FIXED POINT INDEX THEORY FOR DECOMPOSABLE MULTIVALUED MAPS AND APPLICATIONS TO DISCONTINUOUS ϕ-LAPLACIAN PROBLEMS

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Abstract. In this paper, we develop a fixed point index theory for decomposable multivalued maps, that is, compositions of two multivalued nonlinear upper semicontinuous maps. As an application, this fixed point index theory is combined with the method of lower and upper solutions in order to obtain new existence, localization and multiplicity results for ϕ-Laplacian problems with discontinuous nonlinearities and nonlinear functional boundary conditions.

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1. Introduction

In the last decades, topological degree theory has been widely studied in the literature and several authors have attempted to generalize it to larger classes of operators and spaces [1,12,23]. This is due to the fact that it is a crucial tool in nonlinear analysis, in particular in the study of existence and multiplicity of solutions to differential and integral equations. In this direction, we develop a fixed point index theory for the composition of two multivalued maps.

Consider the fixed point problem

$$x \in \Psi\Phi x,$$

where Ψ and Φ are two upper semicontinuous multivalued maps with closed and convex values. If the operator Ψ is nonlinear, the values of $T = \Psi\Phi$ can be non-convex and thus the classical degree and fixed point index theories for multivalued operators (see [17,29]) are not applicable. To overcome this difficulty we will follow mainly the ideas in the papers [25] and [10] on fixed point theory for decomposable maps. A previous study of this matter can be seen in the paper [2], but there the author only considers the composition of a multivalued Nemytskii operator and a single-valued map instead of two multivalued maps as here and, moreover, his approach is completely different from ours.

As an application of the theory, we investigate the existence of Carathéodory solutions to the following class of ϕ-Laplacian problems with functional boundary conditions

$$\begin{cases} (\phi(u'))' = f(t,u) & \text{for a.a. } t \in I = [0,1], \\ u'(0) = 0, \quad u(1) = g\,(u)\,, \end{cases}$$

where the assumptions about ϕ and g are detailed below and f may be discontinuous even with respect to the second variable. Observe that the above ϕ-Laplacian problem is a particular case of the large class of problems considered in [6], but our assumptions about f here are sharper.

Note that the boundary conditions considered here contain mixed two-point conditions, but also other generic nonlinear conditions which may include, for instance, integral and multi-point ones. In addition, our approach allows us to deal in an unified way with

- the *classical* homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$,
- the *singular* homeomorphism $\phi : (-a, a) \to \mathbb{R}$, and
- the *bounded* homeomorphism $\phi : \mathbb{R} \to (-b, b)$.

Due to the weak regularity assumptions on f, we need to study, as an auxiliary problem, the existence of solutions to

$$\begin{cases} (\phi(u'))' \in F(t,u) & \text{for a.a. } t \in I, \\ u'(0) = 0, \quad u\,(1) = g\,(u)\,, \end{cases}$$

where F is a multivalued map which regularizes f. At this point, the fixed point index theory for decomposable maps is the fundamental tool in order to obtain solutions of the corresponding fixed point problem and it will be combined with the method of lower and upper solutions to achieve localization and multiplicity results. Later, an adequate transversality condition on the discontinuities of f allows to prove that the solutions of the inclusion are also solutions to the former single-valued problem.

For existence results concerning multivalued equations governed by the ϕ-Laplacian operator we may refer the reader to [13, 14], where fixed point theory techniques and the method of lower and upper solutions are combined to this aim.

2. FIXED POINT INDEX FOR DECOMPOSABLE MULTIVALUED MAPS

Let X and Y be Banach spaces and $K_X \subset X$, $K_Y \subset Y$ be closed convex sets. We are interested into the class of decomposable maps, i.e., multivalued maps $T : \overline{\Omega} \to 2^{K_X}$, where $\Omega \subset K_X$ is open in K_X, which can be represented as a composition $T = \Psi\Phi$ of two multivalued maps Φ and Ψ with the following properties:

(i): $\Phi : \overline{\Omega} \to 2^{K_Y}$ is upper semicontinuous (usc, for short) with closed convex values and relatively compact range;
(ii): $\Psi : K_Y \to 2^{K_X}$ is usc with compact convex values.

Denote by $\mathcal{F}$ the class of pairs (Φ, Ψ) having the above two properties, where K_X and K_Y are given, $\Omega \subset K_X$ is open in K_X, and $\Psi\Phi$ is fixed point free on the relative boundary $\partial_{K_X}\Omega$ of Ω, i.e., $x \notin \Psi\Phi x$ for all $x \in \partial_{K_X}\Omega$.

To each such a pair of maps $(\Phi, \Psi) \in \mathcal{F}$, we associate the map $\Pi : \overline{\Omega} \times K_Y \to 2^{K_X \times K_Y}$ given by

$$\Pi(x, y) = \Psi y \times \Phi x.$$

Clearly, the set $K := K_X \times K_Y$ is closed convex in $X \times Y$, the set $\Omega \times K_Y$ is open in K, and the map Π is usc with closed convex values. Also Π is fixed point free on the boundary $\partial_K(\Omega \times K_Y)$ of $\Omega \times K_Y$. Indeed, if there exists $(x, y) \in \partial_K(\Omega \times K_Y)$ such that $(x, y) \in \Pi(x, y)$, then $x \in \partial_{K_X}\Omega$, $y \in K_Y$, $x \in \Psi y$ and $y \in \Phi x$. It follows that $x \in \partial_{K_X}\Omega$ and $x \in \Psi\Phi x = Tx$, contrary to our assumption that $x \notin \Psi\Phi x$ for all $x \in \partial_{K_X}\Omega$. In addition, Π is *ultimately compact*. Indeed, one has

$$\Pi(\overline{\Omega} \times K_Y) = \Psi(K_Y) \times \Phi(\overline{\Omega}),$$

and next the set

$$\Pi(\overline{\Omega} \times K_Y \cap \Pi(\overline{\Omega} \times K_Y)) = \Psi\left(\Phi(\overline{\Omega})\right) \times \Phi\left(\overline{\Omega} \cap \Psi(K_Y)\right),$$

is relatively compact since $\Phi\left(\overline{\Omega}\cap\Psi(K_Y)\right)$ is relatively compact by the assumption on Φ, while $\Psi\left(\Phi(\overline{\Omega})\right)$ is relatively compact as the image of a compact set by an usc map with compact values (see [12, Proposition 24.1]). Thus, according to the Fitzpatrick-Petryshyn degree theory for ultimately compact multivalued maps (see [17]), we may speak about the fixed point index over $\Omega\times K_Y$ with respect to K for the map Π, denoted by $i_K\left(\Pi,\Omega\times K_Y\right)$, and we can give the following definition.

