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Braided Crossed Modules and Loday-Pirashvili category

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by

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Contents

Introduction

Monoidal categories were introduced by Jean Bénabou [7] and Saunders Mac Lane [45] in order to generalize the idea of the tensor product in arbitrary categories.

It is well known that, in the case of the usual tensor product for vector spaces, there is a natural isomorphism between $V \otimes W$ and $W \otimes V$. In order to study if this property also holds in an arbitrary monoidal category, i.e. when the tensor product is (not strictly) commutative, Joyal and Street defined in [38] the concept of braiding for monoidal categories as a natural isomorphism $\tau_{A,B} : A \otimes B \to B \otimes A$.

When we try to study the concept of braiding for the simplest case of monoidal categories (categorical monoids or internal categories in the category of monoids, which is the same as strict monoidal small categories) we encounter that not all internal morphisms are internal isomorphisms, so the braiding cannot be an arbitrary internal morphism satisfying simple properties. To avoid this problem we can work with (strict) categorical groups instead of categorical monoids, obtaining immediately the definition of braided categorical group (see [9, 38]).

On the other hand, in 1949 Whitehead [51] introduced the notion of crossed module of groups as an algebraic model for 2-type homotopy spaces (i.e. connected spaces with trivial homotopy groups in dimension > 2). In 1984, Conduché [16] (see also [9]) introduced the notion of braided crossed module of groups as a particular case of 2-crossed module of groups.

It is well known that the categories of crossed modules of groups and categorical groups are equivalent, and Joyal and Street proved in [38] that the notion of braiding for categorical groups provides an equivalent category to the category of braided crossed modules of groups [9, 16].

The notions of crossed modules for associative algebras [18], Lie algebras [40] and Leibniz algebras [43] appear trying to emulate crossed modules of groups, and it was proven that the correspondent categories are equivalent to their respective internal categories.

Keeping in mind what is done for groups, in this thesis we will give definitions of braidings for the aforementioned internal categories and crossed modules. The case of associative algebras is not complex because the associativity allows us to work in a natural way with braidings on semigroupal categories [17]. The notion of braiding for Lie algebras was already given by Ulualan [50]. On the other hand, Ellis [20] defined the notion of 2-crossed module of Lie algebras, also studied by Martins and Picken [46]. We will use a slightly different definition for braiding for crossed modules of Lie algebras than the one given by Ulualan [50], since we want a parallelism between the examples of braided crossed modules of groups and braided crossed modules of Lie algebras, and we also require braided crossed modules to be a particular case of 2-crossed modules, as it happens in the case of groups.

Leibniz algebras appear in mathematics as a "non-antisymmetric" case of Lie algebras. Bearing this in mind, in this thesis, we will show how to extend the idea of braiding for crossed modules and internal categories of Lie algebras to the Leibniz setting. After introducing these notions, we will prove the equivalence between braided crossed modules of Leibniz algebras and braided categorical Leibniz algebras, and we will show the parallelism between its examples and the ones given for groups, associative algebras and Lie algebras.

Although Lie algebras are a subvariety of the variety of Leibniz algebras, Loday and Pirashvili found in [44] that Leibniz algebras can be seen as a full coreflective subcategory of a specific type of Lie objects. They introduced a new tensor product in the category of linear maps of vector spaces and internalised the concept of Lie algebra in a (braided) symmetric monoidal category. This realisation proved to be very handful studying different problems in Leibniz algebras, as Lie theory is much better developed, see [12, 23, 49] for example. We will use this category to extend the concept of braiding from the Lie case to the Leibniz case.

The concept of central extension of groups or Lie algebras is highly relevant in mathematics, and it plays a fundamental role in several areas of physics as well. This notion was extended to crossed modules of groups or Lie algebras. The study of central extensions in the categories of crossed modules was initiated in [48] for groups and in [13] for Lie algebras, and it remains a current research topic, as shown by the different literature tackling this issue.

Since crossed modules of groups and Lie algebras are a generalisation of groups and Lie algebras, it is essential to search, in the category of crossed modules of groups or Lie algebras, extensions of classical results in the theory of groups or Lie algebras.

In [26], Fukushi gave a braided version of the results on universal central extensions of crossed modules of groups provided by Norrie in [48]. He found a natural braiding on the universal central extension of a crossed module of groups which behaves well with one braided crossed module. However, it is not the archetype of universal central extension in the category of braided crossed modules since, in this category, it is necessary to add additional restrictions including the braiding on the notions of centre and commutator.

In this work, we will devise a braided version of the results given by Casas and Ladra in [13] for braided crossed modules of Lie K-algebras; more precisely, we will study universal central extensions in the category of braided Lie crossed modules $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$. For that purpose, we will need the definition of centre and commutator given by Huq in [35] in the braided context.

Note that the framework of Chapter 3 is different from that given in [14], since the category $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ is not a Birkhoff subcategory of $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$.

The study of the internalisation of Lie algebras is also a very handful tool, as we will see in Section 2.4. It also allows proving different properties in several kinds of categories at the same time, such as Lie superalgebras, $\mathbb{Z}$-graded Lie algebras, differential graded Lie algebras or regular Hom-Lie algebras [33]. For example, two important properties that characterise the variety of Lie algebras amongst all the varieties of non-associative algebras, the existence of algebraic exponents [29, 30] or the representability of actions [28], hold also in the categories of Lie objects over certain types of monoidal categories [27, 34].

We want to generalise Loday and Pirashvili construction out of the linear maps category, defining a new tensor product in certain kinds of categories with operations, with the least amount of properties needed to do so, to obtain the Loday-Pirashvili category. Then, we will prove that the Leibniz objects (the internalisation of Leibniz algebras), can be seen as a particular case of Lie objects in the Loday-Pirashvili category. In the particular case of vector spaces, this construction generalises the one given in [44].

Throughout this text, we will suppose that K is a field.

Structure of the thesis

This manuscript is organized as follows. In the preliminaries (Chapter 1), we will recall some basic definitions, and we will give the notion of braiding for semigroupal categories.

In Chapter 2, we will study the braidings for crossed modules and internal categories. We will start showing the first case of braiding, the case of groups (Section 2.1), and we will take it as a base to introduce the notions of braided categorical associative algebra and braided crossed module of associative algebras (Section 2.2). We will show the equivalence of the associative case. Then, in Section 2.3, we will motivate the definition given by Ulualan [50] for braided crossed modules of Lie algebras using our definition of braiding for crossed modules of associative algebras, and we give a simpler definition when $\mathrm{char}(K) \neq 2$. We will also discuss a different definition of braided crossed module of Lie algebras showing its relationship with the associative case. From there, in Section 2.4 we will study the Leibniz algebras case. We show the internalization of a crossed module's notion with a left Lie action of Lie objects in an arbitrary category. We will also define braidings for crossed modules of Lie objects and categorical Lie objects. Then we apply this definition to the Loday-Pirashvili category $\mathcal{LM}_K$, and we will obtain the concepts of braiding for crossed modules of Leibniz algebras and categorical Leibniz algebras. With the new definition of braiding, we will prove the equivalence between braided categories in the Leibniz algebras case, and finally, in Section 2.5, we will see the non-abelian tensor product of groups as an example of a braided crossed module of groups. Furthermore, with our definition of braiding for crossed modules of Lie algebras, we obtain similarly an example of braiding using the non-abelian tensor product of Lie algebras. The same is true for our definition of braiding for crossed modules of Leibniz algebras.

In Chapter 3, we will study two ideas of universal central extensions for braiding crossed modules of Lie algebras. In Section 3.1, we provide the definitions for central extensions in the category of Lie crossed modules $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ and $\boldsymbol{B}$-central extensions in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$, necessary for developing the chapter. In Section 3.2, we construct the universal $\boldsymbol{B}$-central extension for a $\boldsymbol{B}$-perfect braided Lie crossed module and prove that a braided Lie crossed module admits a universal $\boldsymbol{B}$-central extension if and only if it is $\boldsymbol{B}$-perfect. In Section 3.3, we construct the universal $\mathfrak{U}$-central extension for braided crossed modules, which are perfect as Lie crossed modules, where $\mathfrak{U} : \boldsymbol{BX}(\boldsymbol{LieAlg}_K) \longrightarrow \boldsymbol{X}(\boldsymbol{LieAlg}_K)$ is the forgetful functor. In Section 3.4, we study the relation between the universal $\boldsymbol{B}$-central extension and the universal $\mathfrak{U}$-central extension of a braided Lie crossed module. Finally, we prove that both universal extensions exist and coincide for a $\boldsymbol{B}$-perfect braided Lie crossed module.

In Chapter 4, we will define the LP category for different tensor categories, and then we study the Lie objects in some kind of LP categories and their relationship with the Leibniz objects in the base category. In Section 4.1, we will study the different tensor categories: categories with operations, (braided) semigroupal categories and (braided) monoidal categories, and we will construct their Loday-Pirashvili category. In Section 4.2, we will talk about additive categories and we will show that, with some assumptions, we can recover many properties in the maps between the tensor product and the "+" operation. The last section (Section 4.3) is devoted to study the internalization of a Leibniz object and Lie object in a category $\mathcal{C}$, showing that the Liesation functor exists between these categories. Then we will provide a better understanding of Lie objects in the Loday-Pirashvili category of $\mathcal{C}$. To conclude, we will prove that the category of Leibniz objects in $\mathcal{C}$ is a full coreflective subcategory of the Lie objects in the Loday-Pirashvili category of $\mathcal{C}$.

Objectives and hypotheses

This thesis has the following hypothesis and objectives:

Hyp. 1 Strict monoidal categories can be seen as categorical monoids in ***Set***. Similarly, we can think that a strict semigroupal category over an internal category in $\boldsymbol{Vect}_K$ is really an internal associative K-algebra.

Obj. 1 We want to use the idea of braiding in a semigroupal category to make a braiding for categorical associative K-algebras using that the same idea as the fact that the category of crossed modules of groups has a natural idea of braiding utilising the idea of monoidal category.

Hyp. 2 In the category of crossed modules of groups, we can define the concept of braiding. With that construction, we have en equivalence between braided crossed modules of groups and braided categorical groups.

Obj. 2 We want to construct a braiding for crossed modules of associative algebras such that this new category is equivalent to braiding categorical associative K-algebras.

Hyp. 3 The category of associative K-algebras and Lie K-algebras are related with a functor $(-)^{\mathcal{L}}$: $\boldsymbol{AssAlg}_K \to \boldsymbol{LieAlg}_K$, which takes an associative algebra A and gives back a Lie K-algebra $A^{\mathcal{L}}$. This Lie algebra has the same K-vector space as underlying structure and has as multiplication $[x, y] := xy - yx$.

Obj. 3 Use the idea of functor $(-)^{\mathcal{L}}$ to make a new functor from crossed modules of associative K-algebras to crossed modules of Lie K-algebras. We will also define a functor from its braided versions, showing the naturalness when one defines the braiding for the Lie case. We will also show an equivalent (in the sense of categories) definition for braiding crossed modules, which will give us a good example using the non-abelian tensor product.

Hyp. 4 The Loday-Pirashvili category provides us with a way to see Leibniz K-algebras as a particular case of Lie objects when it is well known that Lie K-algebras are a particular case of Leibniz K-algebras.

Obj. 4 Using the Loday-Pirashvili category and internalization, we want to use the definition of braiding for the Lie case to define braiding for crossed modules of Leibniz K-algebras and categorical Leibniz K-algebras. Once done, we will show that these new structures have the Lie case as a particular example. Also, we will have an excellent example of braiding crossed modules of Leibniz K-algebras using the non-abelian tensor product.

Hyp. 5 The braiding for the Lie case gives equivalent categories.

Obj. 5 We want to show that the braidings for the Leibniz case give equivalent categories.

Hyp. 6 The braiding crossed modules of groups have a universal $\mathfrak{U}$-central extension.

Obj. 6 We want to define the $\mathfrak{U}$-central extension for Lie algebras category since many results are true in the group case are true in the Lie case. We also describe the $\boldsymbol{B}$-central extensions using the idea of centre give by Huq [35]. In general, the $\mathfrak{U}$-central extensions and $\boldsymbol{B}$-central extension do not coincide, but we want to show the relationship between the universal ones.

Hyp. 7 The construction of the LP-category given by Loday and Pirashvili can be defined in categories with a small set of properties.

Obj. 7 We want to define the Loday-Pirashvili category using the least properties that are possible. We will do that for categories with operations, (braided) semigroupal categories and (braided) monoidal categories, showing that the tensor product in the Loday-Pirashvili category gives a tensor category of the same type.

Hyp. 8 There is a Liesation functor from Leibniz K-algebras to Lie K-algebras that is a left adjoint to the forgetful functor.

Obj. 8 We want to construct a Liesation functor for Leibniz objects to Lie objects for any category with a small set of properties.

Obj. 9 We will prove that the category of Leibniz objects in $\mathcal{C}$ is a full coreflective subcategory of the Lie objects in the Loday-Pirashvili category of $\mathcal{C}$.

CHAPTER 1

Preliminaries

In this chapter, we will give the basic concepts to delve into the rest of the chapters.

1.1 Internal categories

Definition 1.1.1. Let $\boldsymbol{C}$ be a category with pullbacks.

An *internal category* in $\boldsymbol{C}$ consist of two objects C_1 (*morphisms object*) and C_0 (*objects object*) of $\boldsymbol{C}$, together with the four following morphisms s, t, e, k:

$$\begin{tikzcd} C_0 \arrow[r, "e" description] & C_1 \arrow[l, shift right=2, "t"'] \arrow[l, shift left=2, "s"] & C_1 \times_{C_0} C_1, \arrow[l, "k"'] \end{tikzcd}$$

where $C_1 \times_{C_0} C_1$ is the pullback of t and s.

s is called *source morphism*, t is called *target morphism*, e is called *identity mapping morphism* and k is called *composition morphism*.

In addition, the morphisms must satisfy commutative diagrams that express the usual category laws (see [6]):

(I1)
$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "\mathrm{Id}_{C_0}"'] & C_1 \arrow[d, "s"] \\ & C_0. \end{tikzcd}$$

(I2)
$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "\mathrm{Id}_{C_0}"'] & C_1 \arrow[d, "t"] \\ & C_0. \end{tikzcd}$$

(I3)
$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_1"] \arrow[d, "k"] & C_1 \arrow[d, "s"] \\ C_1 \arrow[r, "s"] & C_0. \end{tikzcd}$$

(I4)
$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "k"] & C_1 \arrow[d, "t"] \\ C_1 \arrow[r, "t"] & C_0. \end{tikzcd}$$

(I5) If $(C_1 \times_{C_0} C_1) \times_{C_0} C_1$ and $C_1 \times_{C_0} (C_1 \times_{C_0} C_1)$ are the pullbacks given by:

$$\begin{tikzcd} (C_1 \times_{C_0} C_1) \times_{C_0} C_1 \arrow[r, "\pi_1'"] \arrow[d, "\pi_2'"'] & C_1 \times_{C_0} C_1 \arrow[d, "t \circ k"] \\ C_1 \arrow[r, "s"'] & C_0, \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "\pi_1''"] \arrow[d, "\pi_2''"'] & C_1 \arrow[d, "t"] \\ C_1 \times_{C_0} C_1 \arrow[r, "s \circ k"'] & C_0, \end{tikzcd}$$

then the following diagram is commutative:

$$\begin{tikzcd} (C_1 \times_{C_0} C_1) \times_{C_0} C_1 \arrow[r, "k \times_{C_0} \mathrm{Id}_{C_1}"] \arrow[d, "i"] & C_1 \times_{C_0} C_1 \arrow[dd, "k"] \\ C_1 \times_{C_0} (C_1 \times_{C_0} C_1) \arrow[d, "\mathrm{Id}_{C_1} \times_{C_0} k"] & \\ C_1 \times_{C_0} C_1 \arrow[r, "k"] & C_1. \end{tikzcd}$$

I6) Let $C_0 \times_{C_0} C_1$ and $C_1 \times_{C_0} C_0$ be the pullbacks given by:

$$\begin{tikzcd} C_0 \times_{C_0} C_1 \arrow[r, "p_1"] \arrow[d, "p_2"'] & C_0 \arrow[d, "\mathrm{Id}_{C_0}"] \\ C_1 \arrow[r, "s"] & C_0, \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} C_0 \arrow[r, "q_1"] \arrow[d, "q_2"'] & C_1 \arrow[d, "t"] \\ C_0 \arrow[r, "\mathrm{Id}_{C_0}"] & C_0. \end{tikzcd}$$

Then we must have the following commutative diagram:

$$\begin{tikzcd} C_0 \times_{C_0} C_1 \arrow[r, "e \times_{C_0} \mathrm{Id}_{C_1}"] \arrow[dr, "p_2"'] & C_1 \times_{C_0} C_1 \arrow[d, "k"] & C_1 \times_{C_0} C_0 \arrow[l, "\mathrm{Id}_{C_1} \times_{C_0} e"'] \arrow[dl, "q_1"] \\ & C_1. & \end{tikzcd}$$

If the conditions are allowed we will refer to the internal category by the 6-tuple (C_1, C_0, s, t, e, k).

Definition 1.1.2. Let $\mathcal{C} = (C_1, C_0, s, t, e, k)$ and $\mathcal{C}' = (C_1', C_0', s', t', e', k')$ be two internal categories in $\boldsymbol{C}$.

An *internal functor* is a pair of morphisms (F_1, F_0), with $F_1 : C_1 \to C_1'$ and $F_0 : C_0 \to C_0'$ such that must satisfy commutative diagrams corresponding to the usual laws satisfied by a functor (see [6]):

(FI1)
$$\begin{tikzcd} C_1 \arrow[r, "s"] \arrow[d, "F_1"] & C_0 \arrow[d, "F_0"] \\ C_1' \arrow[r, "s'"] & C_0'. \end{tikzcd}$$

(FI2)
$$\begin{tikzcd} C_1 \arrow[r, "t"] \arrow[d, "F_1"] & C_0 \arrow[d, "F_0"] \\ C_1' \arrow[r, "t'"] & C_0'. \end{tikzcd}$$

(FI3)
$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[d, "F_0"] & C_1 \arrow[d, "F_1"] \\ C_0' \arrow[r, "e'"] & C_1'. \end{tikzcd}$$

(FI4)
$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "F_1 \times_{F_0} F_1"] \arrow[d, "k"] & C_1' \times_{C_0'} C_1' \arrow[d, "k'"] \\ C_1 \arrow[r, "F_1"] & C_1'. \end{tikzcd}$$

Where $F_1 \times_{F_0} F_1$ is given by:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[dr, dashed, "F_1 \times_{F_0} F_1"] \arrow[drr, bend left, "F_1 \circ \pi_1"] \arrow[ddr, bend right, "F_1 \circ \pi_2"'] & & \\ & C_1' \times_{C_0'} C_1' \arrow[r, "\pi_1'"] \arrow[d, "\pi_2'"] & C_1' \arrow[d, "t'"] \\ & C_1' \arrow[r, "s'"] & C_0'. \end{tikzcd}$$

We denote by $(F_1, F_0) : \mathcal{C} \to \mathcal{C}'$ the internal functor.

Composition of internal functors is defined in the obvious way. This allows us to construct the category of internal categories and internal functors in a category with pullbacks $\boldsymbol{C}$, denoted by $\boldsymbol{ICat}(\boldsymbol{C})$.

An internal category in $\boldsymbol{C}$ will be also called a *categorical object* in $\boldsymbol{C}$.

1.2 Algebras

Definition 1.2.1. Let K be a field and $(M,*)$ be a K-algebra, i.e. a K-vector space together with a K-bilinear multiplication $*$.

A *derivation* over $(M,*)$ is a K-linear map $D\colon\ M\to\ M$ satisfying the *Leibniz rule*, $D(x*y)=D(x)*y+x*D(y),\ x,y\in M$.

Let $(M,*)$ be a K-algebra and $x\in M$. The map $R(x)\colon\ M\to\ M$ defined by $R(x)(y)=y*x$ (right multiplication) is K-linear using the K-bilinearity.

Leibniz algebras are a non-antisymmetric generalization of Lie algebras. They were introduced in 1965 by Bloh in [8], who called them D-algebras, and in 1993 Loday [42] made them famous by studying their (co)homology.

Definition 1.2.2. We say that the K-algebra $(M,*)$ is a *(right) Leibniz K-algebra* if and only if $R(x)$ is a derivation over $(M,*)$ for all $x\in M$. We denote $x*y=\colon [x,y]$ and call the operation $[-,-]$ *Leibniz bracket*.

If in addition, $(M,[-,-])$ is an alternate K-algebra ($[x,x]=0,\ x\in M$) we say that it is a *Lie K-algebra* and we will call the operation $[-,-]$ *Lie bracket*.

Remark 1.2.3. The fact that $R(z)$ is a derivation for all $z\in M$ can be seen in the next identity for any $x,y,z\in M$, called the *Leibniz identity*:

$$[x,[y,z]]=[[x,y],z]-[[x,z],y].$$

If in addition the K-algebra is anticommutative (for example the Lie K-algebras), we can rewrite the equality, obtaining the *Jacobi identity*:

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

Definition 1.2.4. If $(M,*)$ and $(N,\star)$ are K-algebras, a *homomorphism* between them is a K-linear map $M\xrightarrow{f}N$ such that $f(x*y)=f(x)\star f(y)$.

We have the categories $\boldsymbol{AssAlg}_K$, $\boldsymbol{LieAlg}_K$ and $\boldsymbol{LeibAlg}_K$ by taking as objects, associative, Lie and Leibniz K-algebras (respectively), and as morphisms, homomorphisms of K-algebras between them.

We denote by $\boldsymbol{Vect}_K$ and $\boldsymbol{Grp}$ the categories of K-vector spaces and groups, respectively.

Note that it is immediate that the category of Lie algebras is a full subcategory of the category of Leibniz K-algebras.

1.3 Crossed modules

1.3.1 Crossed modules of groups

The crossed modules of groups were introduced by Whitehead in [51].

