 < \overline{a}(1-g(r))\overline{M}_1 + \|y\|_\infty\, M_1 \int_0^\omega \eta(s,r,r)ds. \tag{3.13}$$

Similarly, from $N_2(x,y) = \lambda y$, we obtain

$$1 < \|x\|_\infty \overline{c} M_2 \int_0^\omega \eta(s,r,r)ds. \tag{3.14}$$

Now, adding (3.13) and (3.14) yields

$$2 < \overline{a}(1-g(r))\overline{M}_1 + M_3(\|x\|_\infty + \|y\|_\infty) \int_0^\omega \eta(s,r,r)ds,$$

which in virtue of $\|x\|_\infty + \|y\|_\infty = r$ contradicts our assumption (3.7). Therefore, condition (3.10) is satisfied.

Now, we prove condition (E_2), which reads as follows

$$(x,y) - (N_1(x,y), N_2(x,y)) \ne \lambda(x_0,y_0) \quad \text{for every} \ \ (x,y) \in C,\ \|(x,y)\| = R, \quad \lambda > 0. \tag{3.15}$$

To this aim, we distinguish again three cases:

(a) If $x \equiv 0$ and $y \not\equiv 0$, then $N_1(x,y),\, N_2(x,y) \equiv 0$, so condition (3.15) trivially holds.

(b) If $x \not\equiv 0$ and $y \equiv 0$, then $N_2(x,y) \equiv 0$. Therefore condition (3.15) is satisfied since $0 < \lambda\, y_0$.

(c) Finally, it remains to consider the case when $x,\, y \not\equiv 0$. If condition (3.15) does not hold, then there exists a pair $(x,y) \in C$ with $\|(x,y)\| = R$ such that

$$x(t) > N_1(x,y)(t), \quad y(t) > N_2(x,y)(t) \quad \text{for every} \ \ t \in [0,\omega],$$

where we have used that $\lambda, x_0,\, y_0 > 0$.

On the one hand, for each $t \in [0, \omega]$, using condition (ii) over φ, one has

$$\begin{aligned}
\|x\|_\infty &\geq x(t) > N_1(x,y)(t) \\
&= \int_t^{t+\omega} H_1(t,s)[a(s)x(s)(1-g(x(s))) + \varphi(s,x(s),y(s))x(s)y(s)]ds \\
&\geq \int_t^{t+\omega} H_1(t,s)\varphi(s,x(s),y(s))\,x(s)\,y(s)ds \\
&\geq m_1q_1q_2\|x\|_\infty\|y\|_\infty \int_0^\omega \psi(s,x(s)+y(s))ds \\
&\geq m_1q_1q_2\|x\|_\infty\|y\|_\infty \int_0^\omega \psi(s,\|(x,y)\|)ds,
\end{aligned}$$

which after dividing by $\|x\|_\infty$ yields

$$1 > m_1q_1q_2\|y\|_\infty \int_0^\omega \psi(s,R)ds. \tag{3.16}$$

On the other hand, in a similar way, from $y(t) > N_2(x,y)(t)$ for every $t \in [0,\omega]$, we deduce that

$$1 > m_2\,\underline{c}\,q_1q_2\|x\|_\infty \int_0^\omega \psi(s,R)ds. \tag{3.17}$$

Now, by adding inequalities (3.16), (3.17) and using $\|x\|_\infty + \|y\|_\infty = \|(x,y)\| = R$, we obtain

$$2 > m_3R\int_0^\omega \psi(s,R)ds,$$

which contradicts our assumption (3.8). Thus, condition (3.15) is fulfilled.

Therefore, all the conditions of Theorem 2.1 being satisfied, the operator N has a fixed point $(x,y) \in C$ with $r \leq \|(x,y)\| \leq R$. This fixed point (x,y) is an ω-periodic solution of the Lotka-Volterra type system (1.1). □

Remark 3.1. *There exists a number $r > 0$ such that conditions (3.6) and (3.7) hold, if*

$$g(0) > 1 - \frac{1}{\overline{M}_1\overline{a}}. \tag{3.18}$$

Indeed, by using the continuity of g at 0, if (3.18) is satisfied, then there exists $r_0 > 0$ such that $g(r) \geq 1 - 1/\left(\overline{M}_1\overline{a}\right)$, or equivalently condition (3.6) holds for every $r \in (0, r_0)$.