Definition 2.1. By the *fixed point index* over Ω with respect to K for the pair of maps $(\Phi,\Psi)\in F$, $ind_K(\Phi,\Psi,\Omega)$ for short, we mean the integer

$$ind_K(\Phi,\Psi,\Omega):=i_K\left(\Pi,\Omega\times K_Y\right). \tag{2.1}$$

Remark 2.1. In particular, if $X=Y$, $K_X=K_Y$ and $\Psi=I$, where I is the identity map in K_X, the map Π becomes $\Pi\left(x,y\right)=\{y\}\times\Phi x$, and a pair (x,y) is a fixed point of Π if and only if $x=y$ and x is a fixed point of Φ. In this sense, the pair of maps (Φ,I) can be identified with the single map Φ and our fixed point index over Ω with respect to $K=K_X\times K_X$ for the pair of maps (Φ,I) is equal to the fixed point index over Ω with respect to K_X for the single compact map Φ, that is

$$ind_K(\Phi,I,\Omega)=i_{K_X}\left(\Phi,\Omega\right).$$

Therefore, our index theory for pairs of multivalued maps extends the well-known index theory for compact usc multivalued maps.

The basic properties of the new fixed point index for the class $\mathcal{F}$ are collected by the next theorem.

Theorem 2.1. *The fixed point index $ind_K(\Phi,\Psi,\Omega)$ has the following properties:*

- **(1):** *(Additivity) If $\Omega_1,\Omega_2\subset K_X$ are disjoint open in K_X and $\Psi\Phi$ is fixed point free on $\partial_{K_X}\Omega_1\cup\partial_{K_X}\Omega_2$, then $ind_K(\Phi,\Psi,\Omega_1\cup\Omega_2)=ind_K(\Phi,\Psi,\Omega_1)+ind_K(\Phi,\Psi,\Omega_2)$.*
- **(2):** *(Excision) If $A\subset\overline{\Omega}$ is closed and $\Psi\Phi$ is fixed point free on A, then $ind_K(\Phi,\Psi,\Omega)=ind_K(\Phi,\Psi,\Omega\setminus A)$.*
- **(3):** *(Existence) If $ind_K(\Phi,\Psi,\Omega)\neq 0$, then there exist $x\in\Omega$ and $y\in K_Y$ with $x\in\Psi y$ and $y\in\Phi x$, consequently $x\in\Psi\Phi x$ and $y\in\Phi\Psi y$.*
- **(4):** *(Homotopy) If $\phi:\overline{\Omega}\times[0,1]\to 2^{K_Y}$ is usc with closed convex values and relatively compact range, $\psi:K_Y\times[0,1]\to 2^{K_X}$ is usc with compact convex values, and $x\notin\psi\left(\phi\left(x,\lambda\right),\lambda\right)$ for all $x\in\partial_{K_X}\Omega$ and $\lambda\in[0,1]$, then the index $ind_K\left(\phi\left(\cdot,\lambda\right),\psi\left(\cdot\cdot,\lambda\right),\Omega\right)$ does not depend on λ.*
- **(5) :** *(Normalization) For every $x_0\in\Omega$ and $\Phi:\overline{\Omega}\to 2^{K_Y}$ usc with closed convex values and relatively compact range, one has*

  $$ind_K\left(\Phi,x_0,\Omega\right)=1.$$

  In particular, $ind_K(y_0,x_0,\Omega)=1$ if $x_0\in\Omega$ and $y_0\in K_Y$.

Proof. The additivity and excision properties are direct consequences of the definition of the new index and of the respective properties of the fixed point index for ultimately compact multivalued maps.

(3) Assume that $ind_K(\Phi,\Psi,\Omega)\neq 0$. Then $i_K\left(\Pi,\Omega\times K_Y\right)\neq 0$ and by the existence property of the fixed point index for ultimately compact multivalued maps, there exists $(x,y)\in\Omega\times K_Y$ with $(x,y)\in\Pi(x,y)$ Then $x\in\Omega$, $y\in K_Y$, $x\in\Psi y$ and $y\in\Phi x$. These yield $x\in\Psi\Phi x$ and $y\in\Phi\Psi y$.

(4) The map $\chi : \overline{\Omega} \times K_Y \times [0,1] \to 2^K$ given by

$$\chi(x, y, \lambda) = \psi(y, \lambda) \times \phi(x, \lambda)$$

is usc with closed convex values and also ultimately compact, as follows from

$$\chi(\overline{\Omega} \times K_Y \times [0,1]) = \psi(K_Y \times [0,1]) \times \phi(\overline{\Omega} \times [0,1])$$

and

$$\begin{aligned} & \chi(\overline{\Omega} \times K_Y \times [0,1] \cap \chi(\overline{\Omega} \times K_Y \times [0,1])) \\ = \ & \psi\left(\phi(\overline{\Omega} \times [0,1])\right) \times \phi\left(\overline{\Omega} \cap \psi(K_Y \times [0,1])\right). \end{aligned}$$

Thus χ is an admissible homotopy in the class of ultimately compact multivalued maps, and the conclusion follows from the homotopy property applied to χ.

(5) Let $y_0 \in K_Y$ be arbitrarily fixed and consider the maps $\phi : \overline{\Omega} \times [0,1] \to 2^{K_Y}$ and $\psi : K_Y \times [0,1] \to 2^{K_X}$ given by

$$\begin{aligned} \phi(x, \lambda) &= \lambda \Phi(x) + (1 - \lambda) y_0 \\ \psi(y, \lambda) &= x_0. \end{aligned}$$

Clearly we may apply the homotopy property to deduce that

$$ind_K(\phi(\cdot, 1), \psi(\cdot, 1), \Omega) = ind_K(\phi(\cdot, 0), \psi(\cdot, 0), \Omega),$$

that is

$$ind_K(\Phi, x_0, \Omega) = ind_K(y_0, x_0, \Omega).$$

But $ind_K(y_0, x_0, \Omega) = i_K(\Pi_0, \Omega \times K_Y)$, where Π_0 is the constant map $(x_0, y_0) \in \Omega \times K_Y$. Then the normalization property of the fixed point index of ultimately compact multivalued maps gives $i_K(\Pi_0, \Omega \times K_Y) = 1$, whence the result. □

The generalizations to our context of the Leray-Schauder principle and Nonlinear alternative (see [18, p. 123]) are stated below.

Theorem 2.2 (Leray-Schauder principle). *Assume that* $\phi : \overline{\Omega} \times [0,1] \to 2^{K_Y}$ *is usc with closed convex values and relatively compact range,* $\psi : K_Y \times [0,1] \to 2^{K_X}$ *is usc with compact convex values, and* $x \notin \psi(\phi(x, \lambda), \lambda)$ *for all* $x \in \partial_{K_X} \Omega$ *and* $\lambda \in [0,1]$. *If* $\psi(\cdot, 1) \equiv x_0 \in \Omega$, *then the map* $T_0 = \Psi_0 \Phi_0$, *where* $\Phi_0 = \phi(\cdot, 0)$ *and* $\Psi_0 = \psi(\cdot, 0)$, *has a fixed point.*

Proof. From the homotopy and normalization properties, one has

$$\begin{aligned} ind_K(\Phi_0, \Psi_0, \Omega) &= ind_K(\phi(\cdot, 0), \psi(\cdot, 0), \Omega) = ind_K(\phi(\cdot, 1), \psi(\cdot, 1), \Omega) \\ &= ind_K(\phi(\cdot, 1), x_0, \Omega) = 1. \end{aligned}$$

The conclusion is now given by the existence property of the index. □

Theorem 2.3 (Nonlinear alternative). *Let* $T : \overline{\Omega} \to 2^{K_X}$, $T = \Psi\Phi$, *where the multivalued maps* Φ, Ψ *satisfy conditions* (i) *and* (ii), *and let* $x_0 \in \Omega$. *Then one of the following two properties holds:*

(a): *T has a fixed point;*

(b): *there exists* $x \in \partial_{K_X} \Omega$ *and* $\lambda \in (0,1)$ *such that* $x - (1 - \lambda) x_0 \in \lambda T x$.