Definition 1.3.1. A *crossed module of groups* is a pair $(G \xrightarrow{\partial} H, \cdot)$ where G and H are groups, $\cdot$ is an action of H on G by automorphisms and $\partial : G \to H$ is a group homomorphism satisfying:
- ∂ is H-equivariant map (we suppose the conjugation action of H on itself), i.e.

$$\partial(h \cdot g) = \mathrm{Conj}(h)(\partial(g)) = h\partial(g)h^{-1}, \qquad g \in G,\ h \in H,$$

- *Peiffer identity*:

$$\partial(g) \cdot g' = \mathrm{Conj}(g)(g') = gg'g^{-1}, \qquad g, g' \in G.$$

Example 1.3.2.

1. It is clear from the definitions that if G is a group, then the pair $(G \xrightarrow{\mathrm{Id}_G} G, \mathrm{Conj})$ is a crossed module of groups, where Conj is the conjugation action.

2. Any central extension of groups $G \overset{\partial}{\twoheadrightarrow} H$ is a crossed module, with the action $\partial(g) \cdot g' = gg'g^{-1} = \mathrm{Conj}(g)(g')$. Conversely, a simply connected crossed module (i.e. ∂ is surjective) is a central extension.

   In particular, the map $G \overset{\mathrm{Conj}}{\twoheadrightarrow} \mathrm{IAut}(G)$, $g \mapsto \mathrm{Conj}(g)$, with the group action, $\mathrm{Conj}(g) \cdot g' = gg'g^{-1}$, is a crossed module of groups, where $\mathrm{IAut}(G)$ are the inner automorphisms of a group G.

Definition 1.3.3. A *homomorphism of crossed modules of groups* between the crossed modules $(G \xrightarrow{\partial} H, \cdot)$ and $(G' \xrightarrow{\partial'} H', *)$ is given by a pair of group homomorphisms, $f_1 : G \to G'$ and $f_2 : H \to H'$ such that:

$$f_1(h \cdot g) = f_2(h) * f_1(g), \quad \partial' \circ f_1 = f_2 \circ \partial, \ \ g \in G, h \in H.$$

Remark 1.3.4. There is an equivalence between the categories ***ICat***(***Grp***) and crossed modules of groups (see [6]).

1.3.2 Crossed modules of associative algebras

The definition of action in the associative algebras case is the following one.

Definition 1.3.5. Let N and M be two associative K-algebras.

An *associative action* of N on M is a pair of K-bilinear maps $* = (*_1, *_2)$, where $*_1 : N \times M \to M$ and $*_2 : M \times N \to M$, $(n, m) \mapsto n *_1 m$ and $(m, n) \mapsto m *_2 n$, satisfy:

$$\begin{aligned} n *_1 (mm') &= (n *_1 m)m', & m *_2 (nn') &= (m *_2 n) *_2 n', \\ n *_1 (m *_2 n') &= (n *_1 m) *_2 n', & m(n *_1 m') &= (m *_2 n)m', \\ n *_1 (n' *_1 m) &= (nn') *_1 m, & m(m' *_2 n) &= (mm') *_2 n, \end{aligned}$$

for $m, m' \in M$, $n, n' \in N$ (see [21], cf. [1–4] for commutative algebras).

Remark 1.3.6. If we change the notation of $*_1$ and $*_2$ by $*$ in both cases, where $*$ is the multiplication, the axioms of the associative actions are all possible rewrites of the associativity when we choose two elements in M and one in N or one in M and two N.

In particular, if $*$ is the multiplication of an associative K-algebra, we have that the pair $(*, *)$ is an associative action of M on itself.

Definition 1.3.7. Let M and N be two associative K-algebras and $* = (*_1, *_2)$ an associative action of N on M. We define its *semidirect product*, denoted by $M \rtimes N$, as the K-vector space $M \times N$ with the following operation:

$$(m, n)(m', n') = (mm' + n *_1 m' + m *_2 n', nn').$$

The definition of crossed module of associative algebras was given by Dedecker and Lue in [18].

Definition 1.3.8. A *crossed module of associative K-algebras* is a pair $(M \xrightarrow{\partial}, N, *)$ where M and N are associative K-algebras, $* = (*_1, *_2)$ is an associative action of N on M, and $\partial\colon M \to N$ is an associative K-homomorphism satisfying:
- ∂ is an N-equivariant associative K-homomorphism (we suppose the action of N on itself is the product), i.e.

$$\partial(n *_1 m) = n\partial(m) \quad \text{and} \quad \partial(m *_2 n) = \partial(m)n, \quad m \in M, n \in N,$$

- *Peiffer identity*:

$$\partial(m) *_1 m' = mm' = m *_2 \partial(m'), \quad m, m' \in M.$$

(see [21], cf. [1–4] for commutative algebras).

Example 1.3.9. If M is an associative K-algebra, then $(M \xrightarrow{\mathrm{Id}_M} M, (*, *))$, where "$*$" is its product, is a crossed module of associative K-algebras.

Definition 1.3.10. A *homomorphism of crossed modules of associative K-algebras* between $(M \xrightarrow{\partial} N, \cdot)$ and $(M' \xrightarrow{\partial'} N', *)$ is a pair of associative K-homomorphisms, $f_1\colon M \to M'$ and $f_2\colon N \to N'$ such that for $m \in M, n \in N$:

$$f_1(n \cdot_1 m) = f_2(n) *_1 f_1(m), \;\; f_1(m \cdot_2 n) = f_1(m) *_2 f_2(n), \;\; \partial' \circ f_1 = f_2 \circ \partial.$$

We denote by $\boldsymbol{X}(\boldsymbol{AssAlg}_K)$ the category of crossed modules of associative K-algebras and their homomorphisms.

Now, we state the correspondence between crossed modules of associative algebras and categorical associative algebras (see [22, 41]).

Proposition 1.3.11. *We have an equivalence between the categories $\boldsymbol{ICat}(\boldsymbol{AssAlg}_K)$ and $\boldsymbol{X}(\boldsymbol{AssAlg}_K)$.*

Proof. Below we define two functors, $\mathcal{C}_{\mathfrak{A}}\colon \boldsymbol{X}(\boldsymbol{AssAlg}_K) \to \boldsymbol{ICat}(\boldsymbol{AssAlg}_K)$ and $\mathcal{X}_{\mathfrak{A}}\colon \boldsymbol{ICat}(\boldsymbol{AssAlg}_K) \to \boldsymbol{X}(\boldsymbol{AssAlg}_K)$, and prove that $\mathcal{X}_{\mathfrak{A}} \circ \mathcal{C}_{\mathfrak{A}} \cong \mathrm{Id}_{\boldsymbol{X}(\boldsymbol{AssAlg}_K)}$ and $\mathcal{C}_{\mathfrak{A}} \circ \mathcal{X}_{\mathfrak{A}} \cong \mathrm{Id}_{\boldsymbol{ICat}(\boldsymbol{AssAlg}_K)}$.

Let us begin with the definition of $C_{\mathfrak{A}}$ on objects. Let $(M \xrightarrow{\partial} N, *)$ be a crossed module in $\boldsymbol{AssAlg}_K$. We will take the semidirect product $M \rtimes N$ with $*$. Consider the following diagram:

$$\begin{tikzcd}
(M \rtimes N) \times_N (M \rtimes N) \arrow[r, "\bar{k}"] & M \rtimes N \arrow[r, shift left, "\bar{s}"] \arrow[r, shift right, "\bar{t}"'] & N \arrow[l, bend right=50, "\bar{e}"']
\end{tikzcd}$$

with the following maps: $\bar{s}((m,n)) = b$, $\bar{t}((m,n)) = \partial(m) + n$, $\bar{e}(n) = (0,n)$ and $\bar{k}(((m,n),(m',\partial(m)+n))) = (m+m',n)$ for all $m, m' \in M$, $n \in N$. Note that if $((n,m),(n',m')) \in (M \rtimes N) \times_N (M \rtimes N)$, $n' = \bar{s}((m',n')) = \bar{t}((m,n)) = \partial(m)+n$, so the definition of $\bar{k}$ makes sense. It is necessary to prove that $\bar{s}$, $\bar{t}$, $\bar{e}$ and $\bar{k}$ are morphisms in $\boldsymbol{AssAlg}_K$, that is, they preserve all the operations. Since it is obvious that $\bar{s}$ and $\bar{e}$ preserve the operations, we will focus on sketching how to prove that $\bar{t}$ and $\bar{k}$ preserve the sum and the product. Calculations are quite long, so we will not include them, although we will point out the crucial ideas required to complete them. Regarding $\bar{t}$, it preserves the sum directly from the fact that ∂ preserve it. Furthermore, the fact that ∂ is an N-equivariant associative morphism is the key to prove that $\bar{t}$ preserves the product.

Concerning $\bar{k}$, note that the elements in $(M \rtimes N) \times_N (M \rtimes N)$ are of the form $((m,n),(m',\partial(m)+n))$, with $m, m' \in M$, $n \in N$. Immediately below we will show the calculations required to prove that $\bar{k}$ preserves the sum. Let $((m_i,n_i),(m'_i,\partial(m_i)+n_i)) \in (M \rtimes N) \times_N (M \rtimes N)$ for $i = 1, 2$. On one hand we have that

$$\begin{aligned}
&\bar{k}(((m_1,n_1),(m'_1,\partial(m_1)+n_1)) + ((m_2,n_2),(m'_2,\partial(m_2)+n_2))) \\
&= \bar{k}((m_1,n_1)+(m_2,n_2),(m'_1,\partial(m_1)+n_1)+(m'_2,\partial(m_2)+n_2)) \\
&= ((m_1+m_2,n_1+n_2),(m'_1+m'_2,\partial(m_1)+n_1+\partial(m_2)+n_2)) \\
&= (m_1+m_2+m'_1+m'_2,n_1+n_2),
\end{aligned}$$

On the other hand,

$$\bar{k}(((m_1,n_1),(m'_1,\partial(m_1)+n_1))) + \bar{k}(((m_2,n_2),(m'_2,\partial(m_2)+n_2)))$$

$$= (m_1 + m_1', n_1) + (m_2 + m_2', n_2) = (m_1 + m_1' + m_2 + m_2', n_1 + n_2),$$

by making use of the definition of $\bar{k}$ and the addition in $M \rtimes N$. Hence, $\bar{k}$ preserves the sum. Calculations for the product are similar, but involving distributivity and the Peiffer identity.

Commutativity of the diagrams of the internal categories is easy.

Defining $C_{\mathfrak{A}}$ on morphisms is quite obvious. Given a morphism of crossed modules (f_1, f_2) between $(M \xrightarrow{\partial} N, *)$ and $(M' \xrightarrow{\partial'} N', *')$, its corresponding internal functor is given by $f_1 \times f_2 : M \rtimes N \to M' \rtimes N'$ and $f_2 : N \to N'$, where $f_1 \times f_2((a, b)) = (f_1(a), f_2(b))$. Commutativity of the diagrams for internal functors follows from the definitions of $\bar{s}$, $\bar{s'}$, $\bar{t}$, $\bar{t'}$, $\bar{e}$, $\bar{e'}$, $\bar{k}$ and $\bar{k'}$, along with the equality $f_2 \circ \partial = \partial' \circ f_1$.

$C_{\mathfrak{A}}$ is clearly a functor with the previous assignments for objects and morphisms.

Now let us define the functor $\mathcal{X}_{\mathfrak{A}}$. Let $C = (C_1, C_0, s, t, e, k)$ be an internal category in $\boldsymbol{AssAlg}_K$. Consider $\ker(s)$ and the morphism $t|_{\ker(s)} : \ker(s) \to C_0$. We will write ∂_t in order to ease notation. We define an associative action $({}^{e}*, *^{e})$ with $a\, {}^{e}*x = e(a)x$ and $x *^{e} a = xe(a)$ with $a \in C_0, x \in \ker(s)$. It is easy that the maps are well defined.

It only remains to prove that $(\ker(s) \xrightarrow{\partial_t} C_0, ({}^{e}*, *^{e}))$ satisfies is a crossed module. Given $a \in C_0$ and $x \in \ker(s)$,

$$\partial_t(a^{e} * x) = \partial_t(e(a)x) = t(e(a)x) = t(e(a))t(x) = a\partial_t(x),$$
$$\partial_t(x *^{e} a) = \partial_t(xe(a)) = t(xe(a)) = t(x)t(e(a)) = \partial_t(x)a.$$

Note that we use that in an internal category $t \circ e = \mathrm{Id}_{C_0}$.

To prove the Peiffer identity, let $x_1, x_2 \in \ker(s)$.

$$\partial_t(x_1)\, {}^{e}*x_2 = e(t(x_1))x_2.$$

We need to show that $e(t(x_1))x_2 = x_1x_2$. We will take $z = (e(t(x_1)) - x_1)$.

$t(z) = t(e(t(x_1))) - t(x_1) = t(x_1) - t(x_1) = 0$ so we can compose with $e(0)$ since. $t(z) = 0 = s(0) = s(e(0))$. Since $x_2 \in \ker(s)$, we can take $k((e(0), x_2))$, because $s(y) = 0 = t(0) = t(e(0))$. In addition we have that in an internal category $k((z, e(0))) = z$ and $k((e(0), x_2)) = x_2$.

$$\begin{aligned} 0 &= k((0,0)) = k((ze(0), e(0)x_2)) \\ &= k((z, e(0))(e(0), x_2)) = k((z, e(0)))k((e(0), x_2)) = zx_2. \end{aligned}$$

Finally, we have:

$$0 = zx_2 = (e(t(x_1)) - x_1)x_2 = e(t(x_1))x_2 - x_1x_2,$$

which establishes one of the Peiffer identities. Similar arguments apply to the other Peiffer identity.

Defining $\mathcal{X}_{\mathfrak{A}}$ on morphisms is also quite obvious. Let $C = (C_1, C_0, s, t, e, k)$ and $C' = (C'_1, C'_0, s', t', e', k')$ be two internal categories in $\boldsymbol{AssAlg}_K$ and $F : C \to C'$ an internal functor, with $F_1 : C_1 \to C'_1$ and $F_0 : C_0 \to C'_0$. Its corresponding morphism of crossed modules is given by (F_1^s, F_0), with $F_1^s(x) = F_1(x)$ for $x \in \ker(s)$, which follows from the diagrams of internal functors. It is easy to check that, with the previous assignments, $\mathcal{X}_{\mathfrak{A}}$ is indeed a functor.

$C_{\mathfrak{A}}$ and $\mathcal{X}_{\mathfrak{A}}$ establish an equivalence between the categories where the natural isomorphisms $\mathrm{Id}_{\boldsymbol{X(AssAlg_K)}} \overset{\alpha^{\mathfrak{A}}}{\cong} \mathcal{X}_{\mathfrak{A}} \circ C_{\mathfrak{A}}$ and $\mathrm{Id}_{\boldsymbol{ICat(AssAlg_K)}} \overset{\beta^{\mathfrak{A}}}{\cong} C_{\mathfrak{A}} \circ \mathcal{X}_{\mathfrak{A}}$ are given by:

• if $\mathcal{Z} = (M \xrightarrow{\partial} N, (*_1, *_2))$ is a crossed module of associative K-algebras, then $\alpha^{\mathfrak{A}}_{\mathcal{Z}} = (\alpha^{\mathfrak{A}}_M, \mathrm{Id}_N)$, with $\alpha^{\mathfrak{A}}_M : M \to (M, 0)$ defined as $\alpha_M(m) = (m, 0)$;

• if $\mathcal{D} = (C_1, C_0, s, t, e, k)$ is a categorical associative K-algebra, then $\beta^{\mathfrak{A}}_{\mathcal{D}} = (\beta^{\mathfrak{A}}_s, \mathrm{Id}_{C_0})$, with $\beta^{\mathfrak{A}}_{C_1} : C_1 \to \ker(s) \rtimes C_0$ is defined as $\beta^{\mathfrak{A}}_{C_1}(x) = (x - e(s(x)), s(x))$. $\square$

1.3.3 Crossed modules of Lie algebras

We have an analogous definition for the case of Lie K-algebras. Crossed modules of Lie K-algebras were introduced by Kassel and Loday in [40].

Definition 1.3.12. Let M and N two Lie K-algebras. A *Lie (left-)action of N on M* is a K-bilinear map $\cdot : N \times M \longrightarrow M$, $(n, m) \longmapsto n \cdot m$, satisfying:

$$[n, n'] \cdot m = n \cdot (n' \cdot m) - n' \cdot (n \cdot m),$$

$$n \cdot [m, m'] = [n \cdot m, m'] + [m, n \cdot m'], \qquad n, n' \in N,\ m, m' \in M.$$

If we denote $\cdot = [-,-]$, the two identities are the two possible rewrites of the Jacobi identity by taking two elements in N or two in M.

In particular, if M is a Lie K-algebra and $x \in M$, we have that the adjoint map $\mathrm{ad}(x)\colon M \to M$, $\mathrm{ad}(x)(y) = [x, y]$, is a Lie action of M on itself.

Definition 1.3.13. A *crossed module of Lie K-algebras* is a pair $(M \xrightarrow{\partial} N, \cdot)$ where M and N are Lie K-algebras, $\cdot$ is a Lie action of N on M, and $M \xrightarrow{\partial} N$ is a Lie K-homomorphism satisfying:
- ∂ is an N-equivariant Lie K-homomorphism (we suppose the adjoint action of N on itself), i.e.

$$\partial(n \cdot m) = \mathrm{ad}(n)(\partial(m)) = [n, \partial(m)], \qquad n \in N,\ m \in M,$$

- *Peiffer identity*:

$$\partial(m) \cdot m' = \mathrm{ad}(m)(m') = [m, m'], \qquad m, m' \in M.$$

Example 1.3.14.

1. As in the previous cases we have the example of crossed module of Lie K-algebras $(M \xrightarrow{\mathrm{Id}_M} M, [-,-])$, with the adjoint action, $m \cdot m' = [m, m']$, where M is a Lie K-algebra.

2. Any central extension of Lie algebras $M \overset{\partial}{\twoheadrightarrow} N$ is a crossed module, with the action $\partial(m) \cdot m' = [m, m']$. Conversely, a simply connected crossed module (i.e. ∂ is surjective) is a central extension.

   In particular, $M \overset{\mathrm{ad}}{\twoheadrightarrow} \mathrm{IDer}(M)$, $m \mapsto \mathrm{ad}(m)$, with the action, $\mathrm{ad}(m) \cdot m' = [m, m']$, is a Lie crossed module, where $\mathrm{IDer}(M)$ are the inner derivations of a Lie algebra M.

Definition 1.3.15. A *homomorphism of crossed modules of Lie K-algebras* between $(M \xrightarrow{\partial} N, \cdot)$ and $(M' \xrightarrow{\partial} N', *)$ is a pair of Lie K-homomorphisms, $f_1\colon M \to M'$ and $f_2\colon N \to N'$ such that:

$$f_1(n \cdot m) = f_2(n) * f_1(m), \tag{XLieH1}$$

$$\partial' \circ f_1 = f_2 \circ \partial, \tag{XLieH1}$$

$m \in M, n \in N$.

There is a natural way to correlate the crossed modules of associative K-algebras with the crossed modules of Lie K-algebras. The following results that relate both can be seen in [21].

Lemma 1.3.16. *Let M and N be two associative K-algebras.*

We denote by $A^{\mathcal{L}}$ the Lie K-algebra associated to an associative K-algebra A, i.e. the Lie K-algebra with the operation $[a, a'] = aa' - a'a$.

(i) If $* = (*_1, *_2)$ is an associative action of N on M, then we have that the map $[-,-]_* : N \times M \rightarrow M$, defined as $[n, m]_* = n *_1 m - m *_2 n$, is a Lie action of $N^{\mathcal{L}}$ on $M^{\mathcal{L}}$.

(ii) If $(M \xrightarrow{\partial} N, *)$ is a crossed module of associative K-algebras, then we have that $(M^{\mathcal{L}} \xrightarrow{\partial} N^{\mathcal{L}}, [-,-]_*)$ is a crossed module of Lie K-algebras.

Remark 1.3.17. With the previous property we can see that the examples given for the associative algebras case, $(M \xrightarrow{\mathrm{Id}_M} M, (*, *))$, and for the Lie case for the associate Lie algebra $M^{\mathcal{L}}$, $(M^{\mathcal{L}} \xrightarrow{\mathrm{Id}_{M^{\mathcal{L}}}} M^{\mathcal{L}}, [-,-]_*)$, are related.

We denote by $\boldsymbol{X(LieAlg_K)}$ the category of crossed modules of Lie K-algebras and their homomorphisms.

Remark 1.3.18. The previous lemma gives us a functor

$$(-)^{\mathcal{L}}_{\mathcal{X}} : \boldsymbol{X(AssAlg_K)} \longrightarrow \boldsymbol{X(LieAlg_K)}.$$

We have the next proposition which relates the categorical associative K-algebras with the categorical Lie K-algebras.

Proposition 1.3.19. *If (C_1, C_0, s, t, e, k) is a categorical associative K-algebra, then $(C_1^{\mathcal{L}}, C_0^{\mathcal{L}}, s, t, e, k)$ is a categorical Lie K-algebra.*

Proof. Immediate since $(C_1 \times_{C_0} C_1)^{\mathcal{L}} = C_1^{\mathcal{L}} \times_{C_0^{\mathcal{L}}} C_1^{\mathcal{L}}$. They are the same underlying vector space and have the same operation. □

Remark 1.3.20. The previous proposition gives us a functor

$$(-)^{\mathcal{L}}_{C} : \boldsymbol{ICat}(\boldsymbol{AssAlg}_K) \longrightarrow \boldsymbol{ICat}(\boldsymbol{LieAlg}_K).$$

Remark 1.3.21. As in the case of groups and associative K-algebras, the categories $\boldsymbol{ICat}(\boldsymbol{LieAlg}_K)$ and $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ are equivalent (see [5, 22, 25]).

It is easy to check that the equivalence functors commute with the functors $(-)^{\mathcal{L}}_{\mathcal{X}}$ and $(-)^{\mathcal{L}}_{C}$. We only need to show that $(-)^{\mathcal{L}}$ preserves the semidirect product.