From (3.18), we also have $g(0) > 1 - 2/\left(\overline{M}_1\overline{a}\right)$, or equivalently

$$(1-g(0))\overline{M}_1\overline{a} < 2,$$

which guarantees (3.7) for any small enough $r > 0$.

Notice that condition (3.18) is trivially satisfied when $g(0) = 1$, which is the case of both linear and logistic growth of the prey population.

We consider now the particular expression of g which correspond to the linear or logistic growth of the prey population, and we show how the conditions over $r, R > 0$ in Theorem 3.1 look. Moreover, we study the existence of such numbers r and R, when $\varphi \equiv \varphi_I$ or $\varphi \equiv \varphi_{II}$.

For that purpose, let us start by proving that φ_I and φ_{II} satisfy all the conditions previously required to a general φ. It is clear that both functions belong to $\mathcal{C}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$

and are ω-periodic in the first variable. Concerning conditions (i) and (ii), for function φ_I, we can take

$$\eta = \varphi_I \quad \text{and} \quad \psi \equiv \lambda,$$

while for function φ_{II}, we can set

$$\eta = \varphi_I \quad \text{and} \quad \psi(t,z) = \frac{\lambda(t)}{1+\beta(t)(\lambda(t)+\alpha(t)\,z)z}.$$

Additionally, we fix the following notations

$$\underline{\lambda} := \min_{s\in[0,\omega]} \lambda(s), \quad \overline{\lambda} := \max_{s\in[0,\omega]} \lambda(s), \quad \overline{\alpha} := \max_{s\in[0,\omega]} \alpha(s) \quad \text{and} \quad \overline{\beta} := \max_{s\in[0,\omega]} \beta(s).$$

3.2.1. *Linear growth.*

Corollary 3.1. *Assume that $g \equiv 1$ and conditions* (i), (ii) *over φ are satisfied. If there exist $r, R \in \mathbb{R}$, $0 < r < R$ such that*

$$r \int_0^\omega \eta(s,r,r)ds \le \frac{2}{M_3}, \tag{3.19}$$

and (3.8) *hold, then the system*

$$\begin{cases} x' = a(t)x - \varphi(t,x,y)xy \\ y' = -b(t)y + c(t)\varphi(t,x,y)xy \end{cases} \tag{3.20}$$

has an ω-periodic solution $(x,y) \in C$ such that

$$r \le \|(x,y)\| = \|x\|_\infty + \|y\|_\infty \le R.$$

Next, we give sufficient conditions for (3.19) and (3.8) to hold, for each one of the two particular expressions of φ given in the Introduction.

Case I: When $\varphi = \varphi_I$, conditions (3.19), (3.8) read as

$$r \int_0^\omega (\lambda(s) + \alpha(s)\,r)\;ds \le \frac{2}{M_3}, \quad R \ge \frac{2}{m_3 \int_0^\omega \lambda(s)\,ds} \tag{3.21}$$

and are respectively satisfied provided that

$$r\left(\overline{\lambda} + \overline{\alpha} r\right) \le \frac{2}{M_3\,\omega}, \quad R \ge \frac{2}{m_3\,\omega\,\underline{\lambda}}. \tag{3.22}$$

Therefore, under condition (3.22), which is satisfied for small enough r and sufficiently large R, Corollary 3.1 applies. Note that the existence of small enough $r > 0$ is proved also at the end of Remark 3.1. However, the expression in (3.22) tells us how to choose suitable values of r and R.

Remark 3.2. *We mention that in the particular case of $\alpha \equiv 0$, system* (3.20) *turns into*

$$\begin{cases} x' = a(t)x - \lambda(t)xy \\ y' = -b(t)y + c(t)\lambda(t)xy, \end{cases}$$

which was studied in [13] *by means of index theory. Even in this particular case, our result based on Theorem* 2.1 *gives a better localization of ω-periodic solutions, namely in the annular conical set*

$$C_{r,R} := \{(x,y) \in C : \; r \le \|x\|_\infty + \|y\|_\infty \le R\},$$

where

$$r = \frac{2}{M_3 \int_0^\omega \lambda(s)\, ds}, \qquad R = \frac{2}{m_3 \int_0^\omega \lambda(s)\, ds}.$$