Proof. We can assume that T is fixed point free on $\partial_{K_X}\Omega$, i.e., $(\Phi,\Psi)\in\mathcal{F}$, else we have property (a). Let $\phi:\overline{\Omega}\times[0,1]\to 2^{K_Y}$ and $\psi:K_Y\times[0,1]\to 2^{K_X}$ be defined by

$$\phi\left(x,\lambda\right)=\Phi x,\quad \psi\left(y,\lambda\right)=\left(1-\lambda\right)x_0+\lambda\Psi y.$$

If for some $x\in\partial_{K_X}\Omega$ and $\lambda\in\left(0,1\right)$, one has $x\in\psi\left(\phi\left(x,\lambda\right),\lambda\right)$, then $x-\left(1-\lambda\right)x_0\in\lambda Tx$, that is we have property (b). Otherwise, the homotopy given by the maps ϕ and ψ is admissible and yields $ind_K\left(\phi\left(\cdot,0\right),\psi\left(\cdot,0\right),\Omega\right)=ind_K\left(\phi\left(\cdot,1\right),\psi\left(\cdot,1\right),\Omega\right)$, or equivalently $ind_K\left(\Phi,x_0,\Omega\right)=ind_K\left(\Phi,\Psi,\Omega\right)$. Since $ind_K\left(\Phi,x_0,\Omega\right)=1$, we have $ind_K\left(\Phi,\Psi,\Omega\right)=1$, whence (a). □

Theorem 2.4. *Under the assumptions of Theorem* 2.3, *if in addition* Ω *is convex and* $T\left(\overline{\Omega}\right)\subset\Omega$, *then*

$$ind_K\left(\Phi,\Psi,\Omega\right)=1.$$

Proof. Indeed, the condition $T\left(\overline{\Omega}\right)\subset\Omega$ implies that the homotopy given by ϕ and ψ given in the proof of the nonlinear alternative, is admissible. To prove this, assume that for some $x\in\overline{\Omega}$ and $\lambda\in[0,1]$, one has $x\in\psi\left(\phi\left(x,\lambda\right),\lambda\right)$, that is $x\in\left(1-\lambda\right)x_0+\lambda T\left(x\right)$. Since $x_0,T\left(x\right)\subset\Omega$ and Ω is convex, we have $\left(1-\lambda\right)x_0+\lambda T\left(x\right)\subset\Omega$. Consequently, $x\in\Omega$, which proves that the homotopy is admissible. Finally the conclusion follows as in proof of Theorem 2.3. □

3. Existence results for ϕ-Laplacian problems with functional boundary conditions

Now we shall study the existence of Carathéodory solutions to the problem

$$\left\{\begin{array}{ll}(\phi(u'))'=f(t,u) & \text{for a.a. } t\in I=[0,1],\\ u'(0)=0,\ u(1)=g(u), & \end{array}\right. \tag{3.1}$$

where $\phi:(-a,a)\to(-b,b)$ is an increasing homeomorphism such that $\phi(0)=0$, $0<a,b\le+\infty$, $g:C(I)\to\mathbb{R}$ is continuous (possibly nonlinear) and the function $f:I\times\mathbb{R}\to\mathbb{R}$ may be discontinuous in both variables.

By a Carathéodory solution of problem (3.1) we mean a function $u\in C^1(I)$, with $u'(0)=0$, $u(1)=g(u)$, such that $u'(t)\in(-a,a)$ for all $t\in I$, $\phi\circ u'\in W^{1,1}(I)$ and $(\phi(u'(t)))'=f(t,u(t))$ for a.a. $t\in I$.

Since f is discontinuous in both variables, we transform (3.1) into a similar problem for a differential inclusion, called the *regularization problem* of (3.1), namely

$$\left\{\begin{array}{ll}(\phi(u'))'\in F(t,u) & \text{for a.a. } t\in I,\\ u'(0)=0,\ u(1)=g(u), & \end{array}\right. \tag{3.2}$$

where $F:I\times\mathbb{R}\to 2^{\mathbb{R}}$ is the multivalued map given by

$$F(t,x)=\bigcap_{\varepsilon>0}\overline{\text{co}}f\left(t,\overline{B}_\varepsilon(x)\right), \tag{3.3}$$

with $\overline{B}_\varepsilon(x):=[x-\varepsilon,x+\varepsilon]$. Equivalently, F can be rewritten as

$$F(t,x)=\left[\min\left\{f(t,x),\liminf_{y\to x}f(t,y)\right\},\ \max\left\{f(t,x),\limsup_{y\to x}f(t,y)\right\}\right].$$

Note that $F(t,x)=\{f(t,x)\}$ whenever $f(t,\cdot)$ is continuous at x.

We need the following lemma about Nemytskii's operator associated to the multivalued map F, that is,

$$\mathcal{N}_F(u) = \{v \in L^1(I) \ : \ v(t) \in F(t, u(t)) \ \text{ for a.a. } t \in I\}.$$

Lemma 3.1. *Assume that the function* $f : I \times \mathbb{R} \to \mathbb{R}$ *satisfies the following condition:*

- **(C_1):** *The composed function* $f(\cdot, u(\cdot))$ *is measurable for every* $u \in C(I)$*, and for each* $\mu \geq 0$*, there exists* $M_\mu \in L^1(I)$ *such that* $|f(t, x)| \leq M_\mu(t)$ *for all* $x \in \mathbb{R}$ *with* $|x| \leq \mu$ *and a.a.* $t \in I$.

Then the Nemytskii operator $\mathcal{N}_F : C(I) \to L^1(I)$ *is usc. In addition,*

$$|v(t)| \leq M_\mu(t) \quad \text{for a.a. } t \in I, \ \text{every} \ \ v \in \mathcal{N}_F(u) \ \ \text{and} \ \ u \in C(I) \ \text{with} \ \ |u|_\infty < \mu. \tag{3.4}$$

Proof. The map $F(t, \cdot)$ is usc for a.a. $t \in I$, see [8] or [11]. Then, according to [7, Theorem 1.1], the Nemytskii operator $\mathcal{N}_F$ is usc from $\mathcal{C}(I)$ to $L^1(I)$.

To prove the second part, let $u \in C(I)$ with $|u|_\infty < \mu$ and $v \in \mathcal{N}_F(u)$. Then $v(t) \in \overline{\text{co}} f\left(t, \overline{B}_\varepsilon(u(t))\right)$ for every $\varepsilon > 0$, in particular for $\varepsilon := \mu - |u|_\infty$. Since $\overline{B}_\varepsilon(u(t)) = [u(t) - \varepsilon, u(t) + \varepsilon] \subset [-\mu, \mu]$, one has $|v(t)| \leq M_\mu(t)$ as claimed. □

Next, in the spirit of the papers by Hu [19] and Cid and Pouso [8], our plan is to guarantee first the existence of solutions to problem (3.2), usually called *Krasovskij* solutions of (3.1), and then to prove that under certain conditions on the nonlinearity f, these solutions are Carathéodory solutions of the discontinuous problem (3.1).