Definition 1.3.22. Let M and N be two Lie K-algebras and $\cdot$ a Lie action of N on M. We define its *semidirect product*, denoted by $M \rtimes N$, as the K-vector space $M \times N$ with the following bracket:

$$[(m,n),(m',n')] = ([m,m'] + n \cdot m' - n' \cdot m, [n,n']).$$

Proposition 1.3.23. *Let M and N be associative K-algebras. If $*$ is an associative action of N on M (then $[-,-]_*$ is a Lie action of $N^{\mathcal{L}}$ in $M^{\mathcal{L}}$), then we have that $(M \rtimes N)^{\mathcal{L}} = M^{\mathcal{L}} \rtimes N^{\mathcal{L}}$.*

Proof. Since the underlying vector space is the same, we only need to prove that the bracket is the same.

$$\begin{aligned}
&(m,n)(m',n') - (m',n')(m,n)\\
&= (mm' + n *_1 m' + m *_2 n', nn') - (m'm + n' *_1 m + m' *_2 n, n'n)\\
&= (mm' + n *_1 m' + m *_2 n' - m'm - n' *_1 m - m' *_2 n, nn' - n'n)\\
&= ([m,m'] + [n,m']_* - [n',m]_*, [n,n']),
\end{aligned}$$

where $(m,n),(m',n') \in M \times N$. □

1.3.4 Crossed modules of Leibniz algebras

The definition of crossed modules of Leibniz K-algebras, "non-antisymmetric" case of Lie K-algebras, was introduced by Loday and Pirashvili in [43].

Definition 1.3.24. Let N and M be two Leibniz K-algebras. A Leibniz action of N on M is a pair $\cdot = (\cdot_1, \cdot_2)$ where $\cdot_1 : N\times M \to M$ and $\cdot_2 : M\times N \to M$ are K-bilinear maps and the following properties are satisfied

$$n \cdot_1 [m, m'] = [n \cdot_1 m, m'] - [n \cdot_1 m', m], \tag{ALeib1}$$
$$[m, n \cdot_1 m'] = [m \cdot_2 n, m'] - [m, m'] \cdot_2 n, \tag{ALeib2}$$
$$[m, m' \cdot_2 n] = [m, m'] \cdot_2 n - [m \cdot_2 n, m'], \tag{ALeib3}$$
$$m \cdot_2 [n, n'] = (m \cdot_2 n) \cdot_2 n' - (m \cdot_2 n') \cdot_2 n, \tag{ALeib4}$$
$$n \cdot_1 (m \cdot_2 n') = (n \cdot_1 m) \cdot_2 n' - [n, n'] \cdot_1 m, \tag{ALeib5}$$
$$n \cdot_1 (n' \cdot_1 m) = [n, n'] \cdot_1 m - (n \cdot_1 m) \cdot_2 n'. \tag{ALeib6}$$

$m, m' \in M,\ n, n' \in N$.

Remark 1.3.25. If we change the notation of $\cdot_1$ and $\cdot_2$ by $[-,-]$ in both cases, the axioms of the Leibniz actions are all possible rewritings of the Leibniz identity when we choose two elements in M and one in N (the first three) or one in M and two N (the last three).

In particular, we have that the pair $([-,-],[-,-])$ where $[-,-]$ is the Leibniz bracket of the Leibniz K-algebra M is a Leibniz action of M on itself.

Definition 1.3.26. A crossed module of Leibniz K-algebras is a pair $(M \xrightarrow{\partial} N, \cdot)$ where M and N are Leibniz K-algebras, $\cdot = (\cdot_1, \cdot_2)$ is a Leibniz action of N on M, $\partial : M \to N$ is a Leibniz K-homomorphism, and the following properties are satisfied:
-∂ is an N-equivariant Leibniz K-homomorphism (we suppose that the bracket gives the action in N), i.e.

$$\partial(n \cdot_1 m) = [n, \partial(m)] \text{ and } \partial(m \cdot_2 n) = [\partial(m), n], \qquad n \in N,\ m \in M,$$

- *Peiffer identity*:

$$\partial(m) \cdot_1 m' = [m, m'] = m \cdot_2 \partial(m') \quad m, m' \in M, n \in N.$$

Example 1.3.27. As for the previous cases, we have that if M is a Leibniz K-algebra then $(M \xrightarrow{\mathrm{Id}_M} M, ([-,-],[-,-]))$ is a crossed module of Leibniz K-algebras.

The next immediate propositions give a relation between crossed modules of Lie and Leibniz K-algebras.

Proposition 1.3.28. *Let M and N be two Lie K-algebras. Then, $\cdot$ is a Lie action of N on M if and only if $(\cdot, \cdot^-)$ is a Leibniz action of N on M, where $\cdot^- : M \times N \to M$ is defined by $m \cdot^- n := -n \cdot m$.*

That is, the Lie action is a particular case of a Leibniz action when the action is "anticommutative".

Proposition 1.3.29. *Let M and N be Lie K-algebras. Then, $(M \xrightarrow{\partial} N, \cdot)$ is a crossed module of Lie K-algebras if and only if $(M \xrightarrow{\partial} N, (\cdot, \cdot^-))$ is a crossed module of Leibniz K-algebras.*

Remark 1.3.30. With the previous property we can see that the examples given for the Lie algebra case, $(M \xrightarrow{\mathrm{Id}_M} M, [-,-])$, and for the Leibniz algebra case, taking a Lie algebra M, $(M \xrightarrow{\mathrm{Id}_M} M, ([-,-],[-,-])$, are related, since using anticommutativity $[-,-]^- = [-,-]$.

Definition 1.3.31. Let $(M \xrightarrow{\partial} N, \cdot)$ and $(M' \xrightarrow{\partial'} N', *)$ be crossed modules of Leibniz K-algebras. A homomorphism is a pair of Leibniz K-homomorphisms, $f_1 : M \to M'$ and $f_2 : N \to N'$ such that

$$f_1(n \cdot_1 m) = f_2(n) *_1 f_1(m), \quad f_1(m \cdot_2 n) = f_1(m) *_2 f_2(n), \; n \in N, m \in M,$$

and

$$\partial' \circ f_1 = f_2 \circ \partial.$$

We will denote by $\boldsymbol{X(LeibAlg_K)}$ the category of crossed modules of Leibniz K-algebras and its homomorphisms.

Remark 1.3.32. As in the case of groups and Lie K-algebras, we have an equivalence between the categories $\boldsymbol{X(LeibAlg_K)}$ and $\boldsymbol{ICat(LeibAlg_K)}$. A proof of this can be found in [22].

$\boldsymbol{X(LieAlg_K)}$ can be seen as a full subcategory of the category $\boldsymbol{X(LeibAlg_K)}$ using Proposition 1.3.29 (we actually have a functorial isomorphism between a full subcategory of $\boldsymbol{X(LeibAlg_K)}$ and $\boldsymbol{X(LieAlg_K)}$).

Since the pullbacks in $\boldsymbol{LieAlg}_K$ and $\boldsymbol{LeibAlg}_K$ are the same, it is immediate to show that $\boldsymbol{ICat}(\boldsymbol{LieAlg}_K)$ is a full subcategory of $\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K)$. The equivalence in the Leibniz case generalizes the equivalence in the Lie case since the bracket gives the action in the functors (which was presented in [22]), and then, it is anticommutative when we have Lie K-algebras. We only have to check that the Leibniz semidirect product generalizes the Lie semidirect product, but this is immediate from definition (since $m \cdot_2 n' = -n' \cdot_1 m$ is the Lie case).

Definition 1.3.33. Let M and N be two Leibniz K-algebras and $\cdot$ a Leibniz action of N on M. The semidirect product, denoted by $M \rtimes N$, is the K-vector space $M \times N$ with the bracket

$$[(m,n),(m',n')] := ([m,m'] + n \cdot_1 m' + m \cdot_2 n', [n,n']), \quad m,m' \in M,\ n,n' \in N.$$

1.4 Braided semigroupal Category

A bifunctor is a functor whose source category is a product category.

Let $F : \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{E}$ be a bifunctor. For $A \in \mathrm{Ob}(\boldsymbol{C})$ and $B \in \mathrm{Ob}(\boldsymbol{D})$, we denote by ${}_AF$ and F_B the functors:

$$\begin{aligned} {}_AF : \boldsymbol{D} \to \boldsymbol{E},\ {}_AF(D \xrightarrow{f} D') &= F(A,D) \xrightarrow{F(\mathrm{Id}_A, f)} F(A,D'), \\ F_B : \boldsymbol{C} \to \boldsymbol{E},\ F_B(C \xrightarrow{g} C') &= F(C,B) \xrightarrow{F(g, \mathrm{Id}_B)} F(C',B). \end{aligned}$$

Definition 1.4.1. Given the categories, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$, we have the functor

$$A^{\mathbf{C},\mathbf{D},\mathbf{E}} : (\mathbf{C} \times \mathbf{D}) \times \mathbf{E} \to \mathbf{C} \times (\mathbf{D} \times \mathbf{E})$$

defined as

$$A^{\mathbf{C},\mathbf{D},\mathbf{E}}\Big(((A,B),C) \xrightarrow{\big((f,g),h\big)} \big((A',B'),C'\big)\Big) = \big(A,(B,C)\big) \xrightarrow{(f,(g,h))} \big(A',(B',C')\big),$$

called *associator functor* for the categorical product of the given categories. It is always a functor isomorphism with the obvious inverse.

Crane and Yetter defined in [17] the notion of semigroupal category.

Definition 1.4.2. A *semigroupal category* is a triple $C = (\mathbf{C}, \otimes, a)$ where $\mathbf{C}$ is a category, $\otimes\colon\ \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ is a bifunctor, and $a\colon\ \otimes\circ(\otimes\times\mathrm{Id}_{\mathbf{C}})\to \otimes\circ(\mathrm{Id}_{\mathbf{C}}\times\otimes)\circ A^{\mathbf{C},\mathbf{C},\mathbf{C}}$ is a natural isomorphism called the *associator*, such that for all $X, Y, Z, W \in \mathrm{Ob}(\mathbf{C})$ the following associative coherence diagram (pentagon axiom) holds:

$$
\begin{tikzcd}
 & ((X\otimes Y)\otimes Z)\otimes W \arrow[dl, "a_{X,Y,Z}\otimes\mathrm{Id}_W"'] \arrow[dr, "a_{X\otimes Y,Z,W}"] & \\
(X\otimes(Y\otimes Z))\otimes W \arrow[d, "a_{X,Y\otimes Z,W}"'] & & (X\otimes Y)\otimes(Z\otimes W) \arrow[d, "a_{X,Y,Z\otimes W}"] \\
X\otimes((Y\otimes Z)\otimes W) \arrow[rr, "\mathrm{Id}_X\otimes a_{Y,Z,W}"'] & & X\otimes(Y\otimes(Z\otimes W))
\end{tikzcd}
$$

We say that a semigroupal category is *strict* if the isomorphism a is the identity morphism. In this case we have that $(X\otimes Y)\otimes Z = X\otimes(Y\otimes Z)$.

It is known that the coherence diagram implies that any diagram made in the same way to more tensor products will be commutative.

The definition of monoidal category was given in [7,45].

Definition 1.4.3. A *monoidal category* is a 6-tuple $C = (\mathbf{C}, \otimes, a, I, l, r)$ where $(\mathbf{C}, \otimes, a)$ is a semigroupal category, I is an object of $\mathbf{C}$ (called the *tensor unit*), and the pair $l\colon\ (I\otimes -)\to \mathrm{Id}_{\mathbf{C}}$, $r\colon\ (-\otimes I)\to \mathrm{Id}_{\mathbf{C}}$ are natural isomorphisms (called the *left* and *right unitors*, respectively), such that for all $X, Y \in \mathrm{Ob}(\mathbf{C})$ the unit coherence diagram (triangle equation) holds:

$$
\begin{tikzcd}
(X\otimes I)\otimes Y \arrow[rr, "a_{X,I,Y}"] \arrow[dr, "r_X\otimes\mathrm{Id}_Y"'] & & X\otimes(I\otimes Y) \arrow[dl, "\mathrm{Id}_X\otimes l_Y"] \\
 & X\otimes Y &
\end{tikzcd}
$$

We say that a monoidal category is *strict* if the isomorphisms a, l and r are the identity morphisms. In this case $(X\otimes Y)\otimes Z = X\otimes(Y\otimes Z)$, $X\otimes I = X = I\otimes X$.

Definition 1.4.4. Given two categories $\mathbf{C}$ and $\mathbf{D}$ there is a functor called *commutator functor* defined as $T_{\mathbf{C},\mathbf{D}} : \mathbf{C} \times \mathbf{D} \to \mathbf{D} \times \mathbf{C}$ defined as

$$T_{\mathbf{C},\mathbf{D}}\Big((c,d) \xrightarrow{(f,g)} (c',d')\Big) = (d,c) \xrightarrow{(g,f)} (d',c').$$

Note that $T_{\mathbf{C},\mathbf{D}}$ is a functorial isomorphism with inverse $T_{\mathbf{D},\mathbf{C}}$.

The notion of braided monoidal category was introduced by Joyal and Street in [38].

Definition 1.4.5. Let $\mathcal{C} = (\mathbf{C}, \otimes, a, I, l, r)$ be a monoidal category.

A *braiding* on $\mathcal{C}$ is a natural isomorphism $\tau : \otimes \to \otimes \circ T_{\mathbf{C},\mathbf{C}}$ such that for all $X, Y, Z \in \mathrm{Ob}(\mathbf{C})$ the following associative coherence diagrams (hexagon axioms) commute:

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[r, "\tau_{X\otimes Y,Z}"] \arrow[dl, "a_{X,Y,Z}"'] & Z \otimes (X \otimes Y) & \\
X \otimes (Y \otimes Z) \arrow[dr, "\mathrm{Id}_X \otimes \tau_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \arrow[ul, "a_{Z,X,Y}"'] \\
& X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "\tau_{X,Z} \otimes \mathrm{Id}_Y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "\tau_{X,Y\otimes Z}"] \arrow[dl, "a^{-1}_{X,Y,Z}"'] & (Y \otimes Z) \otimes X & \\
(X \otimes Y) \otimes Z \arrow[dr, "\tau_{X,Y} \otimes \mathrm{Id}_Z"'] & & & Y \otimes (Z \otimes X) \arrow[ul, "a^{-1}_{Y,Z,X}"'] \\
& (Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z). \arrow[ur, "\mathrm{Id}_Y \otimes \tau_{X,Z}"'] &
\end{tikzcd}$$

We will say that $(\mathbf{C}, \otimes, a, I, l, r, \tau)$ is a *braided monoidal category*.

It is known that the coherence diagram implies that any diagram given by concatenations of associators and braidings with more tensor products will be commutative.

We can define the concept of braided semigroupal category by emulating the definition given for the case of monoidal categories.

Definition 1.4.6. A *braiding on a semigroupal category* $\mathcal{C}$ is a natural isomorphism $\tau : \otimes \to \otimes \circ T_{\mathbf{C},\mathbf{C}}$ which satisfies the two associative coherence diagrams given in the previous definition.

CHAPTER 2

Braided crossed modules and Braided Internal objects

In this chapter, we will study the relations between the different types of braidings of K-algebras and their similarities to the case of groups.

2.1 Braiding for categorical groups and crossed modules of groups

We have the following outlined property in [5, 15].

Lemma 2.1.1. *We will suppose that (C_1, C_0, s, t, e, k) is a categorical associative, Lie or Leibniz K-algebra or a categorical group (where the operation in C_1 is denoted by "+"). Then, if $(x, y) \in C_1 \times_{C_0} C_1$, the following rule for the composition is true:*

$$k((x, y)) = x - e(t(x)) + y = x - e(s(y)) + y.$$

Proof. We have that $(x, e(s(y)))$, $(e(t(x)), e(s(y)))$, $(e(t(x)), y) \in C_1 \times_{C_0} C_1$. This is because $t(x) = s(y) = s(e(s(y)))$, $s(e(t(x))) = t(x) = s(y) = s(e(s(y)))$ and $t(e(t(x))) = t(x) = s(y)$.

So, we have the following equality, where we are using the fact that k is K-linear in the case of K-algebras or a homomorphism of groups in the case of groups, and the properties of the internal categories for composition:

$$k((x, y)) = k((x - e(t(x)) + e(t(x)), e(s(y)) - e(s(y)) + y))$$

$$= k((x, e(s(y)))) - k((e(t(x)), e(s(y)))) + k((e(t(x)), y))$$
$$= x - e(s(y)) + y.$$

□

Lemma 2.1.2. *In the categories of categorical associative, Lie or Leibniz K-algebras and in the category of categorical groups all internal morphisms $f \in C_1$ are internal isomorphisms. That is, there exists $f' \in C_1$ such that $k((f, f')) = e(s(f))$ and $k((f', f)) = e(t(f))$.*

Proof. For $f \in C_1$ we take $f' = e(t(f)) - f + e(s(f))$.

We can compose these internal morphisms:

$$s(f') = s(e(t(f)) - f + e(s(f))) = t(f) - s(f) + s(f) = t(f),$$
$$t(f') = t(e(t(f)) - f + e(s(f))) = t(f) - t(f) + s(f) = s(f).$$

Thus, we have that $(f, f'), (f', f) \in C_1 \times_{C_0} C_1$.

Using Lemma 2.1.1 we have the following equalities:

$$k((f, f')) = f - e(t(f)) + f' = f - e(t(f)) + e(t(f)) - f + e(s(f)) = e(s(f)),$$
$$k((f', f)) = f' - e(s(f)) + f = e(t(f)) - f + e(s(f)) - e(s(f)) + f = e(t(f)).$$

□

It is the same to give a strict monoidal category over a small category (internal category in the case of sets) than to give a categorical monoid. The correspondence is given by taking the product of the monoids C_0 and C_1 as the $\otimes$ product. The fact that the morphisms s, t, e and k are homomorphisms of monoids is equivalent to the functoriality of $\otimes$. The unit I is given by the monoid unit 1_{C_0}, and the unit in C_1 is $e(I)$.

Using this idea one can define what is a braiding for a categorical monoid, but if we think in the case of groups (a little more restrictive) we have that all internal morphisms are isomorphisms. Using this, it is only needed to take a family of internal morphisms.

The definition of braiding on a categorical group was introduced by Joyal and Street in [38] and [39]. Later, the notion of braided crossed module over a groupoid was presented by Brown and Gilbert in [9].

Definition 2.1.3. Let $C = (C_1, C_0, s, t, e, k)$ be a categorical group. A *braiding in* C is a map $\tau : C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, satisfying:

$$\tau_{a,b} : ab \to ba, \tag{GrB1}$$

$$\begin{tikzcd} s(x)s(y) \arrow[r, "xy"] \arrow[d, "\tau_{s(x),s(y)}"'] & t(x)t(y) \arrow[d, "\tau_{t(x),t(y)}"] \\ s(y)s(x) \arrow[r, "yx"'] & t(y)t(x), \end{tikzcd} \tag{GrB2}$$

$$\tau_{ab,c} = (\tau_{a,c}e(b))\circ(e(a)\tau_{b,c}), \tag{GrB3}$$

$$\tau_{a,bc} = (e(b)\tau_{a,c})\circ(\tau_{a,b}e(c)), \tag{GrB4}$$

for $a, b, c \in C_0$, $x, y \in C_1$.

We say that $(C_1, C_0, s, t, e, k, \tau)$ is a *braided categorical group*.

Remark 2.1.4. One can see that (GrB2) is the naturalness and (GrB3), (GrB4) are the coherence diagrams.

Definition 2.1.5. A *braided internal functor between two braided categorical groups* is an internal functor (F_1, F_0) satisfying $F_1(\tau_{a,b}) = \tau'_{F_0(a),F_0(b)}$, where τ and τ' are the braidings and $a, b \in C_0$.

We denote by ***BICat***(***Grp***) the category of braided categorical groups and braided internal functors between them.

The definition of braiding in crossed modules of groups was given by Conduché in [16, Equalities (2.12)] and, although in this case the action is superfluous, it can be recovered, as he says previously, as $m \cdot l = l\{\partial(l)^{-1}, m\}$. We will take this action into account, duplicate one of the equalities and use the last two equalities (2.11) of [16] instead of the last two of (2.12). This is consistent because they are equivalent (see [16]).

Definition 2.1.6. Let $(G \xrightarrow{\partial} H, \cdot)$ be a crossed module of groups. A *braiding* (or *Peiffer lifting*) on the crossed module is a map $\{-,-\} : H \times H \to G$ satisfying:

$$\partial\{h, h'\} = [h, h'],$$

$$
\begin{aligned}
\{\partial g, \partial g'\} &= [g, g'],\\
\{\partial g, h\} &= g(h \cdot g^{-1}),\\
\{h, \partial g\} &= (h \cdot g)g^{-1},\\
\{h, h'h''\} &= \{h, h'\}(h' \cdot \{h, h''\}),\\
\{hh', h''\} &= (h \cdot \{h', h''\})\{h, h''\},
\end{aligned}
$$

for $g, g' \in G$, $h, h', h'' \in H$, where $[g, g'] = gg'g^{-1}g'^{-1}$.

We say that $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ is a *braided crossed module of groups*.

Example 2.1.7.

The commutator map $[-,-]$ is a braiding for the crossed module $(G \xrightarrow{\mathrm{Id}_G} G, \mathrm{Conj})$.

Let $(G \overset{\partial}{\twoheadrightarrow} H, \cdot)$ be a simply connected crossed module of groups (i.e, ∂ is a surjective map). There is a canonical braiding on $(G \overset{\partial}{\twoheadrightarrow} H, \cdot)$, given by $\{\partial(g), \partial(g')\} = [g, g']$. In particular, $M \overset{\mathrm{Conj}}{\twoheadrightarrow} \mathrm{IAut}(G)$ is a braided crossed module with the braiding

$$\{\mathrm{Conj}(g), \mathrm{Conj}(g')\} = [g, g'].$$

Definition 2.1.8. A *homomorphism of braided crossed modules of groups* (f_1, f_2) : $(G \xrightarrow{\partial} H, \cdot, \{-,-\}) \longrightarrow (G' \xrightarrow{\partial'} H', *, \{-,-\}')$ is a homomorphism of crossed modules of groups such that

$$f_1(\{h, h'\}) = \{f_2(h), f_2(h')\}', \quad h, h' \in H.$$

We denote by ***BX***(***Grp***) the category of braided crossed modules of groups and their homomorphisms.