Case II: When $\varphi = \varphi_{II}$, conditions (3.19), (3.8) become

$$r\int_0^\omega (\lambda(s)+\alpha(s)r)\, ds \le \frac{2}{M_3}, \quad R\int_0^\omega \frac{\lambda(s)}{1+\beta(s)(\lambda(s)+\alpha(s)R)R} ds \ge \frac{2}{m_3} \tag{3.23}$$

and are respectively satisfied provided that

$$r(\overline{\lambda}+\overline{\alpha}\, r) \le \frac{2}{M_3\, \omega}, \quad R\frac{\underline{\lambda}}{1+\overline{\beta}(\overline{\lambda}+\overline{\alpha}\, R)R} \ge \frac{2}{m_3\, \omega}. \tag{3.24}$$

The first inequality in (3.24) is satisfied for small enough $r > 0$ as it happens in *Case I.*

Next we study the existence of R as required by the second inequality in (3.24). To this aim, we consider the function $f : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$,

$$f(z) := \frac{Az}{1+Bz+Cz^2}, \quad \text{where } A := m_3\, \omega\, \underline{\lambda}, \quad B := \overline{\beta}\, \overline{\lambda}, \quad C := \overline{\beta}\, \overline{\alpha}.$$

Let us prove that there exists $R > 0$ fulfilling the last inequality in (3.24), if and only if

$$A \ge 2\,(2\sqrt{C} + B). \tag{3.25}$$

One can easily prove that $\lim_{z\to 0} f(z) = \lim_{z\to +\infty} f(z) = 0$ and $f(z) > 0$ for every $z > 0$. Additionally, f has a unique critical point at $1/\sqrt{C} > 0$ and the previous properties ensure that f attains a maximum at $z_{\max} := 1/\sqrt{C}$. Therefore if

$$f(z_{\max}) \ge 2,$$

equivalently $A \ge 2\,(2\sqrt{C} + B)$, then it is possible to choose R close enough or equal to $z_{\max}$, such that the required inequality holds. Moreover, under assumption (3.25), we can precise the interval where we can choose R. It is $[z_1, z_2]$, where z_1, z_2 are the solutions of the equation $f(z) = 2$, namely

$$z_1 = \frac{A-2B-\sqrt{(A-2B)^2-4^2C}}{4C}, \qquad z_2 = \frac{A-2B+\sqrt{(A-2B)^2-4^2C}}{4C}. \tag{3.26}$$

3.2.2. *Logistic growth.*

Corollary 3.2. *Assume that* $g(x) \equiv (1 - x/K)$ *and conditions* (i), (ii) *over* φ *are satisfied. If there exist* r, $R \in \mathbb{R}$, $0 < r < R$ *such that*

$$r \le \frac{K}{\overline{a}\,\overline{M_1}}, \tag{3.27}$$

$$\frac{\overline{a}\,\overline{M_1}}{K}\, r + M_3\, r\int_0^\omega \eta(s,r,r)\, ds \le 2 \tag{3.28}$$

and (3.8) *hold, then the system*

$$\begin{cases} x' = a(t)x\left(1-\frac{x}{K}\right) - \varphi(t,x,y)xy \\ y' = -b(t)y + c(t)\varphi(t,x,y)xy, \end{cases} \tag{3.29}$$

has an ω-periodic solution $(x,y)\in C$ such that

$$r \le \|(x,y)\| = \|x\|_\infty + \|y\|_\infty \le R.$$

It is clear how condition (3.27) can be satisfied. In addition, it is not necessary to study again the existence of an $R>0$ fulfilling condition (3.8), since it does not depend on the expression of g, and therefore we can follow the arguments of Subsection 3.2.1. Thus, let us state some sufficient conditions on $r>0$, such that (3.28) holds when $\varphi\equiv\varphi_I$ or $\varphi\equiv\varphi_{II}$. As it is explained before Section 3.2.1, we can consider $\eta=\varphi_I$ for both expressions of φ. Then, for both cases, condition (3.28) reads as

$$\frac{\overline{M}_1\overline{a}}{K}\,r + M_3\,r\int_0^\omega (\lambda(s)+\alpha(s)r)\;ds \le 2,$$

being trivially satisfied provided that

$$\frac{\overline{M}_1\overline{a}}{K}\,r + M_3\,\omega\,r\,\left(\overline{\lambda}+\overline{\alpha}\,r\right) \le 2. \tag{3.30}$$

We know from Remark 3.1, that there exists small enough $r>0$ fulfilling condition (3.28). However, condition (3.30) tells us how to obtain suitable values of $r>0$.