Theorem 3.2. *Let condition* (C_1) *holds. If there are two numbers* $R, \varepsilon > 0$ *such that*

$$|M_{R+\varepsilon}|_{L^1(I)} < b, \tag{3.5}$$

$$\rho_R + \phi^{-1}\left(|M_{R+\varepsilon}|_{L^1(I)}\right) \leq R < b, \tag{3.6}$$

where $\rho_R := \sup_{\substack{u \in \mathcal{C}(I) \\ |u|_\infty \leq R}} |g(u)|$, *then the regularization problem* (3.2) *has a Carathéodory solution* u *with* $|u|_\infty \leq R$.

Remark 3.1. If $b = +\infty$, then (3.5) trivially holds. If in addition g is bounded, i.e., $|g(u)| \leq c$ for all $u \in C(I)$ and some constant $c \geq 0$, then condition (3.6) is also satisfied by any $\varepsilon > 0$ and sufficiently large R, in each one of the following two situations:

- **(a) :** $a < +\infty$;
- **(b):** $M_\mu = M$ does not depend on μ.

Proof of Theorem 3.2. Under the conditions of the theorem, the operator $T : \overline{B}_R(0) \to 2^{\overline{B}_R(0)}$,

$$T(u) = g(u) - \int_\cdot^1 \phi^{-1}\left(\int_0^\tau F(s, u(s))\right),$$

where $\overline{B}_R(0) = \{u \in C(I) : \ |u|_\infty \leq R\}$ and $F(\cdot, u(\cdot))$ stands for the subset $\mathcal{N}_F(u)$ of $L^1(I)$, is well-defined. Indeed, if $u \in \overline{B}_R(0)$, then $|u|_\infty < R + \varepsilon$ and so by (3.4), for any $v \in \mathcal{N}_F(u)$, one has $|v(t)| \leq M_{R+\varepsilon}(t)$ for a.a. $t \in I$. Then $\left|\int_0^\tau v(s)\right| \leq \int_0^1 |v(s)| \leq |M_{R+\varepsilon}|_{L^1(I)} < b$ for each $\tau \in I$. Thus $\phi^{-1}\left(\int_0^\tau v(s)\right)$ has sense. Furthermore, in virtue of (3.6),

$$\left| g(u) - \int_\cdot^1 \phi^{-1}\left(\int_0^\tau v(s)\right)\right|_\infty \leq \rho_R + \phi^{-1}\left(|M_{R+\varepsilon}|_{L^1(I)}\right) \leq R.$$

Hence $T(u) \subset \overline{B}_R(0)$.

However, ϕ^{-1} being nonlinear, the values of T are not necessarily convex and thus no available fixed point theorem for multivalued maps is applicable. To overcome this drawback, we use the idea from paper [3]. It consists in passing from the space $C(I)$ to the product space $C(I) \times \mathbb{R}$, and defining a decomposable map which is the composition of to maps with convex values. More precisely, we denote

$$\Omega = \{(u,x) \in C(I) \times \mathbb{R} : \ |u|_\infty < R, \ |x| \leq \rho_R\},$$

and we define the operator $\mathbb{T} : \overline{\Omega} \to 2^{C(I)\times\mathbb{R}}$, $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2)$, by

$$\begin{aligned} \mathbb{T}_1(u,x) &= T(u) = g(u) - \int_\cdot^1 \phi^{-1}\left(\int_0^\tau F(s,u(s))\right), && (3.7)\\ \mathbb{T}_2(u,x) &= g(u). && (3.8)\end{aligned}$$

Clearly, if $(u,x) \in \overline{\Omega}$ is a fixed point of $\mathbb{T}$, then u solves (3.2). The operator $\mathbb{T}$ is decomposable, namely $\mathbb{T} = \Psi\Phi$, where $\Phi = (\Phi_1, \Phi_2)$, $\Psi = (\Psi_1, \Psi_2)$,

$$\begin{aligned} \Phi_1(u,x) &= \int_0^\cdot F(s,u(s)), \quad \Phi_2(u,x) = g(u), && (3.9)\\ \Psi_1(u,x) &= x - \int_\cdot^1 \phi^{-1}(u), \quad \Psi_2(u,x) = x. \end{aligned}$$

Notice that Ψ is single-valued and Φ is multivalued with convex values. Therefore, it is justified that the theory developed in Section 2 be applied to this setting. Here we take

$$\begin{aligned} X &= Y = C(I) \times \mathbb{R},\\ K_X &= \{(u,x) \in C(I) \times \mathbb{R} : \ |x| \leq \rho_R\},\\ K_Y &= \{(u,x) \in C(I) \times \mathbb{R} : \ |u|_\infty \leq |M_{R+\varepsilon}|_{L^1(I)}, \ |x| \leq \rho_R\}. \end{aligned}$$

It is clear that Ω is open in K_X. Notice that, as a consequence of Lemma 3.1, Φ_1 is usc and, by a direct application of the Ascoli-Arzelá theorem, it maps $\overline{\Omega}$ into a relatively compact subset of $C(I)$. Also, Φ_1 has closed and convex values, see [27, Theorem 2]. Hence, Φ is usc with closed and convex values and maps $\overline{\Omega}$ into a relatively compact subset of K_Y. Moreover, Ψ is a continuous single-valued map and thus the maps Φ, Ψ satisfy conditions (i) and (ii).

We shall deduce the existence of a fixed point of $\mathbb{T}$ from Theorem 2.3. To do this it suffices to show that alternative (b) does not hold. Here $x_0 = (0,0)$. Assume that $(u,x) \in \lambda\mathbb{T}(u,x)$ for some $\lambda \in (0,1)$. Then, there exists $v \in \mathcal{N}_F(u)$ such that

$$\begin{aligned} u &= \lambda\left(g(u) - \int_\cdot^1 \phi^{-1}\left(\int_0^\tau v(s)\right)\right),\\ x &= \lambda g(u). \end{aligned}$$

Then in view of (3.6), we have

$$|u|_\infty \leq \lambda R < R \quad \text{and} \quad |x| \leq \lambda\rho_R \leq \rho_R.$$

Hence $(u,x) \in \Omega$ and consequently alternative (b) does not hold. Therefore $\mathbb{T}$ has a fixed point as wished. □

Notice that the existence of solutions to the differential inclusion (3.2) is interesting itself when the right-hand side in (3.1) is discontinuous in both variables. Several papers in the literature are devoted to study this matter. We refer the reader to the recent papers [5,20,21] and to the classical one [24], which are based on variational techniques, and also to the monograph [16] concerning first-order discontinuous systems. Unlike these works, we look for Carathéodory solutions for (3.1) even in the case of nonlinearities which are discontinuous in both variables. To this aim, we need a *transversality* condition on f in order to guarantee that the solutions of (3.2) are in fact solutions of (3.1).