Remark 2.1.9. We can see in [38, 39] and [31] that the categories ***BICat***(***Grp***) and ***BX***(***Grp***) are equivalent.

2.2 Braiding for categorical associative algebras and crossed modules of associative algebras

In this section we will introduce a definition of braiding for categorical associative K-algebras.

As categorical monoids can be seen as strict monoidal internal categories in ***Set***, we can think that a strict semigroupal category over an internal category in $\boldsymbol{Vect}_K$ is really a categorical associative K-algebra. By the same reasoning we can identify the $\otimes$ product with the second operation and the functoriality is recovered in the same way as the case of groups.

The K-bilinearity of the product is given by the fact that $\otimes$ is an internal bifunctor in $\boldsymbol{Vect}_K$, i.e. it is an internal functor between the respective small categories on each component (fixing an object on left or right), which means, is linear in internal objects and internal morphisms.

With this in mind we can introduce what is braiding on categorical associative K-algebras, emulating the braiding for semigroupal categories.

We will add that $\tau : C_0 \times C_0 \to C_1$ is K-bilinear, but this is obvious since for an internal object $A \in C_0$ we must have the morphisms in $\boldsymbol{Vect}_K$ $\tau_{A,-}, \tau_{-,A} : C_0 \to C_1$ defined by $\tau_{A,-}(B) = \tau_{A,B}$ and $\tau_{-,A}(B) = \tau_{B,A}$.

We remember that the internal categories in which we work all internal morphisms are internal isomorphisms.

Definition 2.2.1. Let $C = (C_1, C_0, s, t, e, k)$ be a categorical associative K-algebra.

A *braiding on* C is a K-bilinear map $\tau : C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, satisfying:

$$\tau_{a,b} : ab \to ba, \tag{AsB1}$$

$$\begin{array}{ccc} s(x)s(y) & \xrightarrow{xy} & t(x)t(y) \\ {\scriptstyle \tau_{s(x),s(y)}}\downarrow & & \downarrow{\scriptstyle \tau_{t(x),t(y)}} \\ s(y)s(x) & \xrightarrow[yx]{} & t(y)t(x), \end{array} \tag{AsB2}$$

$$\tau_{ab,c} = (\tau_{a,c} e(b)) \circ (e(a) \tau_{b,c}), \tag{AsB3}$$

$$\tau_{a,bc} = (e(b) \tau_{a,c}) \circ (\tau_{a,b} e(c)), \tag{AsB4}$$

for $a, b, c \in C_0$, $x, y \in C_1$.

We say that $(C_1, C_0, s, t, e, k, \tau)$ is a *braided categorical associative K-algebra.*

Remark 2.2.2. As in the case of groups, (AsB2) is the naturalness and (AsB3), (AsB4) are the coherence diagrams.

Definition 2.2.3. A *braided internal functor between two braided categorical associative K-algebras* is an internal functor (F_1, F_0) such that $F_1(\tau_{a,b}) = \tau'_{F_0(a),F_0(b)}$, where τ and τ' are the braidings and $a, b \in C_0$.

We denote by $\boldsymbol{BICat}(\boldsymbol{AssAlg}_K)$ the category of braided categorical associative K-algebras and braided internal functors between them.

We will introduce the notion of braiding for crossed modules of associative algebras looking for an equivalence between braided crossed modules and braided internal categories of associative algebras, as it happens in the case of groups.

Definition 2.2.4. Let $(M \xrightarrow{\partial} N, *= (*_1, *_2))$ be a crossed module of associative K-algebras. A *braiding* (or *Peiffer lifting*) is a K-bilinear map $\{-,-\}: N \times N \to M$ satisfying:

$$\partial\{n, n'\} = [n, n'], \tag{BXAs1}$$

$$\{\partial m, \partial m'\} = [m, m'], \tag{BXAs2}$$

$$\{\partial m, n\} = -[n, m]_*, \tag{BXAs3}$$

$$\{n, \partial m\} = [n, m]_*, \tag{BXAs4}$$

$$\{n, n'n''\} = n' *_1 \{n, n''\} + \{n, n'\} *_2 n'', \tag{BXAs5}$$

$$\{nn', n''\} = n *_1 \{n', n''\} + \{n, n''\} *_2 n', \tag{BXAs6}$$

with $m, m' \in M$, $n, n', n'' \in N$.

Here, $[n, m]_* = n *_1 m - m *_2 n$ and $[x, y] = xy - yx$.

$(M \xrightarrow{\partial} N, *, \{-,-\})$ is a *braided crossed module of associative K-algebras*.

Example 2.2.5. The commutator map $[-,-]$ is a braiding on the crossed module $(M \xrightarrow{\partial} M, (*, *))$.

Definition 2.2.6. A *homomorphism of braided crossed modules of associative K-algebras* $(f_1, f_2): (M \xrightarrow{\partial} N, \cdot, \{-,-\}) \longrightarrow (M' \xrightarrow{\partial'} N', *, \{-,-\}')$ is a homomorphism of crossed modules of associative K-algebras such that

$$f_1(\{n, n'\}) = \{f_2(n), f_2(n')\}', \quad n, n' \in N.$$

We denote by $\boldsymbol{BX}(\boldsymbol{AssAlg}_K)$ the category of braided crossed modules of associative K-algebras and their homomorphisms.

Proposition 2.2.7. *Let $\mathcal{X} = (M \xrightarrow{\partial} N, (*_1, *_2), \{-,-\})$ be a braided crossed module of associative K-algebras.*

Then $\mathcal{C}_{\mathcal{X}} := (M \rtimes N, N, \bar{s}, \bar{t}, \bar{e}, \bar{k}, \bar{\tau})$ is a braided categorical associative K-algebra where $\bar{s}, \bar{t}, \bar{e}, \bar{k}$ are defined in Proposition 1.3.11 and the braiding is:

$$\bar{\tau} : N \times N \to M \rtimes N,\ \bar{\tau}_{n,n'} = (-\{n, n'\}, nn').$$

Proof. We only need to check the braiding axioms for this internal category since $(M \rtimes N, N, \bar{s}, \bar{t}, \bar{e}, \bar{k})$ is a categorical associative K-algebra by Proposition 1.3.11.

We will start with AsB1. Let $n, n' \in N$.

$$\begin{aligned}\bar{s}(\bar{\tau}_n, n') &= \bar{s}((-\{n, n'\}, nn')) = nn',\\ \bar{t}(\bar{\tau}_{n,n'}) &= \bar{t}((-\{n, n'\}, nn')) = -\partial\{n, n'\} + nn' = -[n, n'] + nn' = n'n,\end{aligned}$$

where we use (BXAs1).

We will prove now AsB2. Let $x = (m, n), y = (m', n') \in M \rtimes N$.

We need to show that $\bar{\tau}_{t(x),t(y)} \circ xy = yx \circ \bar{\tau}_{s(x),s(y)}$.

$$\begin{aligned}&\bar{\tau}_{t(x),t(y)} \circ xy\\ &= \bar{k}(((m, n)(m', n'), (-\{\bar{t}((m, n)), \bar{t}((m', n'))\}, \bar{t}((m, n))\bar{t}((m', n')))))\\ &= \bar{k}(((m, n)(m', n'), (-\{\partial m + n, \partial m' + n'\}, (\partial m + n)(\partial m' + n'))))\\ &= \bar{k}(((mm' + n *_1 m' + m *_2 n', nn'), (-\{\partial m + n, \partial m' + n'\},\\ &\qquad\qquad\qquad\qquad (\partial m + n)(\partial m' + n'))))\\ &= (mm' + n *_1 m' + m *_2 n' - \{\partial m + n, \partial m' + n'\}, nn')\\ &= (mm' + n *_1 m' + m *_2 n' - \{\partial m, \partial m'\} - \{\partial m, n'\} - \{n, \partial m'\} - \{n, n'\}, nn')\\ &= (mm' + n *_1 m' + m *_2 n' - [m, m'] + [n', m]_* - [n, m']_* - \{n, n'\}, nn')\\ &= (m'm + m' *_2 n + n' *_1 m - \{n, n'\}, nn'),\end{aligned}$$

where we use (BXAs2), (BXAs3) and (BXAs4) in the sixth equality. On the other hand,

$$
\begin{aligned}
& yx \circ \bar{\tau}_{s(x),s(y)} \\
& = \bar{k}(((-\{\bar{s}((m,n)), \bar{s}((m,n'))\}, \bar{s}((m,n))\bar{s}((m',n'))), (m',n')(m,n))) \\
& = \bar{k}(((-\{n,n'\}, nn'), (m',n')(m,n))) \\
& = \bar{k}(((-\{n,n'\}, nn'), (m'm + n' *_1 m + m' *_2 n, n'n))) \\
& = (-\{n,n'\} + m'm + n' *_1 m + m' *_2 n, nn').
\end{aligned}
$$

We will verify AsB3. If $n, n', n'' \in N$, then

$$
\begin{aligned}
& (\bar{\tau}_{n,n''}\bar{e}(n')) \circ (\bar{e}(n)\bar{\tau}_{n',n''}) = \bar{k}((\bar{e}(n)\bar{\tau}_{n',n''}, \bar{\tau}_{n,n''}\bar{e}(n'))) \\
& = \bar{k}((0,n)(-\{n',n''\}, n'n''), (-\{n,n''\}, nn'')(0,n')) \\
& = \bar{k}(-n *_1 \{n',n''\}, n(n'n''), (-\{n,n''\} *_2 n', (nn'')n')) \\
& = (-n *_1 \{n',n''\} - \{n,n''\} *_2 n', n(n'n'')) = (-\{nn',n''\}, (nn')n'') = \bar{\tau}_{nn',n''},
\end{aligned}
$$

where we have used (BXAs6) and associativity.

Finally, we will show that AsB4 is satisfied. If $n, n', n'' \in N$, then

$$
\begin{aligned}
& (\bar{e}(n')\bar{\tau}_{n,n''}) \circ (\bar{\tau}_{n,n'}\bar{e}(n'')) = \bar{k}(\tau_{n,n'}\bar{e}(n''), \bar{e}(n')\tau_{n,n''}) \\
& = \bar{k}((-\{n,n'\}, nn')(0,n''), (0,n')(-\{n,n''\}, nn'')) \\
& = \bar{k}((-\{n,n'\} *_2 n'', (nn')n''), (-n' *_1 \{n,n''\}, n'(nn''))) \\
& = (-n' *_1 \{n,n''\} - \{n,n'\} *_2 n'', (nn')n'') = (-\{n,n'n''\}, n(n'n'')) = \bar{\tau}_{n,n'n''},
\end{aligned}
$$

where we use (BXAs5) along with the associativity in the second equality. □

Proposition 2.2.8. *We have a functor $C_{\mathfrak{A}} : \boldsymbol{BX}(\boldsymbol{AssAlg}_K) \rightarrowtail \boldsymbol{BICat}(\boldsymbol{AssAlg}_K)$ defined by*

$$
C_{\mathfrak{A}}(\mathcal{X} \xrightarrow{(f_1,f_2)} \mathcal{X}') = C_{\mathcal{X}} \xrightarrow{(f_1 \times f_2, f_2)} C_{\mathcal{X}'}
$$

where $C_{\mathcal{X}}$ is described in the previous proposition.

Proof. We know that the pair $(f_1 \times f_2, f_2)$ is an internal functor between the respective internal categories since what we are doing is extending an existing functor (see Proposition 1.3.11) to the braided case. In the same way, as it is an extension, we only need to show that it is well defined, since it satisfies the properties of functor because the composition and identity are the same as in the categories without braiding.

So, to conclude the proof, it is enough to see that $(f_1 \times f_2, f_2)$ is a braided internal functor of braided categorical associative K-algebras.

$$\begin{aligned}(f_1 \times f_2)(\bar{\tau}_{n,n'}) &= (f_1 \times f_2)((-\{n, n'\}, nn')) = (-f_1(\{n, n'\}), f_2(nn')) \\ &= (-\{f_2(n), f_2(n')\}', f_2(n)f_2(n')) = \bar{\tau}'_{f_2(n), f_2(n')},\end{aligned}$$

where we use that (f_1, f_2) is a homomorphism of braided crossed modules of associative algebras. □

Proposition 2.2.9. *Let $C = (C_1, C_0, s, t, e, k, \tau)$ be a braided categorical associative K-algebra.*

Then $\mathcal{X}_C := (\ker(s) \xrightarrow{\partial_t} C_0, ({}^e *, *^e), \{-,-\}_\tau)$ is a braided crossed module of associative K-algebras, where $({}^e *, *^e), \partial_t$ are defined in Proposition 1.3.11 and the braiding is:

$$\{-,-\}_\tau : C_0 \times C_0 \to \ker(s), \ \{a, b\}_\tau := e(ab) - \tau_{a,b}.$$

Proof. We only need to show that $\{-,-\}_\tau$ is a braiding on a crossed module, since under the above assumptions, $(\ker(s) \xrightarrow{\partial_t} C_0, (*^e, {}^e *))$ is a crossed module of associative K-algebras by Proposition 1.3.11.

First, we see that it is well defined, i.e. $\{a, b\}_\tau \in \ker(s)$ for $a, b \in C_0$.

$$s(\{a, b\}_\tau) = s(e(ab) - \tau_{a,b}) = ab - ab = 0,$$

where we use AsB1.

Now, we will check (BXAs1). If $a, b \in C_0$, then

$$\partial_t \{a, b\}_\tau = t(e(ab) - \tau_{a,b}) = ab - ba = [a, b],$$

where we have used (AsB1).

We proceed to check (BXAs2). If $x, y \in \ker(s)$, then

$$\{\partial_t x, \partial_t y\}_\tau = e(\partial_t x \partial_t y) - \tau_{\partial_t x, \partial_t y} = e(t(x)t(y)) - \tau_{t(x),t(y)}.$$

We need to show that $e(t(x)t(y)) - \tau_{t(x),t(y)} = [x, y]$.

By axiom AsB2 we know the equality

$$k((xy, \tau_{t(x),t(y)})) = k((\tau_{s(x),s(y)}, yx)).$$

As $x \in \ker(s)$, we have that $s(x) = 0$ (in the same way for y), and $\tau_{s(x),s(y)} = 0$ by K-bilinearity. We have then that

$$k((\tau_{s(x),s(y)}, yx)) = k((0, yx)),$$

and therefore the equality

$$k((xy, \tau_{t(x),t(y)})) = k((0, yx)).$$

Using now the K-linearity of k in the previous expression, we obtain

$$0 = k((xy, \tau_{t(x),t(y)} - yx)).$$

Since $t(\tau_{t(x),t(y)} - yx) = t(y)t(x) - t(y)t(x) = 0 = s(e(0))$ we can write $k((\tau_{t(x),t(y)} - yx, e(0)))$. Further $k((\tau_{t(x),t(y)} - yx, e(0))) = \tau_{t(x),t(y)} - yx$ by the internal category axioms.

Adding both equalities and using the K-linearity of k, we get

$$k((xy + \tau_{t(x),t(y)} - yx, \tau_{t(x),t(y)} - yx)) = \tau_{t(x),t(y)} - yx.$$

Therefore, by grouping, we have

$$k(([x, y] + \tau_{t(x),t(y)}, \tau_{t(x),t(y)} - yx)) = \tau_{t(x),t(y)} - yx.$$

As $s(\tau_{t(x),t(y)} - yx) = t(x)t(y) + 0 = t(x)t(y)$ (we use that x or y are in $\ker(s)$) it makes sense to talk about the composition $k((e(t(x)t(y)), \tau_{t(x),t(y)} - yx))$, which is equal to $\tau_{t(x),t(y)} - yx$.

Subtracting both equalities and using the K-linearity of k, we obtain

$$k(([x,y]+\tau_{t(x),t(y)}-e(t(x)t(y)),0))=0.$$

Again, using the properties for internal categories, we have

$$\begin{aligned}0&=k(([x,y]+\tau_{t(x),t(y)}-e(t(x)t(y)),0))\\&=k(([x,y]+\tau_{t(x),t(y)}-e(t(x)t(y)),e(0)))\\&=[x,y]+\tau_{t(x),t(y)}-e(t(x)t(y)),\end{aligned}$$

which gives us the required equality.

As an observation to the above, in the part of the proof that is related to $x, y \in \ker(s)$, it is sufficient that one of the two is in that kernel. By using this fact we have the following equalities for $x \in \ker(s)$ and $y \in C_1$:

$$e(t(x)t(y))-\tau_{t(x),t(y)}=[x,y],\qquad e(t(y)t(x))-\tau_{t(y),t(x)}=[y,x].$$

Now with these equalities, we will prove (BXAs3) and (BXAs4).

Let $a \in C_0$ and $x \in \ker(s)$. Then

$$\begin{aligned}\{\partial_t x,a\}_\tau&=e(t(x)t(e(a)))-\tau_{t(x),t(e(a))}=[x,e(a)]=xe(a)-e(a)x=x*^e a-a\,{}^e\!*x,\\\{a,\partial_t x\}_\tau&=e(t(e(a))t(x))-\tau_{t(e(a)),t(x)}=[e(a),x]=e(a)x-xe(a)=a\,{}^e\!*x-x*^e a.\end{aligned}$$

We will see now the last conditions, starting with (BXAs5). Let $a, b, c \in C_0$.

$$\begin{aligned}\{a,bc\}_\tau&=e(a(bc))-\tau_{a,bc}=e(a(bc))-((e(b)\tau_{a,c})\circ(\tau_{a,b}e(c)))\\&=e(a(bc))-e(b)\tau_{a,c}-\tau_{a,b}e(c)+e(t(\tau_{a,b}e(c)))\\&=e((ab)c)-e(b)\tau_{a,c}-\tau_{a,b}e(c)+e((ba)c)\\&=e(b)e(ac)-e(b)\tau_{a,c}+e(ab)e(c)-\tau_{a,b}e(c)\\&=e(b)\{a,c\}_\tau+\{a,b\}_\tau e(c)=b\,{}^e\!*\{a,c\}_\tau+\{a,b\}_\tau *^e c,\end{aligned}$$

where we have used (AsB4), Lemma 2.1.1 and the associativity.

To conclude we will check (BXAs6).

$$\{ab,c\}_\tau=e((ab)c)-\tau_{ab,c}=e((ab)c)-((\tau_{a,c}e(b))\circ(e(a)\tau_{b,c}))$$

$$
\begin{aligned}
&= e((ab)c) - \tau_{a,c}e(b) - e(a)\tau_{b,c} + e(t(e(a)\tau_{b,c}))\\
&= e(a(bc)) - \tau_{a,c}e(b) - e(a)\tau_{b,c} + e(a(cb))\\
&= e(a)e(bc) - e(a)\tau_{b,c} + e(ac)e(b) - \tau_{a,c}e(b)\\
&= e(a)\{b,c\}_\tau + \{a,c\}_\tau e(b) = a \,{}^{e}* \{b,c\}_\tau + \{a,c\}_\tau *^{e} b,
\end{aligned}
$$

where we have used (AsB3), Lemma 2.1.1 and associativity. □

Proposition 2.2.10. *We have a functor $\mathcal{X}_{\mathfrak{A}} : \boldsymbol{BICat}(\boldsymbol{AssAlg}_K) \to \boldsymbol{BX}(\boldsymbol{AssAlg}_K)$ defined by*

$$\mathcal{X}_{\mathfrak{A}}(C \xrightarrow{(F_1,F_0)} C') = \mathcal{X}_C \xrightarrow{(F_1^s,F_0)} \mathcal{X}_{C'},$$

where $\mathcal{X}_C$ is described in the previous proposition and $F_1^s : \ker(s) \to \ker(s')$ is determined by $F_1^s(x) = F_1(x)$, with $x \in \ker(s)$.

Proof. We only need to show that $\mathcal{X}_{\mathfrak{A}}$ can be extended to the braided case since is a functor between the categories without braiding (see Proposition 1.3.11). For this, we have to satisfy the axioms of the homomorphisms of braided crossed modules of associative K-algebras.

$$
\begin{aligned}
F_1^s(\{a,b\}_\tau) &= F_1(e(ab) - \tau_{a,b}) = F_1(e(a,b)) - F_1(\tau_{a,b})\\
&= e'(F_0(ab)) - \tau'_{F_0(a),F_0(b)} = e'(F_0(a)F_0(b)) - \tau'_{F_0(a),F_0(b)}\\
&= \{F_0(a), F_0(b)\}_{\tau'}.
\end{aligned}
$$

□

Remark 2.2.11. Note that, if $(M \xrightarrow{\partial} N, (*_1, *_2), \{-,-\})$ is a braided crossed module of associative K-algebras, then $\ker(\bar{s}) = \{(m,0) \in M \rtimes N \mid m \in M\} =: (M,0)$, where $\bar{s}$ is defined for the functor $\mathcal{C}_{\mathfrak{A}}$.