Remark 3.3 (ω-dependence of r and R)**.** *Since ω-periodic functions are also $n\omega$-periodic, for every natural number $n\ge 2$, it makes sense to ask in what way the numbers r and R depend on the period. For each natural number $n\ge 1$, denote by r_n and R_n the numbers r and R satisfying the conditions* (3.21)*, when ω is replaced by $n\omega$.*

Making some computations it can be shown that

$$r_{n+1} < r_n < R_n < R_{n+1},$$

for every natural number $n\ge 1$. Therefore, by taking a multiple of the period ω, it can happen to localize the same ω-periodic solution for every $n\ge 1$, and the best localization is that for $n=1$.

The same conclusion is true if we refer to conditions (3.23) *instead of* (3.21)*.*

Remark 3.4. *It deserves to mention that the classical Krasnosel'skii compression-expansion fixed point theorem does not apply to the integral operator N given by* (3.9) *and associated to the Lotka-Volterra type system* (1.1)*. We recall that one of the hypotheses of Krasnosel'skii's theorem requires that for some positive number τ,*

$$(x,y)-N(x,y)\notin C \quad \textit{for all}\ \ (x,y)\in C \ \ \textit{with}\ \ \|(x,y)\|=\tau. \tag{3.31}$$

However, in our case, if we choose pairs of the form $(0,y)\in C$, with $\|(0,y)\|=\|y\|_\infty=\tau$, then since $N(0,y)=(0,0)$, one has

$$(0,y)-N(0,y)=(0,y)\in C.$$

Consequently, condition (3.31) *does not hold. This shows the appropriateness of the homotopy version of Krasnosel'skii's theorem for Lotka-Volterra type systems, as compared to the other more popular versions.*

We can easily see that for the classical Lotka-Volterra system with nonconstant coefficients, the vector version of Krasnosels'skii's theorem used in [10] *still cannot be applied.*

3.3. **Nonconstant periodic solutions.** Here, we are interested in sufficient conditions in order that the periodic solution guaranteed by Theorem 3.1 does not reduce to a steady state, letting it the possibility to be a limit cycle.

Theorem 3.2. *Assume that condition* (3.2) *does not hold and*

$$g(x) \neq 0, \qquad \text{for all } \; x > 0. \tag{3.32}$$

Then any ω*-periodic solution* (x, y) *of system* (1.1) *with* $\|(x, y)\| > 0$*, does not reduce to a steady state.*

Proof. In view of the inequality $\|(x, y)\| > 0$, the result follows once we have proved that the only one steady state is $(0, 0)$.

From Proposition 3.1, we know that there are no steady states of the form $(0, y_0)$ with $y_0 > 0$, also that if $g(x) \neq 0$ for all $x > 0$, then there are no steady states of the type $(x_0, 0)$ with $x_0 > 0$, and that if condition (3.2) does not hold, then there are no positive (with both positive components) steady states. Therefore, the unique steady state of the system is $(0, 0)$, as wished. □

In case of linear growth of the prey population, condition (3.32) of Theorem 3.2 trivially holds. Consequently, if there exist numbers $0 < r < R$ satisfying conditions (3.19), (3.8) in Corollary 3.1 and condition (3.3) is not fulfilled, then by means of Theorem 3.2, we can assert that there exists a nonconstant ω-periodic solution $(x, y) \in C$ with $r \leq \|(x, y)\| \leq R$.

In particular, when $\varphi = \varphi_I$, conditions (3.19), (3.8) read as (3.21), which can always be satisfied for small enough r and sufficiently large R. Therefore, if we ensure that condition (3.3) does not hold, then there is an ω-periodic solution which is nonconstant. For instance, if a, b, c, λ are constants, but the cooperation coefficient α is nonconstant, then condition (3.3) is not fulfilled.

However, for the logistic growth of the prey population, Theorem 3.2 does not apply since $g(K) = 0$. Nevertheless, if condition (3.4) does not hold, then the only steady states of system (3.29) are $(0, 0)$ and $(K, 0)$. Recall that we can always consider small enough $r > 0$ such that conditions (3.27), (3.28) in Corollary 3.2 are fulfilled. Therefore, if the exists a number $R > 0$ such that $K > R > r > 0$ and condition (3.8) hold, then the orbit of the ω-periodic solution given by Corollary 3.2 does not reduce to a point.