Lemma 3.3. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions* (C_1) *and*

(C_2)**:** *There is a countable number of functions $\gamma_n \in C^1(I)$, ($n \in \mathbb{N}$), with $\phi \circ \gamma_n' \in W^{1,1}(I)$, and a countable number of closed subintervals I_n of I such that*

$$\{(\phi(\gamma_n'(t)))'\} \cap F(t, \gamma_n(t)) \subset \{f(t, \gamma_n(t))\} \quad \textit{for a.a. } t \in I_n, \ \textit{all } n \in \mathbb{N} \tag{3.10}$$

and

$$f(t, \cdot) \textit{ is continuous on } \mathbb{R} \setminus \bigcup_{\{n:\, t\in I_n\}} \{\gamma_n(t)\} \quad \textit{for a.a. } t \in I.$$

Then the set of solutions to problem (3.2) *coincides with the set of solutions to* (3.1).

Proof. Let u be a solution of (3.2). Then

$$(\phi(u'(t)))' \in F(t, u(t)) \quad \text{for a.a. } \ t \in I. \tag{3.11}$$

We shall prove that u solves the discontinuous problem (3.1). Consider the set

$$J_n := \{t \in I_n \, : \, u(t) = \gamma_n(t)\}\,, \quad n \in \mathbb{N}.$$

By [28, Lemma 6.92],

$$(\phi(u'(t)))' = (\phi(\gamma_n'(t)))' \quad \text{for a.a. } \ t \in J_n.$$

Hence, from (3.11) one follows that

$$(\phi(\gamma_n'(t)))' \in F(t, u(t)) = F(t, \gamma_n(t)) \quad \text{for a.a. } \ t \in J_n.$$

This, joint with condition (3.10), implies

$$(\phi(\gamma_n'(t)))' = f(t, \gamma_n(t)) \quad \text{for a.a. } \ t \in J_n,$$

equivalently

$$(\phi(u'(t)))' = f(t, u(t)) \quad \text{for a.a. } \ t \in J_n.$$

Thus u satisfies the initial discontinuous differential equation a.e. in $J = \bigcup_{n\in\mathbb{N}} J_n$. Finally, one has

$$F(t, u(t)) = \{f(t, u(t))\} \quad \text{for } \ t \in I \setminus J.$$

This together with (3.11) shows that u also satisfies (3.1) a.e. in $I \setminus J$. Therefore u solves (3.1) in I. □

Remark 3.2. The transversality condition (3.10) was recently considered in [26, 27] and it comes from the concept of admissible discontinuity curve presented in [15, 22] for the case $\phi(x) = x$ for all $x \in \mathbb{R}$.

In particular, a sufficient condition for a curve $\gamma : I \to \mathbb{R}$ to satisfy (3.10) on some closed interval $\tilde{I} \subset I$ can be stated as follows: there exist $\delta, \epsilon > 0$ such that

$$(\phi(\gamma'(t)))' + \delta \le f(t, y) \quad \text{for a.a. } t \in \tilde{I} \text{ and all } y \in [\gamma(t) - \epsilon, \gamma(t) + \epsilon]\,, \tag{3.12}$$

or

$$(\phi(\gamma'(t)))' - \delta \ge f(t, y) \quad \text{for a.a. } t \in \tilde{I} \text{ and all } y \in [\gamma(t) - \epsilon, \gamma(t) + \epsilon]\,. \tag{3.13}$$

Observe that (3.12) and (3.13) recall the notion of strict lower and upper solutions for the differential equation $(\phi(u'))' = f(t,u)$.

Remark 3.3. Suppose that for a.a. $t \in I$, the function $f(t,\cdot)$ is discontinuous at a countable number of points $\{x_n\}_{n\in\mathbb{N}}$. In this case, the curves $\gamma_n : I \to \mathbb{R}$ are constant functions $\gamma_n(t) \equiv x_n$. Then $(\phi(\gamma'(t)))' = 0$ and thus condition (3.12) holds whenever

$$\inf_{t\in I,\, x\in\mathbb{R}} f(t,x) > 0. \tag{3.14}$$

Condition (3.14) is similar to that in [4], where $f(t,x) = f(x)$ is assumed to be continuous almost everywhere and to satisfy the inequality $\operatorname{essinf}_{x\in\mathbb{R}} f(x) > 0$.

As a straightforward consequence of Theorem 3.2 and Lemma 3.3, we have the following existence result for problem (3.1).

Theorem 3.4. *Under assumptions of Theorem* 3.2, *if in addition f satisfies condition* (C_2), *then problem* (3.1) *has at least one Carathéodory solution.*

An example regarding p-Laplacian equations is now presented in order to illustrate the previous existence result.

Example 3.5. Let us consider the problem

$$\begin{cases} (|u'|^{p-2}\, u')' = f(t,u) & \text{for a.a. } t \in I, \\ u'(0) = u(1) = 0, \end{cases} \tag{3.15}$$

with $p > 1$ and

$$f(t,x) = |x| \left[\left|\frac{1}{x}\right|\right] + \frac{1}{2\sqrt{t}} \qquad \text{for } x \neq 0,$$

and

$$f(t,0) = 1 + \frac{1}{2\sqrt{t}},$$

where $[x]$ denotes the integer part of x. Here $a = b = +\infty$ and $g\,(u) = 0$, so conditions (3.5) and (3.6) hold with a sufficiently large R. Notice that $f(t,\cdot)$ is continuous on $\mathbb{R}\backslash\bigcup_{n\in\mathbb{N}}\{\pm 1/n\}$ and, as a consequence of Remark 3.3, it is direct to prove that the transversality condition (3.10) holds.

Therefore, Theorem 3.4 implies that problem (3.15) has at least one solution.

4. Three solutions theorem for ϕ-Laplacian problems with nonlinear functional boundary conditions: lower and upper solutions

In this section, we continue to investigate problem (3.1). For a better localization of solutions and for giving multiplicity results, we shall combine the fixed point index theory for decomposable multivalued maps with lower and upper solutions techniques.

Let us introduce the notion of lower and upper solutions for problem (3.1).

Definition 4.1. A function $\alpha \in C^1(I)$ is said to be a *lower solution* for (3.1) if $\phi\circ\alpha' \in W^{1,1}(I)$ and

$$(\phi(\alpha'(t)))' \geq f(t,\alpha(t)) \quad \text{for a.a. } t \in I,\ \alpha'(0) \geq 0,\ g\,(\alpha) \geq \alpha\,(1)\,.$$

Similarly, a function $\beta \in C^1(I)$ with $\phi \circ \beta' \in W^{1,1}(I)$ is an *upper solution* for (3.1) if it satisfies the above inequalities with the reverse order.

Given α and β lower and upper solutions for (3.1), such that $\alpha < \beta$ on I, in the sequel we assume the following condition on g:

(H_g)**:** For every $u, v \in C(I)$ with $\alpha \leq u \leq v \leq \beta$ and $u(1) = v(1)$, one has $g(u) \leq g(v)$.

Remark 4.1. Note that if condition (H_g) holds for a pair (α, β) then it also holds for any other pair (α', β') with $\alpha \leq \alpha' < \beta' \leq \beta$. This remark is implicitely used to obtain multiple solutions, at the end of this section.