Proposition 2.2.12. *The categories $\boldsymbol{BX}(\boldsymbol{AssAlg}_K)$ and $\boldsymbol{BICat}(\boldsymbol{AssAlg}_K)$ are equivalent categories.*

Further, the functors $\mathcal{C}_{\mathfrak{A}}$ and $\mathcal{X}_{\mathfrak{A}}$ are inverse equivalences, where the natural isomorphisms $\mathrm{Id}_{\boldsymbol{BX}(\boldsymbol{AssAlg}_K)} \overset{\alpha^{\mathfrak{A}}}{\cong} \mathcal{X}_{\mathfrak{A}} \circ \mathcal{C}_{\mathfrak{A}}$ and $\mathrm{Id}_{\boldsymbol{BICat}(\boldsymbol{AssAlg}_K)} \overset{\beta^{\mathfrak{A}}}{\cong} \mathcal{C}_{\mathfrak{A}} \circ \mathcal{X}_{\mathfrak{A}}$ are given by:

- *if $\mathcal{Z} = (M \xrightarrow{\partial} N, (*_1, *_2), \{-,-\})$ is a braided crossed module of associative K-algebras, then $\alpha^{\mathfrak{A}}_{\mathcal{Z}} = (\alpha^{\mathfrak{A}}_M, \mathrm{Id}_N)$, with $\alpha^{\mathfrak{A}}_M : M \to (M,0)$ defined as $\alpha_M(m) = (m,0)$;*

• *if* $\mathcal{D} = (C_1, C_0, s, t, e, k, \tau)$ *is a braided categorical associative* K*-algebra, then* $\beta^{\mathfrak{A}}_{\mathcal{D}} = (\beta^{\mathfrak{A}}_s, \mathrm{Id}_{C_0})$*, with* $\beta^{\mathfrak{A}}_{C_1} : C_1 \to \ker(s) \rtimes C_0$ *defined as* $\beta^{\mathfrak{A}}_{C_1}(x) = (x - e(s(x)), s(x))$.

Proof. We only need to show that $\alpha^{\mathfrak{A}}_{\mathcal{Z}}$ and $\beta^{\mathfrak{A}}_{\mathcal{D}}$ are isomorphisms between braided objects since that they are well-defined maps, isomorphisms in the categories without braiding, as well as are natural isomorphisms (see [22]). So, it is sufficient to prove that $\alpha^{\mathfrak{A}}_{\mathcal{Z}}$ and $\beta^{\mathfrak{A}}_{\mathcal{D}}$ satisfy the braided axioms, since the bijective morphisms are isomorphisms in both categories.

Let $\mathcal{Z} = (M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), \{-,-\})$ a braided crossed module of associative K-algebras. We will check that $\alpha^{\mathfrak{A}}_{\mathcal{Z}} = (\alpha^{\mathfrak{A}}_M, \mathrm{Id}_N)$ is a homomorphism.

$$\begin{aligned} \mathrm{Id}_N(\{n,n'\}_{\bar{\tau}}) = \{n,n'\}_{\bar{\tau}} &= \bar{e}(nn') - \bar{\tau}_{n,n'} = (0, nn') - (-\{n,n'\}, nn') \\ &= (\{n,n'\}, 0) = \alpha^{\mathfrak{A}}_M(\{n,n'\}), \qquad \text{where } n, n' \in N. \end{aligned}$$

Let $\mathcal{D} = (C_1, C_0, s, t, e, k, \tau)$ be a braided categorical associative K-algebra. We will check that $\beta^{\mathfrak{A}}_{\mathcal{D}} = (\beta^{\mathfrak{A}}_s, \mathrm{Id}_{C_0})$ is a morphism.

If $a, b \in C_0$, we have

$$\begin{aligned} \mathrm{Id}_{C_0}(\bar{\tau}_{a,b}) = \bar{\tau}_{a,b} &= (-\{a,b\}_\tau, ab) = (\tau_{a,b} - e(ab), ab) \\ &= (\tau_{a,b} - e(s(\tau_{a,b})), s(\tau_{a,b})) = \beta^{\mathfrak{A}}_{C_1}(\tau_{a,b}). \end{aligned}$$

Therefore, the equivalence of categories is obtained since they are morphisms, and we know that they are natural isomorphisms. □

2.3 Braiding for categorical Lie algebras and crossed modules of Lie algebras

In this section, we will show that the definition given by Ulualan in [50] for braided categorical Lie K-algebras appears naturally from the previous one, using the fact that we can transform an associative K-algebra M in a Lie K-algebra $M^{\mathcal{L}}$ with bracket $[x, y] = xy - yx$.

Now, we will suppose that K is a field of $\mathrm{char}(K) \neq 2$ to change a little the definition of braiding. Doing this we will obtain the definition given in [24], where the equivalence is proven with the category of braided crossed modules of Lie K-algebras when $\mathrm{char}(K) \neq 2$.

The notion of braiding for categorical Lie K-algebras was introduced by Ulualan in [50].

Definition 2.3.1 ([50]). Let $\mathcal{C} = (C_1, C_0, s, t, e, k)$ be a categorical Lie K-algebra.

A *braiding on* $\mathcal{C}$ is a K-bilinear map $\tau :\ C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, satisfying:

$$\tau_{a,b} :\ [a, b] \to [b, a], \tag{LieT1}$$

$$\begin{tikzcd} {[s(x), s(y)]} \arrow[r, "{[x,y]}"] \arrow[d, "\tau_{s(x),s(y)}"'] & {[t(x), t(y)]} \arrow[d, "\tau_{t(x),t(y)}"] \\ {[s(y), s(x)]} \arrow[r, "{[y,x]}"'] & {[t(y), t(x)]}, \end{tikzcd} \tag{LieT2}$$

$$\tau_{[a,b],c} = [\tau_{a,c}, e(b)] + [e(a), \tau_{b,c}], \tag{LieB3}$$

$$\tau_{a,[b,c]} = [e(b), \tau_{a,c}] + [\tau_{a,b}, e(c)], \tag{LieB4}$$

for $a, b, c \in C_0$, $x, y \in C_1$.

We say that $(C_0, C_1, s, t, e, k, \tau)$ is a *braided categorical Lie K-algebra*.

Remark 2.3.2. The lack of associativity of the Lie bracket motivates the use of the addition in LieB3 and LieB4 instead of the composition. This choice makes sense since the source and the target are the same using the Jacobi identity.

We want to show that the definition of braiding for associative K-algebras is well related with the definition of braiding for Lie K-algebras.

Proposition 2.3.3. *Let $(C_1, C_0, s, t, e, k, \tau)$ be a braided categorical associative K-algebra, then $(C_1^{\mathcal{L}}, C_0^{\mathcal{L}}, s, t, e, k, \tau^{Lie})$ is a braided categorical Lie K-algebra, where*

$$\tau^{Lie} :\ C_0^{\mathcal{L}} \times C_0^{\mathcal{L}} \to C_1^{\mathcal{L}}, \qquad \tau^{Lie}_{a,b} := \tau_{a,b} - \tau_{b,a}.$$

Proof. It is easy to see that AsB1 implies LieT1 and AsB2 implies LieT2.

By using Lemma 2.1.1 we obtain LieB3 and LieB4 from AsB3 and AsB4, respectively. □

Another definition for braided internal category of Lie K-algebras was given in [24] to make the equivalence with the braided crossed modules of Lie K-algebras. The equivalence was proven for a field with $\mathrm{char}(K) \neq 2$, so we will show that the two definitions are equivalent.

The definition given in [24] is the following one.

Definition 2.3.4 ([24])**.** Let $\mathcal{C} = (C_1, C_0, s, t, e, k)$ be a categorical Lie K-algebra.

A *braiding* on $\mathcal{C}$ is a K-bilinear map $\tau : C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, satisfying LieT1, LieT2 and the following equalities:

$$\tau_{[a,b],c} = \tau_{a,[b,c]} - \tau_{b,[a,c]}, \tag{LieT3}$$

$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \tau_{[a,c],b}, \tag{LieT4}$$

for $a, b, c \in C_0$.

In the following proposition we show that the two definitions are equivalent when $\mathrm{char}(K) \neq 2$.

Proposition 2.3.5. *Let K be a field of* $\mathrm{char}(K) \neq 2$ *and* (C_1, C_0, s, t, e, k) *a categorical Lie K-algebra.*

If $\tau : C_0 \times C_0 \to C_1$ is a K-bilinear map satisfying LieT1 *and* LieT2*, then*

$$\tau_{a,[b,c]} = [e(a), \tau_{b,c}] \qquad \text{and} \qquad \tau_{[b,c],a} = [\tau_{b,c}, e(a)].$$

In particular, by the anticommutativity, we have that $\tau_{a,[b,c]} = -\tau_{[b,c],a}$.

Proof. Using LieT1 and LieT2, we have the following commutative diagram:

$$\begin{tikzcd}
{[a,[b,c]]} \arrow[r, "{[e(a),\tau_{b,c}]}"] \arrow[d, "\tau_{a,[b,c]}"'] & {[a,[c,b]]} \arrow[d, "\tau_{a,[c,b]}"] \\
{[[b,c],a]} \arrow[r, "{[\tau_{b,c},e(a)]}"'] & {[[c,b],a]}.
\end{tikzcd}$$

That is, we have the equality

$$k(([e(a), \tau_{b,c}], \tau_{a,[c,b]})) = k((\tau_{a,[b,c]}, [\tau_{b,c}, e(a)])),$$

and then

$$\begin{aligned} 0 &= k(([e(a), \tau_{b,c}], \tau_{a,[c,b]})) - k((\tau_{a,[b,c]}, [\tau_{b,c}, e(a)])) \\ &= k(([e(a), \tau_{b,c}] - \tau_{a,[b,c]}, \tau_{a,[c,b]} - [\tau_{b,c}, e(a)])) \\ &= k(([e(a), \tau_{b,c}] - \tau_{a,[b,c]}, -\tau_{a,[b,c]} + [e(a), \tau_{b,c}])). \end{aligned}$$

Using now Lemma 2.1.1, we have

$$\begin{aligned} 0 &= [e(a), \tau_{b,c}] - \tau_{a,[b,c]} + (-\tau_{a,[b,c]} + [e(a), \tau_{b,c}]) - e(s(-\tau_{a,[b,c]} + [e(a), \tau_{b,c}])) \\ &= 2([e(a), \tau_{b,c}] - \tau_{a,[b,c]}) - e(-[a, [b, c]] + [a, [b, c]]) = 2([e(a), \tau_{b,c}] - \tau_{a,[b,c]}). \end{aligned}$$

Since $\mathrm{char}(K) \neq 2$, we have the required equality.

The other equality is similar, using the commutative diagram

$$\begin{tikzcd} {[[a,b],c]} \arrow[r, "{[\tau_{a,b},e(c)]}"] \arrow[d, "\tau_{[a,b],c}"'] & {[[b,a],c]} \arrow[d, "\tau_{[b,a],c}"] \\ {[c,[a,b]]} \arrow[r, "{[e(c),\tau_{a,b}]}"'] & {[c,[b,a]].} \end{tikzcd}$$

□

Definition 2.3.6. A *braided internal functor between two braided categorical Lie K-algebras* is an internal functor (F_1, F_0) such that $F_1(\tau_{a,b}) = \tau'_{F_0(a),F_0(b)}$, where τ and τ' are the braidings and $a, b \in C_0$.

We denote by $\boldsymbol{BICat}(\boldsymbol{LieAlg}_K)$ the category of braided categorical Lie K-algebras and braided internal functors between them.

The definition of braiding for crossed modules of Lie K-algebras was given in [24] trying to make a definition for which the Lie bracket was a braiding for the crossed module $(M \xrightarrow{\mathrm{Id}_M} M, [-,-])$ (the Lie bracket it is also known as commutator of the Lie K-algebra) in parallelism to the fact that $(G \xrightarrow{\mathrm{Id}_G} G, \mathrm{Conj}, [-,-])$ is a braided crossed module of groups. That definition was also made to be a particular case of 2-crossed modules of groups, whose definition can be seen in [46].

Another definition can be seen in [50], but that definition does not satisfy the mentioned requirements.

Definition 2.3.7. Let $\mathcal{X} = (M \overset{\partial}{\longrightarrow} N, \cdot)$ be a crossed module of Lie K-algebras.

A *braiding* (or *Peiffer lifting*) on the crossed module $\mathcal{X}$ is given by K-bilinear map $\{-,-\}\colon N \times N \to M$ satisfying:

$$\partial\{n, n'\} = [n, n'], \tag{BXLie1}$$
$$\{\partial m, \partial m'\} = [m, m'], \tag{BXLie2}$$
$$\{\partial m, n\} = -n \cdot m, \tag{BXLie3}$$
$$\{n, \partial m\} = n \cdot m, \tag{BXLie4}$$
$$\{n, [n', n'']\} = \{[n, n'], n''\} - \{[n, n''], n'\}, \tag{BXLie5}$$
$$\{[n, n'], n''\} = \{n, [n', n'']\} - \{n', [n, n'']\}, \tag{BXLie6}$$

for $m, m' \in M$, $n, n', n'' \in N$.

We say that $(M \overset{\partial}{\longrightarrow} N, \cdot, \{-,-\})$ is a *braided crossed module of Lie K-algebras*.

Example 2.3.8.

1. It is clear that $(M \xrightarrow{\mathrm{Id}_M} M, [-,-], [-,-])$ is a braided crossed module of Lie K-algebras.

2. Let $(M \overset{\partial}{\twoheadrightarrow} N, \cdot)$ be a simply connected Lie crossed module. There is a canonical braiding on $(M \overset{\partial}{\twoheadrightarrow} N, \cdot)$, given by $\{\partial(m), \partial(m')\} = [m, m']$.

   In particular, $M \overset{\mathrm{ad}}{\twoheadrightarrow} \mathrm{IDer}(M)$ is a braided crossed module with the braiding $\{\mathrm{ad}(m), \mathrm{ad}(m')\} = [m, m']$.

Definition 2.3.9. A *homomorphism of braided crossed modules of Lie K-algebras* $(f_1, f_2)\colon (M \overset{\partial}{\longrightarrow} N, \cdot, \{-,-\}) \longrightarrow (M' \overset{\partial'}{\longrightarrow} N', *, \{-,-\}')$ is a homomorphism of crossed modules of Lie K-algebras satisfying

$$f_1(\{n, n'\}) = \{f_2(n), f_2(n')\}', \quad n, n' \in N. \tag{BXLieH3}$$

We denote by $\boldsymbol{BX(LieAlg_K)}$ the category of braided crossed modules of Lie K-algebras and their homomorphisms.

Now, we will show the natural relation between the definitions of braidings in the case of crossed modules of associative algebras and crossed modules of Lie algebras.

Proposition 2.3.10. *Let* $\mathrm{char}(K) \neq 2$. *If* $(M \xrightarrow{\partial} N, *, \{-,-\})$ *is a braided crossed module of associative* K*-algebras, then* $\{n,n'\}_{\mathcal{L}} = \frac{\{n,n'\}-\{n',n\}}{2}$ *is a braiding on the crossed module* $(M^{\mathcal{L}} \xrightarrow{\partial} N^{\mathcal{L}}, [-,-]_*)$ *(defined in Lemma 1.3.16).*

Proof. We only need to show that the braiding axioms are satisfied. For that we will take $n, n', n'' \in N, m, m' \in M$.

Axioms (BXLie1), (BXLie2), (BXLie3) and (BXLie4) are proved using (BXAs1), (BXAs2), (BXAs3) and (BXAs4).

$$\begin{aligned}
\partial(\{n,n'\}_{\mathcal{L}}) &= \partial\Big(\frac{\{n,n'\}-\{n',n\}}{2}\Big) = \frac{[n,n']-[n',n]}{2} = [n,n'],\\
\{\partial m, \partial m'\}_{\mathcal{L}} &= \frac{\{\partial m, \partial m'\}-\{\partial m', \partial m\}}{2} = \frac{[m,m']-[m',m]}{2} = [m,m'],\\
\{\partial m, n\}_{\mathcal{L}} &= \frac{\{\partial m, n\}-\{n, \partial m\}}{2} = \frac{-[n,m]_*-[n,m]_*}{2} = -[n,m]_*,\\
\{n, \partial m\}_{\mathcal{L}} &= \frac{\{n, \partial m\}-\{\partial m, n\}}{2} = \frac{[n,m]_*+[n,m]_*}{2} = [n,m]_*.
\end{aligned}$$

With the previous axioms proved, we have that the followings equalities hold.

$$\begin{aligned}
\{n,[n',n'']\}_{\mathcal{L}} &= \{n, \partial\{n',n''\}_{\mathcal{L}}\}_{\mathcal{L}} = [n, \{n',n''\}_{\mathcal{L}}]_*\\
&= -\{\partial\{n',n''\}_{\mathcal{L}}, n\}_{\mathcal{L}} = -\{[n',n''],n\}_{\mathcal{L}}.
\end{aligned}$$

Finally, we will prove that the braiding satisfies the last axioms. We will abuse language and we will denote $* = [-,-]_*$ (in the definition $* = (*_1, *_2)$).

$$\begin{aligned}
\{n,[n',n'']\}_{\mathcal{L}} &= \{n, n'n''\}_{\mathcal{L}} - \{n, n''n'\}_{\mathcal{L}}\\
&= \frac{1}{2}(\{n,n'n''\} - \{n'n'',n\} - \{n,n''n'\} + \{n''n',n\})\\
&= \frac{1}{2}(n' *_1 \{n,n''\} - \{n,n'\} *_2 n'' - n' *_1 \{n'',n\} - \{n',n\} *_2 n''\\
&\quad - n'' *_1 \{n,n'\} - \{n,n''\} *_2 n' + n'' *_1 \{n',n\} + \{n'',n\} *_2 n')\\
&= \frac{1}{2}(n' * \{n,n''\} - n'' * \{n,n'\} + n'' * \{n',n\} - n' * \{n'',n\})
\end{aligned}$$

$$= n' * \{n, n''\}_{\mathcal{L}} - n'' * \{n, n'\}_{\mathcal{L}} = -\{[n, n''], n'\} + \{[n, n'], n''\}.$$

$$\begin{aligned}
\{[n, n'], n''\}_{\mathcal{L}} &= \{nn', n''\}_{\mathcal{L}} - \{n'n, n''\}_{\mathcal{L}} \\
&= \frac{1}{2}(\{nn', n''\} - \{n'', nn'\} - \{n'n, n''\} + \{n'', n'n\}) \\
&= \frac{1}{2}(n *_1 \{n', n''\} + \{n, n''\} *_2 n' - n *_1 \{n'', n'\} - \{n'', n\} *_2 n' \\
&\quad - n' *_1 \{n, n''\} - \{n', n''\} *_2 n + n' *_1 \{n'', n\} + \{n'', n'\} *_2 n) \\
&= \frac{1}{2}(n * \{n', n''\} - n * \{n'', n'\} - n' * \{n, n''\} + n' * \{n''n\}) \\
&= n * \{n', n''\}_{\mathcal{L}} - n' * \{n, n''\}_{\mathcal{L}} = \{n, [n', n'']\}_{\mathcal{L}} - \{n', [n, n'']\}_{\mathcal{L}}. \square
\end{aligned}$$

Remark 2.3.11. Note that the previous construction translates the example given for associative case to the one given in Lie case.

Remark 2.3.12. If $\mathrm{char}(K) \neq 2$, we have an equivalence between the categories $\boldsymbol{BICat}(\boldsymbol{LieAlg}_K)$ and $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$, as for the case of groups (see [24], cf. [50]).

In addition, the relations with the associative case give us two functors. These functors commute in an immediate way with the functors of the equivalence.

2.4 Braiding for categorical Leibniz algebras and crossed modules of Leibniz algebras

In this section, we will use the idea of Loday and Pirashvili ([44]) to see the Leibniz K-algebras as a particular case of a Lie algebra in the monoidal category of linear maps $\mathcal{LM}_K$, also known as the Loday-Pirashvili category ([23, 49]). Using this, we will try to define the concept of braiding in the case of Leibniz algebras taking advantage of the fact that they will be a particular case of braidings for the corresponding ideas over Lie objects in that category.

First, we will introduce some notation.

Let $\boldsymbol{C}$ be a category with coproducts. If we have $A \xrightarrow{f} C \xleftarrow{g} B$ we will denote the unique morphism given by the universal property of the coproduct as the morphism $[f, g]\colon A \oplus B \to C$. $\oplus$ is used for the coproduct functor.

In the following sections, we will abuse the language denoting $0 : A \to B$ for the zero morphism when it gives no confusion.

The definition of the category $\mathcal{LM}_K$ can be seen in [44].

Definition 2.4.1. The category $\mathcal{LM}_K$ is a monoidal category with the following data:

- As objects we take the K-linear maps.
- If $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ and $\begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ are two K-linear maps, a morphism is a pair of K-linear maps $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 : M \to L$ and $\alpha_2 : N \to H$ of K-linear maps such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "\alpha_1"] \arrow[d, "f"] & L \arrow[d, "g"] \\ N \arrow[r, "\alpha_2"] & H. \end{tikzcd}$$

- The tensor product $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array} \otimes \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array} := \begin{array}{c} (M \otimes H) \oplus (N \otimes L) \\ \downarrow{\scriptstyle [f \otimes \mathrm{Id}_H, \mathrm{Id}_N \otimes g]} \\ N \otimes H \end{array}$, where $\otimes$ between K-vector spaces is the usual tensor product and $\oplus$ is the direct sum of vector spaces.

  In morphisms the tensor product is given by

$$(f_1, f_2) \otimes (g_1, g_2) = ((f_1 \otimes g_2) \oplus (f_2 \otimes g_1), g_1 \otimes g_2).$$

This is a braided monoidal category with the braiding given by

$\mathcal{T}_{f,g} = (\mathcal{T}^1_{f,g}, \mathcal{T}^2_{f,g}) : \begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array} \otimes \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array} \to \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array} \otimes \begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$, with $\mathcal{T}^2_{f,g} = \mathcal{T}_{N,H} : N \otimes H \to H \otimes N$, the usual braiding for the K-vector tensor product, and

$$\mathcal{T}^1_{f,g} : (M \otimes H) \oplus (N \otimes L) \to (L \otimes N) \oplus (H \otimes M)$$

given by $\mathcal{T}^1_{f,g}((m \otimes h) + (n \otimes l)) = (l \otimes n) + (h \otimes m)$.

Remark 2.4.2. $\mathcal{LM}_K$ is an additive category with $\begin{array}{c}\{0\}\\ \downarrow{\scriptstyle 0}\\ \{0\}\end{array}$ as zero object, where we have

$\begin{array}{c}M\\ \downarrow{\scriptstyle f}\\ N\end{array} \times \begin{array}{c}L\\ \downarrow{\scriptstyle g}\\ H\end{array} := \begin{array}{c}M\times L\\ \downarrow{\scriptstyle f\times g}\\ N\times H\end{array}$ as product and the usual abelian group structure on morphisms.