As regards the possibility to have the inequality $R < K$, we mention that in the particular case when $\varphi = \varphi_I$ or $\varphi = \varphi_{II}$, such a choice is possible provided that

$$K > \frac{2}{m_3 \int_0^\omega \lambda(s)\, ds} \qquad \text{or} \qquad z_1 < K,$$

respectively, where z_1 is given in (3.26).

In particular, when $\varphi = \varphi_I$, if a, b, c, λ are constants and α is nonconstant, then condition (3.4) does not hold. Therefore, for every

$$K > \frac{2}{m_3 \omega \lambda},$$

the ω-periodic solution of system (3.29) given by Corollary 3.2, is nonconstant.

Let us finish this subsection by an example where all the coefficients, except c, are nonconstant.

Example 3.1. *Let the coefficients of system* (3.29)*, when* $\varphi = \varphi_I$*, be*

$$a(t) := \sin^2 \pi t, \quad b(t) := \cos^2 \pi t, \quad c(t) := c \in (0,1),$$

$$\lambda(t) := \theta a(t) + (1-\theta) b(t) \quad (\theta \in (0,1)), \qquad \alpha(t) := (1-b(t))b(t).$$

We start by giving a brief interpretation of this particular system, that could be suitable to model the interplay between prey and predator populations by taking into account the following factors: ecological seasonal effects; the prey growth rate a *attains its maximum value, when the predator mortality rate* b *reaches its minimum and vice versa; the attack rate of predators* λ *is a convex combination of the prey growth and the predator mortality rates, and the hunting cooperation coefficient* α *vanishes when the mortality rate of predators attains its maximum or minimum.*

For this particular system, condition (3.4) *does not hold. Indeed, we can show that there is no any constant* $k > 0$ *satisfying condition* (3.5)*. For* $t = 0$ *and any* $k > 0$*, we obtain*

$$a(0) = \sin^2 0 = 0 \neq \frac{k}{c} = k\,\frac{\cos^2 0}{c} = k\,\frac{b(0)}{c}.$$

So, condition (3.5) *does not hold for* $t = 0$ *and, consequently, condition* (3.4) *is not fulfilled. Therefore, if*

$$K > \frac{2}{m_3 \int_0^1 \lambda(s)ds} = \frac{4e(\sqrt{e}-1)}{c},$$

then the 1*-periodic solution given by Corollary* 3.2 *is nonconstant.*

3.4. **Positiveness of periodic solutions.** Theorem 3.1 yields the existence of an ω-periodic solution (x,y) such that

$$x(t) \geq q_1 \|x\|_\infty, \quad y(t) \geq q_2 \|y\|_\infty \quad \text{for every} \quad t \in [0,\omega], \tag{3.33}$$

and

$$r \leq \|x\|_\infty + \|y\|_\infty \leq R.$$

However, in this way, it is not guaranteed that x and y do not vanish. The next result shows that, when (x,y) is not a steady state, this is indeed the case.

Theorem 3.3. *Assume in addition that* $\varphi(t,\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ $(t \in \mathbb{R})$ *and* $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *are locally Lipschitz. Then for every nonconstant* ω*-periodic solution* $(x,y) \in C$ *of system* (1.1) *such that*

$$\|(x,y)\| = \|x\|_\infty + \|y\|_\infty > 0, \tag{3.34}$$

one has

$$x(t) > 0 \quad \textit{and} \quad y(t) > 0 \quad \textit{for all} \quad t \in [0,\omega].$$

Proof. First note that, the local Lipschitz property of φ and g guarantees the uniqueness of the solution to any Cauchy problem associated to system (1.1).

Let $(x,y) \in C$ be a nonconstant ω-periodic solution of system (1.1) such that condition (3.34) is fulfilled. According to (3.33), we have to prove that $\|x\|_\infty > 0$ and $\|y\|_\infty > 0$.

Assume that $\|x\|_\infty = 0$. Then from (3.34), one has $\|y\|_\infty > 0$, so there is a $t_0 \in [0,\omega]$ such that $y(t_0) > 0$. Also, when $\|x\|_\infty = 0$, i.e., $x \equiv 0$, the second equation in system (1.1) becomes

$$y'(t) = -b(t)y(t), \quad t \in \mathbb{R}$$

and gives the expression of y, namely

$$y(t)=y(t_0)e^{-\int_{t_0}^{t} b(s)ds}, \quad t\in\mathbb{R}.$$

Since y is periodic and for $y\left(t_0\right)\neq 0$ the function in the right-hand side is not periodic, the above equality yields a contradiction. Thus, $\|x\|_\infty>0$.