Furthermore, we define the bounded functional $\widetilde{g} : C(I) \to \mathbb{R}$ by

$$\widetilde{g}(u) = \varphi(1, g(u))), \tag{4.1}$$

where

$$\varphi(t, x) = \max\{\min\{x, \beta(t)\}, \alpha(t)\} \quad \text{for} \quad (t, x) \in I \times \mathbb{R}.$$

Note that for every $(t, x) \in I \times \mathbb{R}$, one has $\alpha(t) \leq \varphi(t, x) \leq \beta(t)$, more exactly, $\varphi(t, x) = \alpha(t)$ if $x < \alpha(t)$; $\varphi(t, x) = x$ if $x \in [\alpha(t), \beta(t)]$; and $\varphi(t, x) = \beta(t)$ if $x > \beta(t)$. Consequently, for every $u \in C(I)$, one has

$$\alpha(1) \leq \widetilde{g}(u) \leq \beta(1).$$

Hence

$$|\widetilde{g}(u)| \leq \rho := \max\{|\alpha(1)|, |\beta(1)|\} \quad \text{for every} \quad u \in C(I).$$

Looking for solutions between α and β, it is enough that weaker conditions on f are required. More exactly we shall assume the following conditions:

(H_1)**:** The composed function $f(\cdot, u(\cdot))$ is measurable for every $u \in C(I)$, and there exists $M \in L^1(I)$ such that $|f(t, x)| \leq M(t)$ for all $x \in [\alpha(t), \beta(t)]$ and a.a. $t \in I$,

$$|M|_{L^1(I)} < b,$$

$$\rho + \phi^{-1}\left(|M|_{L^1(I)}\right) < b. \tag{4.2}$$

(H_2)**:** There is a countable number of functions $\gamma_n \in C^1(I)$ $(n \in \mathbb{N})$ with $\phi \circ \gamma_n' \in W^{1,1}(I)$, and a countable number of closed subintervals I_n of I such that (3.10) holds and

$$f(t, \cdot) \text{ is continuous on } [\alpha(t), \beta(t)] \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\} \quad \text{for a.a. } t \in I.$$

Theorem 4.1. *Let α and β be lower and upper solutions for* (3.1)*, with $\alpha < \beta$ on I, and let conditions* (H$_g$), (H$_1$) *and* (H$_2$) *hold. Then problem* (3.1) *has at least one Carathéodory solution u such that $\alpha \leq u \leq \beta$.*

Proof. Consider the modified problem

$$\begin{cases} (\phi(u'))' \in \tilde{F}(t, u) & \text{for a.a. } t \in I, \\ u'(0) = 0, \ u(1) = \widetilde{g}(u), \end{cases} \tag{4.3}$$

which is the regularization of the problem

$$\begin{cases} (\phi(u'))' = \tilde{f}(t, u) & \text{for a.a. } t \in I, \\ u'(0) = 0, \ u(1) = \widetilde{g}(u), \end{cases} \tag{4.4}$$

where

$$\tilde{f}(t, x) = f(t, \varphi(t, x)) \quad \text{for } (t, x) \in I \times \mathbb{R},$$

and $\tilde{F}$ is defined as F in (3.3) just by replacing f by $\tilde{f}$. Note that $\tilde{f}$ like f satisfies (H$_1$) and also

$$\tilde{F}(t, x) \subset F(t, x) \quad \text{for all } x \in [\alpha(t), \beta(t)], \ t \in I.$$

Indeed, for $x \in (\alpha(t), \beta(t))$, clearly one has $\tilde{F}(t,x) = F(t,x)$. Let $x = \alpha(t)$. Then

$$\begin{aligned} \tilde{F}(t, \alpha(t)) &= \bigcap_{0<\varepsilon\leq\beta(t)-\alpha(t)} \overline{\text{co}}\widetilde{f}\left(t, \overline{B}_\varepsilon(\alpha(t))\right) = \bigcap_{0<\varepsilon\leq\beta(t)-\alpha(t)} \overline{\text{co}} f(t, [\alpha(t), \alpha(t)+\varepsilon]) \\ &\subset \bigcap_{0<\varepsilon\leq\beta(t)-\alpha(t)} \overline{\text{co}} f(t, [\alpha(t)-\varepsilon, \alpha(t)+\varepsilon]) = F(t, \alpha(t)). \end{aligned}$$

Similarly $\tilde{F}(t, \beta(t)) \subset F(t, \beta(t))$.

Denote by $\tilde{\mathbb{T}}$, $\tilde{\Phi}$ and $\tilde{\Psi}$ the operators defined as in (3.7), (3.8) and (3.9) just by replacing F by $\tilde{F}$. We follow the proof of Theorem 3.2, where this time for a fixed number R with

$$\rho + \phi^{-1}\left(|M|_{L^1(I)}\right) < R < b, \tag{4.5}$$

we take

$$\begin{aligned} \Omega_R &= \{(u,x) \in C(I) \times \mathbb{R} : |u|_\infty < R, \ |x| \leq R\}, \\ K_X &= \{(u,x) \in C(I) \times \mathbb{R} : |x| \leq R\}, \\ K_Y &= \left\{(u,x) \in C(I) \times \mathbb{R} : |u|_\infty \leq |M|_{L^1(I)}, \ |x| \leq R\right\}. \end{aligned}$$

It is easy to see that the condition (4.2) guarantees that $\tilde{\mathbb{T}}\left(\overline{\Omega}_R\right) \subset \Omega_R$ and so from Theorem 2.4

$$ind_K\left(\tilde{\Phi}, \tilde{\Psi}, \Omega_R\right) = 1.$$

Thus $\tilde{\mathbb{T}}$ has a fixed point in Ω_R. Let us show that if $(u,x) \in \Omega_R$ is a fixed point of $\tilde{\mathbb{T}}$, then $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$. To prove this, first observe that

$$u(1) = \widetilde{g}(u) = \varphi(1, g(u))) \in [\alpha(1), \beta(1)].$$

So, if $u \ngeq \alpha$, then there exists $t_0 \in [0,1)$ such that

$$u(t_0) - \alpha(t_0) = \min_{t\in I}(u(t) - \alpha(t)) < 0,$$

and $u(t_0) - \alpha(t_0) < u(t) - \alpha(t)$ for all $t \in (t_0, 1]$. If $t_0 \neq 0$, then we have that $u'(t_0) = \alpha'(t_0)$ and

$$\forall r > 0 \ \exists\, t_r \in (t_0, t_0 + r) \text{ such that } \alpha'(t_r) < u'(t_r).$$

On the other hand, by the continuity of $u-\alpha$, there exists $\varepsilon > 0$ such that for all $t \in (t_0, t_0+\varepsilon)$ we have $u(t) - \alpha(t) < 0$. Then $\tilde{f}(t, u(t)) = f(t, \alpha(t))$ for all $t \in [t_0, t_0 + \varepsilon)$. Now, since u is a solution to (4.3), it follows that

$$(\phi(u'(t)))' = f(t, \alpha(t)) \quad \text{for a.e. } t \in [t_0, t_0 + \varepsilon),$$

and thus, for $t \in [t_0, t_0 + \varepsilon)$,

$$\begin{aligned} \phi(u'(t)) - \phi(\alpha'(t)) &= \int_{t_0}^t \left((\phi(u'(s)))' - (\phi(\alpha'(s)))'\right) ds \\ &= \int_{t_0}^t \left(f(s, \alpha(s)) - (\phi(\alpha'(s)))'\right) ds \leq 0, \end{aligned}$$

which, joint to the fact that ϕ is increasing, implies that $u'(t) \leq \alpha'(t)$ for $t \in [t_0, t_0 + \varepsilon)$, a contradiction. If $t_0 = 0$, then $u - \alpha$ has a minimum at $t_0 = 0$, and thus $u'(0) - \alpha'(0) \geq 0$. In addition, $u'(0) = 0$ and $\alpha'(0) \geq 0$, so $u'(0) = \alpha'(0)$ and the proof follows as in the case $t_0 \neq 0$. In a similar way we can see that $u \leq \beta$ on I.