Analogously to [44], we can categorify the idea of Lie K-algebra and define it in the $\mathcal{LM}_K$ category. We generalize this scheme to a semigroupal category.

Definition 2.4.3. Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal category where $\mathbf{C}$ is an additive category. We say that a pair (L, μ) with $L \in \mathrm{Ob}(\mathbf{C})$ and $\mu :\ L \otimes L \to\ L$ is a *Leibniz object* in $\mathcal{C}$ if it satisfies the *Leibniz identity*, i.e.,

$$\mu\circ(\mu \otimes \mathrm{Id}_L) = \mu\circ(\mu \otimes \mathrm{Id}_L)\circ a_L^{-1}\circ(\mathrm{Id}_L \otimes\tau_{L,L})\circ a_L + \mu\circ(\mathrm{Id}_L \otimes\mu)\circ a_L, \qquad \text{(Lb)}$$

where a_L denotes $a_{L,L,L}$.

We will say that a morphism $f :\ L\to\ M$ is a *Leibniz morphism* between (L, μ) and (M, η) if satisfies the following diagram:

$$\begin{array}{ccc} L\otimes L & \xrightarrow{\mu} & L\\ \downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle f}\\ M\otimes M & \xrightarrow{\eta} & M. \end{array}$$

The composition in $\mathbf{C}$ of Leibniz morphisms and the identity morphism of a Leibniz object in $\mathbf{C}$ are Leibniz morphisms. Using this, we can define the category ***Leib***$(\mathcal{C})$.

Example 2.4.4. If we take the $\mathcal{C}$ as the category of vector spaces with the usual tensor product, the previous definition recovers the definition of Leibniz Algebra.

Definition 2.4.5. Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal category where $\mathbf{C}$ is an additive category. We say that a Leibniz object (L, μ) of $\mathcal{C}$ is a *Lie object* in $\mathcal{C}$ if it also satisfies the *anticommutativity identity*, i.e.,

$$\mu + \mu\circ\tau_{L,L} = 0. \qquad \text{(AC)}$$

We define the category ***Lie***$(\mathcal{C})$ as the full subcategory of ***Leib***$(\mathcal{C})$ whose objects are the Lie objects of $\mathcal{C}$.

Example 2.4.6. If we take the C as the category of vector spaces with the usual tensor product, the previous definition recovers the definition of Lie Algebra if the characteristic of the prefixed field is not 2. Since the generalization is only true for $\mathrm{char}(K) \neq 2$, we will assume it for the rest of the section.

We want to explain what are an object and a morphism in $\boldsymbol{Lie}(\mathcal{LM}_K)$.

Definition 2.4.7. Let V be a K-vector space and M be a Lie K-algebra.

We say that $(V, \cdot)$ is a *right M-module* if $\cdot : V \times M \to V$ is a K-bilinear map $(v, m) \mapsto v \cdot m$ such that:

$$v \cdot [m_1, m_2] = (v \cdot m_1) \cdot m_2 - (v \cdot m_2) \cdot m_1,$$

for $v \in V$, $m_1, m_2 \in M$.

We say that $(V, \cdot)$ is a *left M-module* if $\cdot : M \times V \to V$ is a K-bilinear map $(m, v) \mapsto m \cdot v$ such that:

$$[m_1, m_2] \cdot v = m_1 \cdot (m_2 \cdot v) - m_2 \cdot (m_1 \cdot v),$$

for $v \in V$, $m_1, m_2 \in M$.

Definition 2.4.8. Let $\alpha : M \to N$ be a Lie K-homomorphism. Let $(V, \cdot)$ be a right (resp. left) M-module and $(W, *)$ a right (resp. left) N-module. A K-linear map $V \xrightarrow{f} W$ is $(\alpha : M \to N, \cdot, *)$-equivariant if we have that

$$f(v \cdot m) = f(v) * \alpha(m) \quad (\text{resp. } f(m \cdot v) = \alpha(m) * f(v)), \qquad \text{for } v \in V,\ m \in M.$$

When $N = M$ and $\alpha = \mathrm{Id}_M$ we said that f is $(M, \cdot, *)$-equivariant.

Let $(V, \cdot)$ be a left M-module and $(W, *)$ a right N-module. A K-linear map $V \xrightarrow{f} W$ is $(\alpha : M \to N, \cdot, *)$-equivariant if we have that

$$f(m \cdot v) = -f(v) * \alpha(m), \quad \text{for } v \in V,\ m \in M.$$

When $N = M$ and $\alpha = \mathrm{Id}_M$ we said that f is $(M, \cdot, *)$-equivariant.

Remark 2.4.9. It is easy to check that if M is a Lie K-algebra, then it is a right and left M-module.

Moreover, if $\cdot$ is a Lie action of N in M, we have that $(M,\cdot)$ is a left N-module.

Using this, we can see in [44] that a Lie object in $\mathcal{LM}_K$ is the following data:

Definition 2.4.10. A Lie object in $\mathcal{LM}_K$ is a triple $\left(\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}, *_N^M, [-,-]_N\right)$ where

- $(N,[-,-]_N)$ is a Lie K-algebra.
- $*_N^M : M\times N \to M$ is such that $(M,*_N^M)$ is an $(N,[-,-]_N)$-module.
- f is $((N,[-,-]_N),*_N^M,[-,-]_N)$-equivariant.

As in the case of Lie K-algebras, we will denote a Lie object in $\mathcal{LM}_K$ using the K-linear map on which it is defined when there is no confusion.

Remark 2.4.11. The “anticommutative” property of Lie object for $\mathcal{LM}_K$ allows to recover the Lie product $\mu = (\mu_1,\mu_2)$ for $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ with the maps $\mu_2 = [-,-]_N$ and $\mu_1 : (M\otimes N)\oplus(N\otimes M)\to M$, with $\mu_1((m\otimes n)+(n'\otimes m')) = m *_N^M n - m' *_N^M n'$.

Definition 2.4.12. Let $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ and $\begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ be Lie objects. A Lie morphism in $\mathcal{LM}_K$ between them is an $\mathcal{LM}_K$ morphism (α_1,α_2) such that:

- $\alpha_2 : N\to H$ is a Lie K-homomorphism.
- $\alpha_1 : M\to L$ is an $(\alpha_2 : N\to H, *_N^M, *_H^L)$-equivariant map.

In [44] is shown a way to see the Leibniz K-algebras as a particular case of Lie objects in $\mathcal{LM}_K$. We show it in the next example.

Example 2.4.13. Let M be a Leibniz K-algebra.

We denote for I_M the ideal generated by elements of the form $[x,x]$ with $x\in M$. It is evident that the quotient Leibniz K-algebra is a Lie K-algebra. We will denote its Lie bracket as $\overline{[-,-]}$, and the elements of the quotient as $\overline{m}$ with $m\in M$.

$\mathrm{Lie}(M) := \frac{M}{I_M}$ is known as Liesation (note that if M is a Lie K-algebra, then $\mathrm{Lie}(M)$ is trivially naturally isomorphic to M), and it is functorial.

We consider the following Lie object in $\mathcal{LM}_K$:

We take $\begin{array}{c} M \\ \downarrow{\scriptstyle \pi_M} \\ \mathrm{Lie}(M) \end{array}$ where $\pi(m) = \overline{m}$ is the natural map. It is a Lie object in $\mathcal{LM}_K$ with the following data:

- $m *^M_{\mathrm{Lie}(M)} \overline{m'} = [m, m']$,
- $[\overline{m}, \overline{m'}]_{\mathrm{Lie}(M)} = \overline{[\overline{m}, \overline{m'}]} := [m, m']$.

It is evident that π is $(\mathrm{Lie}(M), *^M_{\mathrm{Lie}(M)}, [-,-]_{\mathrm{Lie}(M)})$-equivariant.

So, we have a functor $\Phi :$ ***LeibAlg***$_K \to$ ***Lie***$(\mathcal{LM}_K)$, that is trivially full.

This functor is also injective on objects and morphisms, because there is a functor $\Psi :$ ***Lie***$(\mathcal{LM}_K) \to$ ***LeibAlg***$_K$ such that $\Psi \circ \Phi = \mathrm{Id}_{\boldsymbol{LeibAlg}_K}$ (see [44]). The functor Ψ on objects is described in the following proposition.

Proposition 2.4.14 ([44]). *Let* $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ *be a Lie object in* $\mathcal{LM}_K$*. Then* $(M, [-,-])$*, where* $[m, m'] := m *^M_N f(m')$*, is a Leibniz* K*-algebra.*

In [23], we can see that the previous construction can be extended to crossed modules of Lie algebras in $\mathcal{LM}_K$. They did a crossed module with a right action. In this paper, we will define which is a crossed module with a left action, or simply a crossed module of Lie objects.

Definition 2.4.15. Let $\mathcal{C} = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ be a braided semigroupal category where $\boldsymbol{C}$ is an additive category.

If (A, μ_A) and (B, μ_B) are Lie objects, then a (left) Lie action of (B, μ_B) on (A, μ_A) is a morphism $p : B \otimes A \to A$ such that

$$p \circ (\mu_B \otimes \mathrm{Id}_A) = p \circ (\mathrm{Id}_B \otimes p) \circ a_{B,B,A} \circ (\mathrm{Id}_{(B \otimes B) \otimes A} - (\tau_{B,B} \otimes \mathrm{Id}_A)),$$
$$p \circ (\mathrm{Id}_B \otimes \mu_A) \circ a_{B,A,A} = \mu_A \circ (p \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_{(B \otimes B) \otimes A} - (a^{-1}_{B,A,A} \circ (\mathrm{Id}_B \otimes \tau_{A,A}) \circ a_{B,A,A})).$$

We said that $((A,\mu_A)\xrightarrow{\partial}(B,\mu_B),p)$ is a crossed module of Lie objects if p is a Lie action of (B,μ_B) on (A,μ_A) and $\partial:(A,\mu_A)\to(B,\mu_B)$ is a Lie morphism such that

$$\partial\circ p=\mu_B\circ(\mathrm{Id}_B\otimes\partial),$$
$$\mu_A=p\circ(\partial\otimes\mathrm{Id}_A).$$

A morphism between two crossed modules of Lie objects $((A,\mu_A)\xrightarrow{\partial}(B,\mu_B),p,)$ and $((C,\mu_C)\xrightarrow{\delta}(D,\mu_D),q)$ is a pair of Lie morphisms (α,β), $\alpha:(A,\mu_A)\to(C,\mu_C)$ and $\beta:(B,\mu_B)\to(D,\mu_D)$, which satisfies the following diagrams:

$$\begin{tikzcd} B\otimes A \arrow[r, "p"] \arrow[d, "\beta\otimes\alpha"] & A \arrow[d, "\alpha"] \\ D\otimes C \arrow[r, "q"] & C, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D. \end{tikzcd}$$

We have the category $\boldsymbol{XLie}(\mathcal{C})$ with the usual composition in $\boldsymbol{C}\times\boldsymbol{C}$ for pairs of morphisms of Lie morphisms.

Example 2.4.16. We have that $\boldsymbol{XLie}(\boldsymbol{Vect}_K)$ and $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ are isomorphic categories with the usual tensor product in $\boldsymbol{Vect}_K$ (we assume $\mathrm{char}(K)\neq 2$).

Now, we describe the category $\boldsymbol{XLie}(\mathcal{LM}_K)$.

Definition 2.4.17. Let $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ and $\begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ be Lie objects in $\mathcal{LM}_K$. A (left) Lie action of $\begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ on $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ in $\mathcal{LM}_K$ is a triple $\bar{\cdot}=(\cdot_1,\cdot_2,\xi_\cdot)$ where

- $\cdot_1: H\times M\to M$ is a K-bilinear map such that $(M,\cdot_1)$ is a left H-module;
- $\cdot_2: H\times N\to N$ is a Lie action of H on N;
- $\xi_\cdot: L\times N\to M$ is a K-bilinear map;

such that the following properties are satisfied:

- $\cdot_1$ and $\cdot_2$ are compatible actions with $*_N^M$. That is, for $h \in H$, $n \in N$, $m \in M$, we have
$$h \cdot_1 (m *_N^M n) = (h \cdot_1 m) *_N^M n + m *_N^M (h \cdot_2 n);$$

- f is an $(H, \cdot_1, \cdot_2)$-equivariant map;

- $\xi_.$ satisfies, for $l \in L$, $n, n' \in N$, $h \in H$, the following equalities
$$\begin{aligned} f(\xi_.(l,n)) &= g(l) \cdot_2 n, \\ \xi_.(l *_H^L h, n) &= \xi_.(l, h \cdot_2 n) - h \cdot_1 \xi_.(l,n), \\ \xi_.(l, [n,n']_N) &= \xi_.(l,n) *_N^M n' - \xi_.(l,n') *_N^M n. \end{aligned}$$

Remark 2.4.18. An action is, in fact, a pair $\bar{\cdot} = (\bar{\cdot}_1, \bar{\cdot}_2)$, with the two maps
$$\bar{\cdot}_1 \colon (L \otimes N) \oplus (H \otimes N) \to M \qquad \text{and} \qquad \bar{\cdot}_2 \colon H \otimes N \to N$$
satisfying the general properties, but we can easily obtain the previous definition taking $\cdot_2 := \bar{\cdot}_2$ and recovering $\bar{\cdot}_1((l \otimes n) + (h \otimes m)) =: \xi_.(l,n) + h \cdot_1 m$.

Definition 2.4.19. A crossed module of Lie objects in $\mathcal{LM}_K$ is a pair $\left(\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array} \xrightarrow{\partial} \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}, \bar{\cdot} \right)$ where $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$ and $\begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ are Lie objects in $\mathcal{LM}_K$, $\bar{\cdot}$ is a Lie action of $\begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ on $\begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array}$, and $\partial = (\partial_1, \partial_2) \colon \begin{array}{c} M \\ \downarrow{\scriptstyle f} \\ N \end{array} \to \begin{array}{c} L \\ \downarrow{\scriptstyle g} \\ H \end{array}$ is a Lie morphism in $\mathcal{LM}_K$ such that

- $(N, H, \cdot_2, \partial_2)$ is a crossed module of Lie K-algebras;

- ∂_1 is an $(H, \cdot_1, *_H^L)$-equivariant map;

- $\partial_1(\xi_.(l,n)) = l *_N^L \partial_2(h)$ and $\xi_.(\partial_1(m), n) = m *_N^M n = -\partial_2(n) \cdot_1 m, \quad h \in H$, $l \in L$, $m \in M$, $n \in N$.

Definition 2.4.20. Let $\left(\begin{array}{c}M\\ \big\downarrow{\scriptstyle f}\\ N\end{array}\xrightarrow{\partial}\begin{array}{c}L\\ \big\downarrow{\scriptstyle g}\\ H\end{array},\bar{\cdot}\right)$ and $\left(\begin{array}{c}X\\ \big\downarrow{\scriptstyle k}\\ Y\end{array}\xrightarrow{\delta}\begin{array}{c}V\\ \big\downarrow{\scriptstyle h}\\ W\end{array},\bar{\star}\right)$ be crossed modules of Lie objects in $\mathcal{LM}_K$. A morphism of crossed modules of Lie objects in $\mathcal{LM}_K$ is a pair (α,β) of Lie morphisms $\alpha=(\alpha_1,\alpha_2):\begin{array}{c}M\\ \big\downarrow{\scriptstyle f}\\ N\end{array}\to\begin{array}{c}X\\ \big\downarrow{\scriptstyle k}\\ Y\end{array}$ and $\beta=(\beta_1,\beta_2):\begin{array}{c}L\\ \big\downarrow{\scriptstyle g}\\ H\end{array}\to\begin{array}{c}V\\ \big\downarrow{\scriptstyle h}\\ W\end{array}$ such that

- $(\alpha_2,\beta_2):(N,H,\cdot_2,\partial_2)\to(Y,W,\star_2,\delta_2)$ is an homomorphism of crossed modules of Lie K-algebras;
- $\alpha_1(\xi_{\cdot}(l,n))=\xi_\star(\beta_1(l),\alpha_2(n))$, for $l\in L$, $n\in N$;
- α_1 is an $(H\xrightarrow{\beta_2}W,\cdot_1,\star_1)$-equivariant map;
- $\beta_1\circ\partial_1=\delta_1\circ\alpha_1$.

As in the case of Leibniz K-algebras we want to have a pair of functors between the categories ***XLie***$(\mathcal{LM}_K)$ and ***XLeibAlg***$_K$. For this purpose, we give the following propositions of which we omit their proofs because they are immediate. The first is symmetrical to the construction we can see in [23] for crossed modules with right actions.

Proposition 2.4.21. *Let $(M\xrightarrow{\partial}N,(\cdot_1,\cdot_2))$ be a crossed module of Leibniz K-algebras. Then* $\left(\begin{array}{c}M\\ \big\downarrow{\scriptstyle \pi_M}\\ \frac{M}{[M,N]_x}\end{array},\begin{array}{c}N\\ \big\downarrow{\scriptstyle \pi_N}\\ \mathrm{Lie}(N)\end{array},\bar{\bar{\cdot}},\overline{\partial}\right)$ *is a crossed module of Lie objects in $\mathcal{LM}_K$, where*

- *$\frac{M}{[M,N]_x}$ is the Lie K-algebra quotient of M by the ideal $[M,N]_x$ whose generators are $[m,m]$ for $m\in M$ and $n\cdot_1 m+m\cdot_2 n$ for $n\in N$, $m\in M$; we denote the natural map by $\pi_M: M\to\frac{M}{[M,N]_x}$, and the elements of $\frac{M}{[M,N]_x}$ by $\overline{m}$,*
- *$\bar{\cdot}_1:\ \mathrm{Lie}(N)\times M\to M$, $(\overline{n},m)\mapsto -m\cdot_2 n$,*
- *$\bar{\cdot}_2:\ \mathrm{Lie}(N)\times\frac{M}{[M,N]_x}\to\frac{M}{[M,N]_x}$, $(\overline{n},\overline{m})\mapsto\overline{n\cdot_1 m}=\overline{-m\cdot_2 n}$,*
- *$\xi_{\bar{\cdot}}:\ N\times\frac{M}{[M,N]_x}\to M$, $(n,\overline{m})\to n\cdot_1 m$,*
- *$\overline{\partial}_1:\ M\to N$, $m\mapsto\partial(m)$,*

- $\overline{\partial}_2 : \frac{M}{[M,N]_x} \to \mathrm{Lie}(N),\ \overline{m} \mapsto \overline{\partial m}.$

Remark 2.4.22. We will say that the bottom part $(\frac{M}{[M,N]_x} \xrightarrow{\overline{\partial}_2} \mathrm{Lie}(N), \bar{\cdot}_2)$ is the Liesation of the crossed module of Leibniz K-algebras. In this way we found a similar relation with the Leibniz and Lie object case.

This Liesation satisfies again that applied on a crossed module of Lie K-algebras, thought as a crossed module of Leibniz K-algebras with the action $(\cdot, \cdot^{-})$, is naturally isomorphic to itself. That occurs because, in the quotient, the second generators are null too:

$$n \cdot_1 m + m \cdot_2 n = n \cdot m + m \cdot^{-} n = n \cdot m - n \cdot m = 0.$$

Proposition 2.4.23. *Let* $\left(\begin{array}{ccc} M & \xrightarrow{\partial} & L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N & & H \end{array}, \bar{\cdot}\right)$ *be a crossed module of Lie objects in* $\mathcal{LM}_K$*, then* $(M \xrightarrow{\partial_1} L, (\tilde{\cdot}_1, \tilde{\cdot}_2))$ *is a crossed module of Leibniz* K*-algebras, where*

- *The Leibniz brackets are given by:* $[m,m'] = m *_N^M f(m')$*, for* $m, m' \in M$ *and* $[l,l'] = l *_H^L g(l')$*, for* $l, l' \in M$*;*
- $\tilde{\cdot}_1 : L \times N \to M$ *is defined by* $l \tilde{\cdot}_1 m = \xi_{\cdot}(l, f(m))$ *for* $l \in L$*,* $m \in M$*;*
- $\tilde{\cdot}_2 : M \times L \to M$ *is defined by* $m \tilde{\cdot}_2 l = -g(l) \cdot_1 m$ *for* $l \in L$*,* $m \in M$*.*

We have the functors $\boldsymbol{X(LeibAlg_K)} \underset{X\Psi}{\overset{X\Phi}{\rightleftarrows}} \boldsymbol{XLie}(\mathcal{LM}_K)$ satisfying $X\Psi \circ X\Phi = \mathrm{Id}_{\boldsymbol{X(LeibAlg_K)}}$, and so, the functor $X\Phi$ is a full inclusion functor.

2.4.1 Braiding for crossed modules of Lie objects in $\mathcal{LM}_K$ and crossed modules of Leibniz algebras

We want to define the notion of braiding for crossed modules of Leibniz algebras. We will use the idea that the braiding for crossed module of Leibniz K-algebras must be a particular case of braiding for Lie objects in $\mathcal{LM}_K$, satisfying symmetrical properties to the previous ones.

Definition 2.4.24. Let $\mathcal{C} = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ be a braided semigroupal category where $\boldsymbol{C}$ is an additive category. Let $\mathcal{X} = ((A, \mu_A) \xrightarrow{\partial} (B, \mu_B), p)$ be a crossed module of Lie objects in $\mathcal{C}$.