Assume next that $\|y\|_\infty=0$. Then the first equation in (1.1) reads as

$$x'(t)=a(t)x(t)g(x(t)), \quad t\in\mathbb{R}. \tag{3.35}$$

We claim that

$$x(t)g(x(t))\neq 0 \quad \text{for every} \;\; t\in\mathbb{R}. \tag{3.36}$$

To prove this, we start by showing that

$$x(t)>0 \quad \text{for all} \;\; t\in\mathbb{R}.$$

Indeed, if there exists $t_0\in\mathbb{R}$ such that $x(t_0)=0$, then $(x,0)$ is a solution of the Cauchy problem with the initial conditions $x(t_0)=y(t_0)=0$. But $(0,0)$ is also a solution of this Cauchy problem, and by the uniqueness of solution, we must have $x\equiv 0$, which is excluded by our assumption (3.34). Hence $x(t)>0$ for every $t\in\mathbb{R}$ as wished. Next, we prove that

$$g(x(t))\neq 0 \quad \text{for every} \;\; t\in\mathbb{R}.$$

Indeed, assuming the contrary, there exists $t_1\in\mathbb{R}$ such that $g(x(t_1))=0$. Then, as shown by Proposition 3.1 (b), $(x(t_1),0)$ is a steady state of system (1.1). Therefore, the steady state $(x(t_1),0)$ and $(x,0)$ solve the same Cauchy problem with the initial conditions $x(t_1)=x(t_1)$ and $y(t_1)=0$. Again by the uniqueness of the solution of the Cauchy problem, one has $(x,0)\equiv(x(t_1),0)$, i.e., $(x,0)$ is a constant solution of the system, which is excluded from the hypothesis. Thus, $g(x(t))\neq 0$ for every $t\in\mathbb{R}$ and our claim is proved.

Now using (3.36), we can put equation (3.35) under the equivalent form

$$\frac{x'(t)}{x(t)g(x(t))}=a(t), \quad t\in\mathbb{R},$$

which by integration gives

$$\int_0^t \frac{x'(s)}{x(s)g(x(s))}ds=\int_0^t a(s)ds, \quad t\in\mathbb{R}. \tag{3.37}$$

Let $D:=\{\tau\in\mathbb{R}_+\setminus\{0\}: \; g(\tau)\neq 0\}$ and $G:D\longrightarrow\mathbb{R}$ be such that

$$G'(\tau)=\frac{1}{\tau\, g(\tau)}.$$

Then, from (3.37), we have

$$G(x(t))=G(x(0))+\int_0^t a(s)ds, \quad t\in\mathbb{R}.$$

As x is periodic, $G(x)$ is also periodic, while the function in the right-hand side is not periodic, since $a\not\equiv 0$. This contradiction shows that $\|y\|_\infty>0$, and the proof is finished. □

3.5. **Case of a not time-dependent φ.** In case that φ does not depend on t, then η and ψ do not depend on t as well and we can obtain a better localization of a solution.

Theorem 3.4. *Let φ, η and ψ in conditions* (i) *and* (ii) *do not depend on t. If there exist r, $R \in \mathbb{R}$, $0 < r < R$ such that condition* (3.6) *is satisfied and*

$$\overline{M}_1\overline{a}\,(1 - g(r)) + \overline{M}_3\, r\, \eta(r, r) \leq 2, \tag{3.38}$$

$$R\,\psi(R) \geq \frac{2}{\underline{m}_3}, \tag{3.39}$$

then system (1.1) *has an ω-periodic solution $(x, y) \in C$ such that*

$$r \leq \|(x, y)\| = \|x\|_\infty + \|y\|_\infty \leq R.$$

Proof. The proof is similar to that of Theorem 3.1, except the step (c) for proving conditions (3.10) and (3.15). For instance, when proving (3.10), we can replace the estimations in (3.12) by the following ones:

$$\begin{aligned}\|x\|_\infty &= x(t_0) < \lambda x(t_0) = N_1(x, y)(t_0)\\ &= \int_{t_0}^{t_0+\omega} H_1(t_0, s)[a(s)\, x(s)(1 - g(x(s))) + \varphi(x(s), y(s))\, x(s)\, y(s)]ds\\ &\leq \overline{a}\|x\|_\infty(1 - g(\|x\|_\infty))\overline{M}_1 + \|x\|_\infty\, \|y\|_\infty\, \eta(\|x\|_\infty, \|y\|_\infty) \int_{t_0}^{t_0+\omega} H_1(t_0, s)ds\\ &\leq \overline{a}\|x\|_\infty(1 - g(\|x\|_\infty))\overline{M}_1 + \|x\|_\infty\, \|y\|_\infty\, \eta(\|x\|_\infty, \|y\|_\infty)\overline{M}_1,\end{aligned}$$

which gives

$$1 < \overline{a}\,(1 - g(r))\,\overline{M}_1 + \|y\|_\infty\, \eta(r, r)\,\overline{M}_1.$$

Similarly, instead of (3.14), we obtain

$$1 < \|x\|_\infty\, \overline{c}\,\overline{M}_2\, \eta(r, r).$$

So, adding the two last inequalities, we find that

$$2 < \overline{M}_1\, \overline{a}\,(1 - g(r)) + r\, \eta(r, r)\,\overline{M}_3,$$

which now contradicts our assumption (3.38).

The sufficiency of condition (3.39) for proving (3.15) can be shown in a similar way. □

For the linear growth of the prey population, when φ does not depend on t, we have the following result that improves Corollary 3.1.

Corollary 3.3. *Assume that $g \equiv 1$ and φ, η, ψ fulfill conditions* (i), (ii) *without depending on $t \in \mathbb{R}$. If there exist $r, R \in \mathbb{R}$, $0 < r < R$ such that*

$$r\, \eta(r, r) \leq \frac{2}{\overline{M}_3}, \qquad R\, \psi(R) \geq \frac{2}{\underline{m}_3},$$

then system (3.20) *has an ω-periodic solution $(x, y) \in C$ such that*

$$r \leq \|(x, y)\| = \|x\|_\infty + \|y\|_\infty \leq R.$$

For the logistic growth of the prey population, when φ does not depend on t, we improve Corollary 3.2 as follows:

Corollary 3.4. *Assume that* $g(x)=(1-x/K)$ *and* φ, η, ψ *fulfill conditions* (i), (ii) *without depending on* $t\in\mathbb{R}$. *If there exist* r, $R\in\mathbb{R}$, $0<r<R$ *such that*

$$r\le \frac{K}{\overline{a}\,\overline{M}_1},\qquad \frac{\overline{M}_1\,\overline{a}}{K}\,r+\overline{M}_3\,r\,\eta(r,r)\le 2,$$

$$R\,\psi(R)\ge \frac{2}{\underline{m}_3},$$

then system (3.29) *has an* ω*-periodic solution* $(x,y)\in C$ *such that*

$$r\le \|(x,y)\|=\|x\|_\infty+\|y\|_\infty\le R.$$

3.6. **Case of constant coefficients.** Under the conditions required to φ in Subsection 3.5, we now assume that a, b and c are constant. Then

$$H_1(t,s)=\frac{e^{-a(s-t)}}{1-e^{-a\,\omega}},\qquad H_2(t,s)=\frac{e^{b(s-t)}}{e^{b\,\omega}-1}.$$

Moreover, for every $t\in\mathbb{R}$, one can compute

$$\begin{aligned}&\overline{M}_1=\underline{m}_1=\int_t^{t+\omega}H_1(t,s)\,ds=\frac{1}{a},\quad \overline{M}_2=\underline{m}_2=\int_t^{t+\omega}H_2(t,s)\,ds=\frac{1}{b},\\ &\underline{m}_3=q_1q_2\min\left\{\frac{1}{a},\frac{c}{b}\right\},\quad \overline{M}_3=\max\left\{\frac{1}{a},\frac{c}{b}\right\}.\end{aligned}$$

As a direct consequence of Theorem 3.4, we have the following result, where the conditions over r, R look much more simpler.