Let us now prove that $u(1) = g(u)$. This, in virtue of the equality $u(1) = \widetilde{g}(u) = \varphi(1, g(u))$, is equivalent to showing that $\varphi(1, g(u)) = g(u)$, but this happens if and only if $g(u) \in [\alpha(1), \beta(1)]$. Suppose that $g(u) < \alpha(1)$, then $\varphi(1, g(u)) = \alpha(1) = u(1)$, and by condition (H_g) and the definition of the lower solution, we obtain

$$\alpha(1) > g(u) \geq g(\alpha) \geq \alpha(1),$$

a contradiction. Analogously, we can show that $g(u) \leq \beta(1)$. Therefore $u(1) = g(u)$.

Next observe that from $\alpha \leq u \leq \beta$, we have $\widetilde{F}(t, u(t)) \subset F(t, u(t))$ for $t \in I$. Therefore u solves (3.2). Finally, Lemma 3.3 implies that u is a Carathéodory solution of (3.1). □

Remark 4.2. In the proof of the previous theorem, it was shown that any fixed point of $\tilde{\mathbb{T}}$ belongs to $\overline{\Omega}$, where

$$\Omega := \{(u, x) \in C(I) \times \mathbb{R} : \alpha < u < \beta, \ |x| \leq R\}.$$

Consequently, if $\tilde{\mathbb{T}}$ is fixed point free on the relative boundary of Ω, then using the excision property of the fixed point index gives

$$ind_K\left(\tilde{\Phi}, \tilde{\Psi}, \Omega\right) = ind_K\left(\tilde{\Phi}, \tilde{\Psi}, \Omega_R\right) = 1. \tag{4.6}$$

To guarantee that $\tilde{\mathbb{T}}$ is fixed point free on the relative boundary of Ω, making possible formula (4.6), and to obtain multiple solutions, we need the concept of *strict lower and upper solutions* for (3.1).

Definition 4.2. A function $\alpha \in C^1(I)$ is said to be a *strict lower solution* for (3.1) if $\phi \circ \alpha' \in W^{1,1}(I)$ and there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}&(\phi(\alpha'(t)))' \geq f(t, x) \quad \text{for a.e. } t \in I \text{ and all } \ x \in [\alpha(t), \alpha(t) + \varepsilon_0],\\ &\alpha'(0) \geq 0 \ \text{ and } \ g(\alpha) > \alpha(1).\end{aligned}$$

Similarly, a function $\beta \in C^1(I)$ is said to be a *strict upper solution* for (3.1) if $\phi \circ \beta' \in W^{1,1}(I)$ and there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}&(\phi(\beta'(t)))' \leq f(t, x) \quad \text{for a.e. } t \in I \text{ and all } \ x \in [\beta(t) - \varepsilon_0, \beta(t)],\\ &\beta'(0) \leq 0 \ \text{ and } \ g(\beta) < \beta(1).\end{aligned}$$

Lemma 4.2. *Let α, β be strict lower and upper solutions and let u be a solution for problem* (3.1)*. Then $\alpha \leq u$ implies that $\alpha < u$ and $u \leq \beta$ implies that $u < \beta$.*

Proof. Let $\alpha \leq u$ and $u(t_0) = \alpha(t_0)$ at $t_0 \in (0, 1)$. Moreover, suppose that $\alpha(t) < u(t)$ for all $t \in (t_0, 1]$. Then $u'(t_0) = \alpha'(t_0)$ and for every $r > 0$ there exists $t_r \in (t_0, t_0 + r)$ such that $\alpha'(t_r) < u'(t_r)$, and so that $\phi(\alpha'(t_r)) < \phi(u'(t_r))$, since ϕ is increasing.

We can choose $r > 0$ such that for every $t \in (t_0, t_r)$ we have $u(t) \leq \alpha(t) + \varepsilon_0$. Hence, for a.a. $t \in (t_0, t_r)$,

$$(\phi(\alpha'(t)))' \geq f(t, u(t)),$$

which leads to the contradiction

$$\begin{aligned}\phi(u'(t_r)) - \phi(\alpha'(t_r)) &= \int_{t_0}^{t_r} \left((\phi(u'(s)))' - (\phi(\alpha'(s)))'\right) ds\\ &= \int_{t_0}^{t_r} \left(f(s, u(s)) - (\phi(\alpha'(s)))'\right) ds \leq 0.\end{aligned}$$

If $u(0)=\alpha(0)$, there are two possibilities: either $\alpha'(0)>0$ or $\alpha'(0)=0$. The condition $\alpha'(0)>0=u'(0)$ implies that $\alpha\not\leq u$, a contradiction. On the other hand, if $\alpha'(0)=0=u'(0)$, the reasoning follows as in the case $t_0\in(0,1)$.

Finally, if $u(1)=\alpha(1)$, the following contradiction is reached by condition (H_g),

$$\alpha\left(1\right)=u\left(1\right)=g\left(u\right)\geq g\left(\alpha\right)>\alpha\left(1\right).$$

Analogously, it can be seen that $u\leq\beta$ implies $u<\beta$. □

Remark 4.3. Notice that even in the case of Carathéodory nonlinearities Lemma 4.2 does not hold if the strict lower and upper solutions are defined just by taking strict inequalities in Definition 4.1, see [9, Chapter III-1].

Remark 4.4. According to Lemma 4.2, if in Theorem 4.1, α is a strict lower solution, then the guaranteed solution u satisfies $\alpha<u$, while if β is a strict upper solution, then $u<\beta$.

The localization conclusion derived from the previous result allows us to complement Example 3.5.

Example 4.3. Consider the problem (3.15) in Example 3.5. One may easily check that

$$\alpha(t)=(2p-2)t^{(2p-1)/(2p-2)}/(2p-1)-2$$

is a strict lower solution for problem (3.15) and that any $\beta(t)=c$ with $c>0$ is a strict upper solution for it. Hence, according to Remark 4.4, problem (3.15) has a nontrivial solution u such that $\alpha<u\leq0$.

As a consequence of the previous results, a three solutions theorem in the line of that by Amann [1] can be easily derived by assuming the existence of two pairs of strict lower and upper solutions for (3.1) satisfying certain order relations.