A braiding (or Peiffer lifting) on $\mathcal{X}$ is a morphism $\mathfrak{T} : B \otimes B \to A$ satisfying:

$$
\begin{aligned}
\partial \circ \mathfrak{T} &= \mu_B, \\
\mathfrak{T} \circ (\partial \otimes \partial) &= \mu_A, \\
-\mathfrak{T} \circ (\partial \otimes \mathrm{Id}_B) &= p \circ \mathcal{T}_{A,B}, \\
\mathfrak{T} \circ (\mathrm{Id}_B \otimes \partial) &= p, \\
\mathfrak{T} \circ (\mathrm{Id}_B \otimes \mu_B) \otimes a_{B,B,B} &= \mathfrak{T} \circ (\mu_B \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_{(B \otimes B) \otimes B} - (a^{-1}_{B,B,B} \circ (\mathrm{Id}_B \otimes \mathcal{T}_{B,B}) \circ a_{B,B,B})), \\
\mathfrak{T} \circ (\mu_B \otimes \mathrm{Id}_B) &= \mathfrak{T} \circ (\mathrm{Id}_B \otimes \mu_B) \circ a_{B,B,B} \circ (\mathrm{Id}_{(B \otimes B) \otimes B} - (\mathcal{T}_{B,B} \otimes \mathrm{Id}_B)).
\end{aligned}
$$

$((A, \mu_A) \xrightarrow{\partial} (B, \mu_B), p, \mathfrak{T})$ will be called a braided crossed module of Lie objects in $\mathcal{C}$.

A morphism $(\alpha, \beta) : ((A, \mu_A) \xrightarrow{\partial} (B, \mu_B), p, \mathfrak{T}) \to ((C, \mu_C) \xrightarrow{\delta} (D, \mu_D), q, \mathfrak{Y})$ of braided crossed modules of Lie objects is a morphism of crossed modules of Lie objects in the category $\mathcal{C}$ satisfying the following commutative diagram

$$
\begin{tikzcd}
B \otimes B \arrow[r, "\mathfrak{T}"] \arrow[d, "\beta \otimes \beta"] & A \arrow[d, "\alpha"] \\
D \otimes D \arrow[r, "\mathfrak{Y}"] & B.
\end{tikzcd}
$$

We denote this new category as $\boldsymbol{BXLie}(\mathcal{C})$.

Example 2.4.25. As in the previous cases, we have that $\boldsymbol{BXLie}(\boldsymbol{Vect}_K)$ and $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ are isomorphic, taking in $\boldsymbol{Vect}_K$ the usual tensor product.

$\boldsymbol{BXLie}(\mathcal{LM}_K)$ is described in the following definitions.

Definition 2.4.26. Let $\mathcal{X} = \left(\begin{array}{c} M \\ \downarrow f \\ N \end{array} \xrightarrow{\partial} \begin{array}{c} L \\ \downarrow g \\ H \end{array}, \cdot \right)$ be a crossed module of Lie objects in $\mathcal{LM}_K$.

A braiding (or Peiffer lifting) for $\mathcal{X}$ is given by a triple of maps

$$T_{\{-,-\}} = (\{-,-\}_{LH}, \{-,-\}_{HL}, \{-,-\}_2)$$

where

- $\{-,-\}_2 : H \times H \to N$ is a K-bilinear map such that $(N, H, \cdot_2, \partial_2, \{-,-\}_2)$ is a braided crossed module of Lie K-algebras.

- $\{-,-\}_{LH} : L \times H \to M$ and $\{-,-\}_{HL} : H \times L \to M$ are K-bilinear maps, which with $\{-,-\}_2$ satisfy the following properties for $l \in L$, $h, h' \in H$, $m \in M$, $n \in N$:

$$
\begin{aligned}
f(\{l,h\}_{LH}) &= \{g(l),h\}_2, & f(\{h,l\}_{HL}) &= \{h,g(l)\}_2,\\
\partial_1\{l,h\}_{LH} &= l *^L_H h, & \partial_1\{h,l\}_{HL} &= -l *^L_H h,\\
\{\partial_1(m),\partial_2(n)\}_{LH} &= m *^M_N n, & \{\partial_2(n),\partial_1(m)\}_{HL} &= -m *^M_N n,\\
\{\partial_1(m),h\}_{LH} &= -h \cdot_1 m, & \{\partial_2(n),l\}_{HL} &= -\xi_{\cdot}(l,n),\\
\{l,\partial_2(n)\} &= \xi_{\cdot}(l,n), & \{h,\partial_1(m)\} &= h \cdot_1 m,
\end{aligned}
$$

$$
\begin{aligned}
\{l,[h,h']_H\}_{LH} &= \{l *^L_H h, h'\}_{LH} - \{l *^L_H h', h\}_{LH},\\
\{[h,h']_H, l\}_{HL} &= -\{h, l *^L_H h'\}_{HL} - \{l *^L_H h, h'\}_{LH},\\
\{l,[h,h']_H\}_{LH} &= \{l *^L_H h, h'\}_{LH} + \{h, l *^L_H h'\}_{HL},\\
\{[h,h']_H, l\}_{HL} &= -\{h, l *^L_H h'\}_{HL} + \{h', l *^L_H h\}_{HL}.
\end{aligned}
$$

We will say that $\left(\begin{smallmatrix} M & \xrightarrow{\partial} & L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N & & H \end{smallmatrix}, \bar{\cdot}, T_{\{-,-\}}\right)$ is a braided crossed module of Lie objects in $\mathcal{LM}_K$.

Remark 2.4.27. A braiding is a pair $T_{\{-,-\}} = (T^1_{\{-,-\}}, T^2_{\{-,-\}})$, but for simplicity we denote $T^1_{\{-,-\}} : (L \otimes H) \oplus (H \otimes L) \to M$ with $T^1_{\{-,-\}}((l \otimes h) + (h' \otimes l')) = \{l,h\}_{LH} + \{h',l'\}_{HL}$ and $T^2_{\{-,-\}}(h,h') = \{h,h'\}_2$.

Definition 2.4.28. Let $\left(\begin{smallmatrix} M & \xrightarrow{\partial} & L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N & & H \end{smallmatrix}, \bar{\cdot}, T_{\{-,-\}}\right)$ and $\left(\begin{smallmatrix} X & \xrightarrow{\delta} & V \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} \\ Y & & W \end{smallmatrix}, \bar{\star}, T_{\{-,-\}'}\right)$ be two braided crossed modules of Lie objects in $\mathcal{LM}_K$. A morphism of braided crossed modules of Lie objects in $\mathcal{LM}_K$ is a morphism (α, β) of crossed modules of Lie objects in $\mathcal{LM}_K$ satisfying:

- $(\alpha_2, \beta_2) : (N, H, \cdot_2, \partial_2, \{-,-\}_2) \to (Y, W, \star_2, \delta_2, \{-,-\}'_2)$ is an morphism of braided crossed modules of Lie K-algebras,

- $\alpha_1(\{l,h\}_{LH}) = \{\beta_1(l), \beta_2(h)\}'_{VW}$, for $l \in L$, $h \in H$,
- $\alpha_1(\{h,l\}_{HL}) = \{\beta_2(h), \beta_1(l)\}'_{WV}$, for $l \in L$, $h \in H$.

We want to use the concept of braiding on crossed modules of Lie objects in $\mathcal{LM}_K$ to obtain a definition for crossed modules of Leibniz K-algebras. For that, we will take a braiding on $\left(\begin{array}{c} M \\ \downarrow{\scriptstyle \pi_M} \\ \frac{M}{[M,N]_x} \end{array} \xrightarrow{\bar{\partial}} \begin{array}{c} N \\ \downarrow{\scriptstyle \pi_N} \\ \mathrm{Lie}(N) \end{array}, \bar{\bar{\cdot}} \right)$. If we try to take one K-bilinear map $\{-,-\}$ we would find problems with the way of defining the corresponding maps because we have that the first properties add one more quotient that we would like to be trivial for Lie K-algebras, or if we take it to be trivial, the rest of properties prevent it from being made for the general case of Leibniz K-algebras (if we take $\{n, \overline{n'}\}_{N\,\mathrm{Lie}(N)} = \{n, n'\} = \{\overline{n}, n'\}_{\mathrm{Lie}(N)N}$ for example, the third and fourth property leads us to prove that M must be Lie K-algebra).

For this, as in the case of the two actions, we will take for braiding two K-bilinear maps $\{-,-\}, \langle -,- \rangle \colon N \times N \to M$, and define $\{n, \overline{n'}\}_{N\,\mathrm{Lie}(N)} = \{n, n'\}$, $\{\overline{n}, n'\}_{\mathrm{Lie}(N)N} = -\langle n', n \rangle$ and $\{\overline{n}, \overline{n'}\}_2 = \overline{\{n, n'\}} = \overline{-\langle n', n \rangle}$, where we can see that we introduce a new quotient in M.

Definition 2.4.29. Let $\mathcal{X} = (M \xrightarrow{\partial} N, (\cdot_1, \cdot_2))$ be a crossed module of Leibniz K-algebras.

A braiding (or Peiffer lifting) on $\mathcal{X}$ is a pair $(\{-,-\}, \langle -,- \rangle)$ of K-bilinear maps $\{-,-\}, \langle -,- \rangle \colon N \times N \to M$, $(n, n') \mapsto \{n, n'\}$ and $(n, n') \mapsto \langle n, n' \rangle$, satisfying:

$$\partial\{n, n'\} = [n, n'] = \partial\langle n, n' \rangle, \tag{BXLeib1}$$

$$\{\partial m, \partial m'\} = [m, m'] = \langle \partial m, \partial m' \rangle, \tag{BXLeib2}$$

$$\{\partial m, n\} = m \cdot_2 n = \langle \partial m, n \rangle, \tag{BXLeib3}$$

$$\{n, \partial m\} = n \cdot_1 m = \langle n, \partial m \rangle, \tag{BXLeib4}$$

$$\{n, [n', n'']\} = \{[n, n'], n''\} - \{[n, n''], n'\}, \tag{BXLeib5}$$

$$\langle n, [n', n''] \rangle = \{[n, n'], n''\} - \langle [n, n''], n' \rangle, \tag{BXLeib6}$$

$$\{n, [n', n'']\} = \{[n, n'], n''\} - \langle [n, n''], n' \rangle, \tag{BXLeib7}$$

$$\langle n, [n', n''] \rangle = \langle [n, n'], n'' \rangle - \langle [n, n''], n' \rangle, \tag{BXLeib8}$$

for $n, n', n'' \in N,\ m, m' \in M$.

In this case, we say that $(M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,- \rangle))$ is a *braided crossed module of Leibniz K-algebras*.

Definition 2.4.30. An *homomorphism* (f_1, f_2) of braided crossed modules of Leibniz K-algebras

$$(M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,- \rangle)) \xrightarrow{(f_1, f_2)} (M' \xrightarrow{\partial'} N', (*_1, *_2), (\{-,-\}', \langle -,- \rangle'))$$

is an morphism between the corresponding crossed modules of Leibniz K-algebras satisfying:

$$f_1(\{n, n'\}) = \{f_2(n), f_2(n')\}', \tag{BXLeibH1}$$

$$f_1(\langle n, n' \rangle) = \langle f_2(n), f_2(n') \rangle', \qquad n, n' \in N. \tag{BXLeibH2}$$

We denote the category of braided crossed modules of Leibniz K-algebras and its homomorphisms by $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)$.

We want to know how to introduce the braided crossed modules of Lie K-algebras as a particular case. The next two properties answer this question:

Proposition 2.4.31. *Let $(M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,- \rangle))$ be a braided crossed module of Leibniz K-algebras.*

If for all $n, n' \in N$ it is satisfied that $\{n, n'\} = -\langle n', n \rangle$, then we have the following properties:

- $m \cdot_2 n = -n \cdot_1 m$.
- $(M \xrightarrow{\partial} N, \cdot_1, \{-,-\})$ *is a braided crossed module of Lie K-algebras.*

Proof. We will check first that M and N are Lie K-algebras.

By using (BXLeib1), we have that for all $n, n' \in N$, $\partial \langle n, n' \rangle = [n, n']$. Then, if we use that $\langle n, n' \rangle = -\{n', n\}$ we obtain, again for (BXLeib1):

$$[n, n] = \partial \langle n, n' \rangle = -\partial \{n', n\} = -[n, n'].$$

We conclude that N is a Lie K-algebra (we are working in a field of $\mathrm{char}(K) \neq 2$).

Now we take $m, m' \in M$. By (BXLeib2) we have that $\langle \partial m, \partial m' \rangle = [m, m']$. Using again $\langle \partial m, \partial m' \rangle = -\{\partial m', \partial m\}$ and (BXLeib2) we have

$$[m, m'] = \langle \partial m, \partial m' \rangle = -\{\partial m', \partial m\} = -[m', m].$$

We will check that $m \cdot_2 n = -n \cdot_1 m$, for $m \in M$, $n \in N$. We have

$$m \cdot_2 n = \langle \partial m, n \rangle = -\{n, \partial m\} = -n \cdot_1 m,$$

where we used (BXLeib3) in the first equality and (BXLeib4) in the third.

Now, we know that $(M \xrightarrow{\partial} N, \cdot_1)$ is a crossed module of Lie K-algebras using Proposition 1.3.29.

We will prove the equivalences for the axioms of braiding.

The first equality of properties (BXLeib1)–(BXLeib4) coincides, respectively, with (BXLie1)–(BXLie4) (in the case of (BXLeib3) remember that $m \cdot_2 n = -n \cdot_1 m$).

The second identity of (BXLeib1) and (BXLeib2) is immediate because of the anticommutativity of the bracket, while the second (BXLeib3) is equivalent to (BXLie4) and the second equality of (BXLeib4) is to (BXLie3) (again using that $n \cdot_1 m = -m \cdot_2 n$).

It is clear that (BXLeib5) and (BXLie5) are identical, and it is straightforward to prove that (BXLeib8) is equivalent to (BXLie6).

To see the last equivalences, we must prove an earlier property, which is satisfied for both braidings under our assumptions:

If $n, n', n'' \in N$, then $\{[n, n'], n''\} = -\{n'', [n, n']\}$.

We will start in the Lie case (we suppose we have an action $\cdot$).

$$\{[n, n'], n''\} = \{\partial\{n, n'\}, n''\} = -n'' \cdot \{n, n'\} = -\{n'', \partial\{n, n'\}\} = -\{n'', [n, n']\},$$

where we use (BXLie1), (BXLie3) and (BXLie4).

For the Leibniz case it is not true in general. We need to use $m \cdot_2 n = -n \cdot_1 m$.

$$\begin{aligned} \{[n, n'], n''\} &= \{\partial\{n, n'\}, n''\} = \{n, n'\} \cdot_2 n'' \\ &= -n'' \cdot_1 \{n, n'\} = -\{n'', \partial\{n, n'\}\} = -\{n'', [n, n']\}, \end{aligned}$$

where we use (BXLeib1), (BXLeib3) and (BXLeib4).

With this new property we prove that (BXLeib6) is equivalent to (BXLie6), and (BXLeib7) is equivalent to (BXLie5). In particular $(M, N, \cdot_1, \partial\{-,-\})$ is a braided crossed module of Lie K-algebras. $\square$

The next two propositions are immediate, and the second one gives the construction of the functor.

Proposition 2.4.32. *Let M and N be Lie K-algebras. Then, $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is a crossed module of Lie K-algebras if and only if $(M \xrightarrow{\partial} N, (\cdot, \cdot^{-}), (\{-,-\}, \{-,-\}^{-}))$ is a crossed module of Leibniz K-algebras.*

$\cdot^{-} : M \times N \to N$ and $\{-,-\}^{-} : N \times N \to M$ are defined as $m \cdot^{-} n = -n \cdot m$ and $\{n, n'\}^{-} = -\{n', n\}$.

Proposition 2.4.33. *Let $(M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,-\rangle))$ be a braided crossed module of Leibniz K-algebras.*

Then $\left(\begin{array}{c} M \\ \downarrow{\scriptstyle \pi_M} \\ \frac{M}{\{M,N\}_x} \end{array} \xrightarrow{\overline{\partial}} \begin{array}{c} N \\ \downarrow{\scriptstyle \pi_N} \\ \mathrm{Lie}(N) \end{array}, \bar{\bar{\cdot}}, \overline{\partial}, (\{-,-\}_{N\,\mathrm{Lie}(N)}, \{-,-\}_{\mathrm{Lie}(N)N}, \{-,-\}_2) \right)$ gives us a braided crossed module of Lie objects in $\mathcal{LM}_K$, where

- *$\frac{M}{\{M,N\}_x}$ is the Lie K-algebra quotient of M by the ideal $\{M, N\}_x$ whose generators are $[x, x]$ for $x \in M$, $n \cdot_1 m + m \cdot_2 n$ for $n \in N$, $m \in M$, and $\{n, n'\} + \langle n', n\rangle$ for $n, n' \in N$; we denote the natural map by $\pi_M : M \to \frac{M}{\{M,N\}_x}$, and the elements of $\frac{M}{\{M,N\}_x}$ by $\overline{m}$,*
- *$\bar{\cdot}_1 : \mathrm{Lie}(N) \times M \to M$, $(\overline{n}, m) \mapsto -m \cdot_2 n$,*
- *$\bar{\cdot}_2 : \mathrm{Lie}(N) \times \frac{M}{\{M,N\}_x} \to \frac{M}{\{M,N\}_x}$, $(\overline{n}, \overline{m}) \mapsto \overline{n \cdot_1 m} = \overline{-m \cdot_2 n}$,*
- *$\xi_{\bar{\cdot}} : N \times \frac{M}{\{M,N\}_x} \to M$, $(n, \overline{m}) \to n \cdot_1 m$,*
- *$\overline{\partial}_1 : M \to N$, $m \mapsto \partial(m)$,*
- *$\overline{\partial}_2 : \frac{M}{\{M,N\}_x} \to \mathrm{Lie}(N)$, $\overline{m} \mapsto \overline{\partial m}$,*

- $\{-,-\}_{N\,\mathrm{Lie}(N)}\colon N\times \mathrm{Lie}(N)\to M$, $(n,\overline{n'})\mapsto \{n,n'\}$,

- $\{-,-\}_{\mathrm{Lie}(N)N}\colon \mathrm{Lie}(N)\times N\to M$, $(\overline{n},n')\mapsto -\langle n',n\rangle$,

- $\{-,-\}_2\colon \mathrm{Lie}(n)\times \mathrm{Lie}(N)\to M$, $(\overline{n},\overline{n'})\mapsto \overline{\{n,n'\}}=\overline{-\langle n',n\rangle}$.

Remark 2.4.34. As in the previous cases, $(\frac{M}{\{M,N\}_x}\xrightarrow{\overline{\partial}_2}\mathrm{Lie}(N),\bar{\cdot}_2,\{-,-\}_2)$ will be called Liesation, and it is functorial.

Applying this Liesation on a crossed module of Lie K-algebras, thought as a crossed module of Leibniz K-algebras (Proposition 2.4.32), the third generators are null too:

$$\{n,n'\}+\langle n',n\rangle=\{n,n'\}+\{n',n\}^-=\{n,n'\}-\{n,n'\}=0.$$

In the Lie case, we obtain a natural isomorphism to itself after doing the Liesation.

Proposition 2.4.35. *Let* $\left(\begin{smallmatrix} M & \xrightarrow{\partial} & L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N & & H\end{smallmatrix},\bar{\cdot},T_{\{-,-\}}\right)$ *be a braided crossed module of Lie objects in* $\mathcal{LM}_K$.

Then $(M\xrightarrow{\partial_1}L,(\tilde{\cdot}_1,\tilde{\cdot}_2),(\{-,-\}_{T_{\{-,-\}}},\langle -,-\rangle_{T_{\{-,-\}}}))$ *is a braided crossed module of Leibniz* K*-algebras, where*

- $[m,m']=m*^M_N f(m')$*, for* $m,m'\in M$ *and* $[l,l']=l*^L_H g(l')$*, for* $l,l'\in M$ *are the Leibniz brackets;*

- $\tilde{\cdot}_1\colon L\times N\to M$ *is defined by* $l\,\tilde{\cdot}_1 m=\xi_{\cdot}(l,f(m))$ *for* $l\in L$, $m\in M$*;*

- $\tilde{\cdot}_2\colon M\times L\to M$ *is defined as* $m\tilde{\cdot}_2 l=-g(l)\cdot_1 m$ *for* $l\in L$, $m\in M$*;*

- $\{-,-\}_{T_{\{-,-\}}}\colon L\times L\to M$ *is defined as* $\{l,l'\}_{T_{\{-,-\}}}=\{l,g(l')\}_{LH}$ *for* $l,l'\in L$*;*

- $\langle -,-\rangle_{T_{\{-,-\}}}\colon L\times L\to M$ *is defined as* $\langle l,l'\rangle_{T_{\{-,-\}}}=-\{g(l'),l\}_{HL}$ *for* $l,l'\in L$.

Thus, we have the functors $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)\underset{BX\Psi}{\overset{BX\Phi}{\rightleftarrows}}\boldsymbol{BXLie}(\mathcal{LM}_K)$ satisfying the identity $BX\Psi\circ BX\Phi=\mathrm{Id}_{\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)}$, and so, the functor $BX\Phi$ is a full inclusion functor.

2.4.2 Braiding for categorical Lie objects in $\mathcal{LM}_K$ and categorical Leibniz algebras

We also want to define a braiding for categorical Leibniz K-algebras. As in the crossed module case, we will use the idea of the category $\mathcal{LM}_K$. We only need to show that $\mathcal{LM}_K$ is a category with pullbacks.

Remark 2.4.36. $\mathcal{LM}_K$ is a category with pullbacks.

If we have the morphisms $\begin{array}{c} A \\ \downarrow{\scriptstyle f} \\ B \end{array} \xrightarrow{\alpha} \begin{array}{c} X \\ \downarrow{\scriptstyle h} \\ Y \end{array} \xleftarrow{\beta} \begin{array}{c} C \\ \downarrow{\scriptstyle g} \\ D \end{array}$, then $\left(\begin{array}{c} A \times_X C \\ \downarrow{\scriptstyle f \times_h g} \\ B \times_Y D \end{array}, (\pi_A, \pi_B), (\pi_C, \pi_D)\right)$ is their pullback.

It is easy to check that ***Lie***$(\mathcal{LM}_K)$ has the same pullback with the operations $[(b,d),(b',d')]_{B\times_Y D} := ([b,b']_B, [d,d']_D)$ and $(a,c) *_{B\times_Y D}^{A\times_X C} (b,d) := (a *_B^A b, c *_D^C d)$. So, we can speak about categorical Lie objects in $\mathcal{LM}_K$.