Corollary 3.5. *Let* a, b, c *be constant and* φ, η, ψ *in conditions* (i),(ii) *do not depend on time. If there exist* r, $R\in\mathbb{R}$, $0<r<R$ *such that*

$$g(r)\ge 0,\quad r\eta(r,r)\max\left\{\frac{1}{a},\frac{c}{b}\right\}-g(r)\le 1,\tag{3.40}$$

$$R\,\psi(R)\ge \frac{2}{q_1q_2\min\left\{\frac{1}{a},\frac{c}{b}\right\}},$$

then the system

$$\begin{cases}x'=axg(x)-\varphi(x,y)xy\\ y'=-by+c\varphi(x,y)xy,\end{cases}\tag{3.41}$$

has an ω*-periodic solution* $(x,y)\in C$ *such that*

$$r\le \|(x,y)\|=\|x\|_\infty+\|y\|_\infty\le R.$$

Notice that, in particular if $g\equiv 1$ (linear growth on prey), condition (3.40) becomes

$$r\eta(r,r)\le \frac{2}{\max\left\{\frac{1}{a},\frac{c}{b}\right\}}$$

and we have the following remark about the classical Lotka-Volterra system:

Remark 3.5. *In the particular case where* $\varphi \equiv \lambda > 0$, *system* (3.41) *reduces to the classical Lotka-Volterra model* (1.2). *Then, one can consider* η, $\psi = \varphi = \lambda$, *and the conditions over* r *and* R *reduce to*

$$\begin{aligned} r &\le \frac{2}{\max\left\{\frac{1}{a}, \frac{c}{b}\right\}\lambda} = 2\min\left\{a, \frac{b}{c}\right\}\frac{1}{\lambda}, \\ R &\ge \frac{2}{q_1 q_2 \min\left\{\frac{1}{a}, \frac{c}{b}\right\}\lambda} = 2\max\left\{a, \frac{b}{c}\right\}\frac{1}{q_1 q_2 \lambda}. \end{aligned}$$

It is easy to see that the non-trivial steady state $(x_*, y_*) := (b/(c\lambda), a/\lambda)$ *satisfies*

$$\begin{aligned} r &\le 2\min\left\{a, \frac{b}{c}\right\}\frac{1}{\lambda} \le \|(x_*, y_*)\| = \left(\frac{b}{c} + a\right)\frac{1}{\lambda} \le 2\max\left\{a, \frac{b}{c}\right\}\frac{1}{\lambda} \\ &\le 2\max\left\{a, \frac{b}{c}\right\}\frac{1}{q_1 q_2 \lambda} \le R. \end{aligned}$$

Therefore, it may happen that the localized solution given by Corollary 3.5 *is in fact the steady state* (x_*, y_*).

In case that $g(x) = (1 - x/K)$ (logistic growth on prey), condition (3.40) becomes

$$r \le K, \quad \max\left\{\frac{1}{a}, \frac{c}{b}\right\} r\,\eta(r, r) + \frac{r}{K} \le 2$$

and we can make the following remark about the classical Lotka-Volterra model with logistic growth of the prey population:

Remark 3.6. *In the particular case where* $\varphi \equiv \lambda > 0$, *system* (3.41) *reduces to the classical Lotka-Volterra model with logistic growth on prey*

$$\begin{cases} x' = ax\left(1 - \frac{x}{K}\right) - \lambda xy \\ y' = -by + c\lambda xy. \end{cases}$$

Then, one can consider η, $\psi = \varphi = \lambda$, *and the conditions over* r *and* R *become*

$$r \le K, \qquad r \le \frac{2}{\frac{1}{K} + \max\left\{\frac{1}{a}, \frac{c}{b}\right\}\lambda},$$

$$R \ge 2\max\left\{a, \frac{b}{c}\right\}\frac{1}{\lambda}. \tag{3.42}$$

If one has

$$K > 2\max\left\{a, \frac{b}{c}\right\}\frac{1}{\lambda},$$

then (3.42) *holds for* $R = 2\max\{a, b/c\}/\lambda < K$. *Therefore, we are not localizing the steady state* $(K, 0)$, *but it may also happen that the localized solution given by Corollary* 3.5 *is in fact the steady state*

$$(x^*, y^*) = \left(\frac{b}{c\lambda}, \frac{a}{\lambda}\left(1 - \frac{b}{c\lambda K}\right)\right),$$

where x^*, $y^* > 0$.

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Instituto de Matematicas, Facultade de Matematicas, Campus Vida, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

Department of Mathematics, Babeş–Bolyai University, 400084 Cluj-Napoca, Romania