Theorem 4.4. *Assume that there exist α_1,α_2 strict lower solutions and β_1,β_2 strict upper solutions for* (3.1) *such that*

$$\alpha_1<\beta_1<\beta_2,\quad \alpha_1<\alpha_2<\beta_2,\quad \alpha_2(\tau)>\beta_1(\tau)\ \textit{ for some } \tau\in I,$$

and conditions (H_g), (H_1)-(H_2) hold with $\alpha=\alpha_1$ and $\beta=\beta_2$. Then the problem (3.1) *has at least three solutions u_1, u_2 and u_3 such that*

$$\alpha_1<u_1<\beta_1,\quad \alpha_2<u_2<\beta_2,$$

and there exist $t_1,t_2\in I$ with

$$u_3(t_1)<\alpha_2(t_1)\quad \textit{and}\quad u_3(t_2)>\beta_1(t_2).$$

Proof. Let $R>0$ satisfy (4.5), where $\rho=\max\left\{\left|\alpha_1\left(1\right)\right|,\left|\beta_2\left(1\right)\right|\right\}$. Consider the open sets

$$\begin{aligned}&\Omega=\{(u,x)\in\mathcal{C}(I)\times\mathbb{R}\,:\,\alpha_1<u<\beta_2 \text{ on } I,\ |x|\leq R\},\\&\Omega_1=\{(u,x)\in\Omega\,:\,\alpha_1<u<\beta_1 \text{ on } I\},\\&\Omega_2=\{(u,x)\in\Omega\,:\,\alpha_2<u<\beta_2 \text{ on } I\}.\end{aligned}$$

By Remarks 4.2 and 4.4,

$$ind_K\left(\widetilde{\Phi},\widetilde{\Psi},\Omega\right)=ind_K\left(\widetilde{\Phi},\widetilde{\Psi},\Omega_1\right)=ind_K\left(\widetilde{\Phi},\widetilde{\Psi},\Omega_2\right)=1.$$

Hence, the operator $\widetilde{\mathbb{T}}$ has two different fixed points: u_1 and u_2 such that $\alpha_1<u_1<\beta_1$ and $\alpha_2<u_2<\beta_2$. Note that they are distinct due to the fact that there exists $\tau\in I$ such that

$\alpha_2(\tau) > \beta_1(\tau)$ which implies that $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$. Moreover, by the additivity property of the fixed point index we have

$$\begin{aligned} ind_K\left(\widetilde{\Phi}, \widetilde{\Psi}, \Omega \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})\right) &= ind_K\left(\widetilde{\Phi}, \widetilde{\Psi}, \Omega\right) - ind_K\left(\widetilde{\Phi}, \widetilde{\Psi}, \Omega_1\right) - ind_K\left(\widetilde{\Phi}, \widetilde{\Psi}, \Omega_2\right) \\ &= -1. \end{aligned}$$

Therefore, the operator $\widetilde{\mathbb{T}}$ has at least one fixed point (u_3, x_3) in $\Omega \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$, hence with $u_3(t_1) < \alpha_2(t_1)$ and $u_3(t_2) > \beta_1(t_2)$ for some $t_1, t_2 \in I$. Furthermore, since for $\alpha_1 \leq u \leq \beta_2$, one has $\widetilde{F}(t, u(t)) \subset F(t, u(t))$ for $t \in I$, we infer that the three functions u_1, u_2 and u_3 are solutions of (3.2), and finally, based on Lemma 3.3, that they are Carathéodory solutions of (3.1). $\square$

Next we give an example of second–order problem which has three solutions and it is not covered by the previous literature, as far as the authors are aware.

Example 4.5. Consider the second–order problem

$$\begin{cases} u'' = f(t,u) \quad \text{for a.a. } t \in I, \\ u'(0) = 0, \ \max_{t\in I} u(t) = 10\cos u(1), \end{cases} \tag{4.7}$$

with

$$f(t,x) = 2\cos x + h(x - 2t^2) \quad \text{for every } (t,x) \in I \times \mathbb{R},$$

where the function h is defined below.

Note that in this case $g(u) = -10\cos u(1) + u(1) + \max_I u$, which clearly is continuous and satisfies condition (H_g).

Let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of all rational numbers. Now the function $h : \mathbb{R} \to \mathbb{R}$ is given by

$$h(x) = \sum_{n\,:\,q_n<x} 2^{-n}.$$

It is continuous at the irrational numbers and discontinuous at the rational ones. Besides, $h(x) \in (0,1)$ for every $x \in \mathbb{R}$.

Observe that $\alpha_1 \equiv -\pi$ and $\alpha_2 \equiv \pi$ are strict lower solutions for the problem (4.7) and $\beta_1 \equiv 0$ and $\beta_2 \equiv 2\pi$ are strict upper solutions for (4.7).

In addition, for a.a. $t \in I$ the function $x \mapsto f(t,x)$ is continuous on $[-\pi, 2\pi]\setminus\bigcup_{\{n\,:\,t\in I_n\}}\{\gamma_n(t)\}$, where for each $n \in N$,

$$\gamma_n(t) = 2t^2 + q_n \quad \text{for all } t \in I_n = [0,1].$$

For every $n \in \mathbb{N}$ and for all $t \in [0,1]$ and $x \in \mathbb{R}$, we have

$$f(t,x) = 2\cos(x) + h(x - 2t^2) \leq 3 < 4 = \gamma_n''(t),$$

so the transversality condition (H_2) is satisfied since the sufficient condition (3.13) given in Remark 3.2 holds.

Therefore, Theorem 4.4 ensures that problem (4.7) has at least three nontrivial solutions u_1, u_2 and u_3 satisfying that $-\pi < u_1 < 0$, $\pi < u_2 < 2\pi$ and there exist $t_1, t_2 \in I$ such that $u_3(t_1) < \pi$ and $u_3(t_2) > 0$.

To finish this section, we emphasize that the fixed point formulation is simpler if the functional $g : C(I) \to \mathbb{R}$ is linear and continuous with $g(1) \neq 1$.

In that case, the existence of solutions to problem (3.1) is equivalent to the existence of fixed points to the operator $T : C(I) \to 2^{C(I)}$ given by

$$T\left(u\right)\left(t\right) = \frac{g\left(S\right)}{g\left(1\right) - 1} - S(t),$$

where

$$S(t) = \int_t^1 \phi^{-1}\left(\int_0^s f(r, u(r))\, dr\right) ds.$$

Therefore, in order to study the problem

$$\begin{cases} (\phi(u'))' \in F(t, u) & \text{for a.a. } t \in I, \\ u'(0) = 0, \quad u\left(1\right) = g\left(u\right), & \end{cases}$$

we can simply consider the multivalued operator $\mathbb{T} : C(I) \to 2^{C(I)}$ as the composition $\mathbb{T} = \Psi\Phi$, where $\Phi : C(I) \to 2^{C(I)}$ is defined as

$$\left(\Phi u\right)\left(t\right) = \int_0^t F(r, u(r))\, dr,$$

and $\Psi : C(I) \to C(I)$ is given by

$$\left(\Psi v\right)\left(t\right) = \frac{1}{g\left(1\right) - 1} g\left(\int_\cdot^1 \phi^{-1}\left(v(s)\right) ds\right) - \int_t^1 \phi^{-1}(v(s))\, ds.$$

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