As in the crossed module case, we have the following result.

Proposition 2.4.37. *Let (C_1, C_0, s, t, e, k) be a categorical Leibniz K-algebra.*

Then $\left(\begin{array}{c} C_1 \\ \downarrow{\scriptstyle \pi_{C_1}} \\ \mathrm{Lie}(C_1) \end{array}, \begin{array}{c} C_0 \\ \downarrow{\scriptstyle \pi_{C_0}} \\ \mathrm{Lie}(C_0) \end{array}, (s, \mathrm{Lie}(s)), (t, \mathrm{Lie}(t)), (e, \mathrm{Lie}(e)), (k, \overline{k})\right)$ *is a categorical Lie object in $\mathcal{LM}_K$, where* $\mathrm{Lie} :$ ***LeibAlg***$_K \to$ ***LieAlg***$_K$ *is the Liesation functor and the morphism* $\overline{k} : \mathrm{Lie}(C_1) \times_{\mathrm{Lie}(C_0)} \mathrm{Lie}(C_1) \to \mathrm{Lie}(C_1)$ *is defined as* $\overline{k}(\overline{x}, \overline{y}) = \overline{\mathring{k}(x,y)}$*, where* $\mathring{k} : C_1 \times C_1 \to C_1$ *is the extension of k to the product defined as* $\mathring{k}(x,y) = x + y - e(s(y))$.

Remark 2.4.38. $\mathring{k}$ is an extension of k, since the same formula is satisfied for composition, as can be seen in Lemma 2.1.1.

One can ask why not to extend k as $\mathring{k}'(x,y) = x + y - e(t(x))$, which is not identical to $\mathring{k}$ in the general case. But, in this case we have that the result is the same

$$\overline{\mathring{k}(x,y)} = \overline{x} + \overline{y} - \mathrm{Lie}(e)(\mathrm{Lie}(s)(\overline{y})) = \overline{x} + \overline{y} - \mathrm{Lie}(e)(\mathrm{Lie}(t)(\overline{x})) = \overline{\mathring{k}'(x,y)},$$

since $(\overline{x}, \overline{y}) \in \mathrm{Lie}(C_1) \times_{\mathrm{Lie}(C_0)} \mathrm{Lie}(C_1)$ implies $\mathrm{Lie}(s)(\overline{y}) = \mathrm{Lie}(t)(\overline{x})$.

Remark 2.4.39. We again have in the bottom part the Liesation, and in the case of Lie K-algebras thought as Leibniz K-algebras, we obtain the identity.

Proposition 2.4.40. *If* $\left(\begin{array}{c} C_1 \\ \downarrow f_1 \\ D_1 \end{array}, \begin{array}{c} C_0 \\ \downarrow f_0 \\ D_0 \end{array}, s, t, e, k\right)$ *is a categorical Lie object in* $\mathcal{LM}_K$*, then* $(C_1, C_0, s_1, t_1, e_1, k_1)$ *is a categorical Leibniz* K*-algebra, where* $[x, y]_{C_1} = x *_{D_1}^{C_1} y$ *and* $[a, b]_{C_0} = a *_{D_0}^{C_0} b$*, for* $x, y \in C_1, a, b \in C_0$.

Analogously to the crossed modules case, we have once again for internal categories the pair of functors $\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K) \underset{I\Psi}{\overset{I\Phi}{\rightleftarrows}} \boldsymbol{ICat}(\boldsymbol{Lie}(\mathcal{LM}_K))$ satisfying the identity $I\Psi \circ I\Phi = \mathrm{Id}_{\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K)}$, and so, the functor $I\Phi$ is a full inclusion functor.

This new inclusion functor allows us to define a braiding on categorical Leibniz K-algebras using the idea of braiding of Lie objects in $\mathcal{LM}_K$.

Proposition 2.4.41. *Let* $C = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ *be a braided semigroupal category where* $\boldsymbol{C}$ *is an additive category with pullbacks. Then* $\boldsymbol{Lie}(C)$ *has pullbacks.*

Proof. If we have two Lie morphisms

$$\begin{tikzcd} & (A, \mu_A) \arrow[d, "f"] \\ (B, \mu_B) \arrow[r, "g"] & (C, \mu_C), \end{tikzcd}$$

the pullback is given by $((A \times_C B, \mu_{A \times_C B}), \pi_A, \pi_B)$, where $A \times_C B$ is the pullback in the category $\boldsymbol{C}$

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C, \end{tikzcd}$$

and $\mu_{A \times_C B}$ is the unique morphism such that $\pi_X \circ \mu_{A \times_C B} = \mu_X \circ (\pi_X \otimes \pi_X)$ taking $X \in \{A, B\}$ constructed by the universal property of pullbacks in $\mathbf{C}$ in the following diagram:

$$\begin{tikzcd}
(A \times_C B) \otimes (A \times_C B) \arrow[rr, "\pi_A \otimes \pi_A"] \arrow[dr, dashed, "\mu_{A \times_C B}"] \arrow[dd, "\pi_B \otimes \pi_B"'] & & A \otimes A \arrow[d, "\mu_A"] \\
& A \times_C B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] & A \arrow[d, "f"] \\
B \otimes B \arrow[r, "\mu_B"] & B \arrow[r, "g"] & C.
\end{tikzcd}$$

It is straightforward to see that $\mu_{A \times_C B}$ is well defined. Now, we will prove that $(A \times_C B, \mu_{A \times_C B})$ is a Lie object checking the first axiom. For simplicity of notation, we will denote $D := A \times_C B$. Let $X \in \{A, B\}$. Using universal properties we have:

$$\pi_X \circ (-\mu_D \circ \mathcal{T}_{D,D}) = -\pi_X \circ \mu_D \circ \mathcal{T}_{D,D} = -\mu_X \circ (\pi_X \otimes \pi_X) \circ \mathcal{T}_{D,D}.$$

Since $\mathcal{T}$ is a natural isomorphism and that (X, μ_X) is a Lie object, we get

$$\pi_X \circ (-\mu_D \circ \mathcal{T}_{D,D}) = -\mu_X \circ \mathcal{T}_{X,X} \circ (\pi_X \otimes \pi_X) = \mu_X \circ (\pi_X \otimes \pi_X).$$

We conclude that $\mu_D = -\mu_D \circ \mathcal{T}_{D,D}$ because μ_D is the unique morphism that satisfies the previous equality for $X \in \{A, B\}$.

Now, we will check the second axiom of Lie object.

For the first summand, we have:

$$\begin{aligned}
&\pi_X \circ \mu_D \circ (\mathrm{Id}_D \otimes \mu_D) \circ a_{D,D,D} = \mu_X \circ (\pi_X \otimes \pi_X) \circ (\mathrm{Id}_D \otimes \mu_D) \circ a_{D,D,D} \\
&= \mu_X \circ (\pi_X \otimes (\pi_X \circ \mu_D)) \circ a_{D,D,D} = \mu_X \circ (\pi_X \otimes (\mu_X \circ (\pi_X \otimes \pi_X))) \circ a_{D,D,D} \\
&= \mu_X \circ (\mathrm{Id}_X \otimes \mu_X) \circ (\pi_X \otimes (\pi_X \otimes \pi_X)) \circ a_{D,D,D}.
\end{aligned}$$

Using that a is a natural isomorphism, we have

$$\pi_X \circ \mu_D \circ (\mathrm{Id}_D \otimes \mu_D) \circ a_{D,D,D} = \mu_X \circ (\mathrm{Id}_X \otimes \mu_X) \circ a_{X,X,X} \circ ((\pi_X \otimes \pi_X) \otimes \pi_X).$$

Doing the same for the second and third summands (the naturalness of a gives the same naturalness to a^{-1}), we have that:

$$\begin{aligned}
&\pi_X \circ \mu_D \circ (\mu_D \otimes \mathrm{Id}_D) \circ a^{-1}_{D,D,D} \circ (\mathrm{Id}_D \otimes \mathcal{T}_{D,D}) \circ a_{D,D,D} \\
&= \mu_X \circ (\mu_X \otimes \mathrm{Id}_X) \circ a^{-1}_{X,X,X} \circ (\mathrm{Id}_X \otimes \mathcal{T}_{X,X}) \circ a_{X,X,X} \circ ((\pi_X \otimes \pi_X) \otimes \pi_X),
\end{aligned}$$

$$\pi_X\circ(-\mu_D\circ(\mu_D \otimes \mathrm{Id}_D)) = -\mu_X\circ(\mu_X \otimes \mathrm{Id}_X)\circ((\pi_X \otimes \pi_X) \otimes \pi_X).$$

Adding the three last equalities and using the distributivity of the composition, we have $\pi_X\circ\mathcal{L}_D = \mathcal{L}_X\circ((\pi_X \otimes \pi_X) \otimes \pi_X)$, where denote by $\mathcal{L}_Y$ the morphism that is in the first term of the equality of the second axiom for a Lie object (Y, μ_Y). Since (X, μ_X) is a Lie object, we get $\mathcal{L}_X = {}_{(X\otimes X)\otimes X}0_X$, and so $\pi_X\circ\mathcal{L}_D = {}_{(D\otimes D)\otimes D}0_X$.

Now, by the universal property, we have that $\mathcal{L}_D = {}_{(D\otimes D)\otimes D}0_D$ and therefore (D, μ_D) is a Lie object.

To conclude the proof it is enough to check that the morphism given by the pullback in $\boldsymbol{C}$ is a Lie morphism, but this is a routine verification. □

Definition 2.4.42. Let $\mathcal{C} = (\boldsymbol{C}, \otimes, a, \mathcal{T})$ be a braided semigroupal category where $\boldsymbol{C}$ is an additive category with pullbacks.

Let $\mathfrak{C} = ((C_1, \mu_{C_1}), (C_0, \mu_{C_0}), s, t, e, k)$ be a categorical Lie object in $\boldsymbol{Lie}(\mathcal{C})$.

A braiding on $\mathfrak{C}$ is a morphism $\tau\colon\ C_0 \otimes C_0 \to C_1$ satisfying:

- $s\circ\tau = \mu_{C_0}$ and $t\circ\tau = \mu_{C_0}\circ\mathcal{T}_{C_0,C_0}$,
- We define $C_0\otimes C_0 \xrightarrow{\mu_{C_1}\times_{C_0}(\tau\circ(t\otimes t)),(\tau\circ(s\otimes s))\times_{C_0}(\mu_{C_1}\circ\mathcal{T})} C_1\times_{C_0} C_1$ as the two unique morphisms which satisfy the universal property, respectively, in the following diagrams:

$$\begin{tikzcd}
C_1 \otimes C_1 \arrow[dr, dashed] \arrow[drr, bend left, "\mu_{C_1}"] \arrow[ddr, bend right, "\tau\circ(t\otimes t)"'] & & \\
& C_1 \times_{C_0} C_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & C_1 \arrow[d, "t"] \\
& C_1 \arrow[r, "s"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \otimes C_1 \arrow[dr, dashed] \arrow[drr, bend left, "\tau\circ(s\otimes s)"] \arrow[ddr, bend right, "\mu_{C_1}\circ\mathcal{T}_{C_1,C_1}"'] & & \\
& C_1 \times_{C_0} C_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & C_1 \arrow[d, "t"] \\
& C_1 \arrow[r, "s"] & C_0
\end{tikzcd},$$

and the equality

$$k\circ(\mu_{C_1} \times_{C_0} (\tau\circ(t \otimes t))) = k\circ((\tau\circ(s \otimes s)) \times_{C_0} (\mu_{C_1}\circ\mathcal{T})).$$

- It must satisfy

$$\tau\circ(\mathrm{Id}_{C_0}\otimes\mu_{C_0})\otimes a_{C_0} = \tau\circ(\mu_{C_0}\otimes \mathrm{Id}_{C_0})\circ(\mathrm{Id}_{(C_0\otimes C_0)\otimes C_0} - (a_{C_0}^{-1}\circ(\mathrm{Id}_{C_0}\otimes \mathcal{T}_{C_0})\circ a_{C_0})),$$

$$\tau\circ(\mu_{C_0}\otimes \mathrm{Id}_{C_0}) = \tau\circ(\mathrm{Id}_{C_0}\otimes\mu_{C_0})\circ a_{C_0}\circ(\mathrm{Id}_{(C_0\otimes C_0)\otimes C_0} - (\mathcal{T}_{C_0}\otimes \mathrm{Id}_{C_0})).$$

  We denote $a_{C_0,C_0,C_0} =: a_{C_0}$ and $\mathcal{T}_{C_0,C_0} =: \mathcal{T}_{C_0}$.

We will say that $((C_1,\mu_{C_1}),(C_0,\mu_{C_0}),s,t,e,k,\tau)$ is a braided categorical Lie object in $\mathcal{C}$.

An internal functor

$$((C_1,\mu_{C_1}),(C_0,\mu_{C_0}),s,t,e,k,\tau)\xrightarrow{(F_1,F_0)}((C_1',\mu_{C_1'}),(C_0',\mu_{C_0'}),s',t',e',k',\tau')$$

is said to be a braided internal functor of braided categorical Lie objects in $\mathcal{C}$ if it satisfies the following diagram:

$$\begin{array}{ccc} C_0\otimes C_0 & \xrightarrow{\tau} & C_1 \\ \downarrow{\scriptstyle F_0\otimes F_0} & & \downarrow{\scriptstyle F_1} \\ C_0'\otimes C_0' & \xrightarrow{\tau'} & C_1'. \end{array}$$

We denote this new category as $\boldsymbol{BICat}(\boldsymbol{Lie}(\mathcal{C}))$.

Example 2.4.43. We have that the categories $\boldsymbol{BICat}(\boldsymbol{Lie}(\boldsymbol{Vect}_K))$ and $\boldsymbol{BICat}(\boldsymbol{LieAlg}_K)$ are isomorphic, taking in $\boldsymbol{Vect}_K$ the usual tensor product (we assume $\mathrm{char}(K)\neq 2$).

Definition 2.4.44. Let $\mathcal{C} = \left(\begin{array}{c} C_1 \\ \downarrow{\scriptstyle f_1} \\ D_1 \end{array}, \begin{array}{c} C_0 \\ \downarrow{\scriptstyle f_0} \\ D_0 \end{array}, s,t,e,k\right)$ be a categorical Lie object in $\mathcal{LM}_K$.

A braiding on $\mathcal{C}$ is a triple $\tau = (\tau^{C_0,D_0},\tau^{D_0,C_0},\tau^2)$ where

- $\tau^2 : D_0\times D_0 \to D_1$ is a K-bilinear map such that (D_1,D_0,s,t,e,k,τ_2) is a braided crossed module of Lie K-algebras,
- $\tau^{D_0,C_0} : D_0\times C_0 \to C_1$ and $\tau^{C_0,D_0} : C_0\times D_0 \to C_1$ are K-bilinear maps which, with τ^2, satisfy the following properties for $c\in C_0$, $d,d'\in D_0$, $x\in C_1$, $y\in D_1$:

$$f_1(\tau^{C_0,D_0}_{c,d}) = \tau^2_{f_0(c),d} \qquad \text{and} \qquad f_1(\tau^{D_0,C_0}_{d,c}) = \tau^2_{d,f_0(c)},$$

$$\tau^{C_0,D_0}_{c,d}:\ c *^{C_0}_{D_0} d \to -c *^{C_0}_{D_0} d \qquad \text{and} \qquad \tau^{D_0,C_0}_{d,c}:\ -c *^{C_0}_{D_0} d \to c *^{C_0}_{D_0} d.$$

The following diagrams are satisfied in the internal category:

$$\begin{tikzcd}
s_1(x) *^{C_0}_{D_0} s_2(y) \arrow[r, "x *^{C_1}_{C_0} y"] \arrow[d, "\tau^{C_0,D_0}_{s_1(x),s_2(y)}"] & t_1(x) *^{C_0}_{D_0} t_2(y) \arrow[d, "\tau^{C_0,D_0}_{t_1(x),t_2(y)}"] \\
-s_1(x) *^{C_0}_{D_0} s_2(y) \arrow[r, "-x *^{C_1}_{D_1} y"] & -t_1(x) *^{C_0}_{D_0} t_2(y)
\end{tikzcd},$$

$$\begin{tikzcd}
-s_1(x) *^{C_0}_{D_0} s_2(y) \arrow[r, "-x *^{C_1}_{C_0} y"] \arrow[d, "\tau^{D_0,C_0}_{s_2(y),s_1(x)}"] & -t_1(x) *^{C_0}_{D_0} t_2(y) \arrow[d, "\tau^{D_0,C_0}_{t_2(y),t_1(x)}"] \\
s_1(x) *^{C_0}_{D_0} s_2(y) \arrow[r, "x *^{C_1}_{D_1} y"] & t_1(x) *^{C_0}_{D_0} t_2(y)
\end{tikzcd}.$$

Moreover, we have the following properties:

$$\begin{aligned}
\tau^{C_0,D_0}_{c,[d,d']_{D_0}} &= \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d,d'} - \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d',d},\\
\tau^{D_0,C_0}_{[d,d']_{D_0},c} &= -\tau^{D_0,C_0}_{d,c *^{C_0}_{D_0} d'} - \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d,d'},\\
\tau^{C_0,D_0}_{c,[d,d']_{D_0}} &= \tau^{C_0,D_0}_{c *^{C_0}_{D_0} d,d'} + \tau^{D_0,C_0}_{d,c *^{C_0}_{D_0} d'},\\
\tau^{D_0,C_0}_{[d,d']_{D_0},c} &= -\tau^{D_0,C_0}_{d,c *^{C_0}_{D_0} d'} + \tau^{D_0,C_0}_{d',c *^{C_0}_{D_0} d}.
\end{aligned}$$

We will say that $\left(\begin{smallmatrix} C_1 \\ \downarrow f_1 \\ D_1 \end{smallmatrix}, \begin{smallmatrix} C_0 \\ \downarrow f_0 \\ D_0 \end{smallmatrix}, s, t, e, k, \tau\right)$ is a braided categorical Lie object in $\mathcal{LM}_K$.

Remark 2.4.45. A braiding is a pair $\tau = (\tau_1, \tau_2)$ but, for simplicity, we take for the definition $\tau_2(d, d') = \tau^2_{d,d'}$ and $\tau_1 : (C_0 \otimes D_0) \oplus (D_0 \otimes C_0) \to C_1$ by the expression $\tau_1((c \otimes d) + (d' \otimes c')) = \tau^{C_0,D_0}_{c,d} + \tau^{D_0,C_0}_{d',c'}$.

Definition 2.4.46. Let $\left(\begin{smallmatrix} C_1 \\ \downarrow f_1 \\ D_1 \end{smallmatrix}, \begin{smallmatrix} C_0 \\ \downarrow f_0 \\ D_0 \end{smallmatrix}, s, t, e, k, \tau\right)$ and $\left(\begin{smallmatrix} C'_1 \\ \downarrow g_1 \\ D'_1 \end{smallmatrix}, \begin{smallmatrix} C'_0 \\ \downarrow g_0 \\ D'_0 \end{smallmatrix}, s', t', e', k', \psi\right)$ be braided categorical Lie objects in $\mathcal{LM}_K$. A braided internal functor between categorical Lie objects in $\mathcal{LM}_K$ is an internal functor $((F_1^1, F_1^0), (F_0^1, F_0^0))$ between the respective categorical Lie objects which satisfies:

- $(F_1^0, F_0^0)\colon (D_1, D_0, s_2, t_2, e_2, k_2, \tau_2) \to (D_1', D_0', s_2', t_2', e_2', k_2', \psi_2)$ is a braided internal functor between categorical Lie K-algebras.
- $F_1^1(\tau_{c,d}^{C_0,D_0}) = \psi_{F_0^1(c),F_0^0(d)}^{C_0',D_0'}$ for $c \in C_0$, $d \in D_0$.
- $F_1^1(\tau_{d,c}^{D_0,C_0}) = \psi_{F_0^0(d),F_0^1(c)}^{D_0',C_0'}$ for $c \in C_0$, $d \in D_0$.

To introduce a braiding for the categorical Leibniz K-algebras with the previous scheme, we will use two K-bilinear maps $\tau, \psi\colon C_0 \times C_0 \to C_1$, as in the case of a braiding of crossed modules of Leibniz K-algebras. Consider for the inclusion Lie object in $\mathcal{LM}_K$ the braiding $\bar{\tau}$ defined by $\bar{\tau}_{a,\bar{b}}^{C_0,\mathrm{Lie}(C_0)} = \tau_{a,b}$, $\bar{\tau}_{\bar{a},b}^{\mathrm{Lie}(C_0),C_0} = -\psi_{b,a}$ and $\bar{\tau}_{\bar{a},\bar{b}}^{2} = \overline{\tau_{a,b}} = \overline{-\psi_{b,a}}$, where we introduce a quotient in C_1 whose elements we will denote as $\overline{x}$.

Definition 2.4.47. A braiding for the categorical Leibniz K-algebra (C_1, C_0, s, t, e, k) is a pair (τ, ψ) of K-bilinear maps $\tau, \psi\colon C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$ and $(a, b) \mapsto \psi_{a,b}$, satisfying:

$$\tau_{a,b}\colon [a, b] \to -[a, b] \quad \text{and} \quad \psi_{a,b}\colon [a, b] \to -[a, b], \tag{LeibT1}$$

$$\begin{array}{ccc} [s(x), s(y)] & \xrightarrow{[x,y]} & [t(x), t(y)] \\ \downarrow{\scriptstyle \tau_{s(x),s(y)}} & & \downarrow{\scriptstyle \tau_{t(x),t(y)}} \\ -[s(x), s(y)] & \xrightarrow{-[x,y]} & -[t(x), t(y)], \end{array} \qquad \begin{array}{ccc} [s(x), s(y)] & \xrightarrow{[x,y]} & [t(x), t(y)] \\ \downarrow{\scriptstyle \psi_{s(x),s(y)}} & & \downarrow{\scriptstyle \psi_{t(x),t(y)}} \\ -[s(x), s(y)] & \xrightarrow{-[x,y]} & -[t(x), t(y)], \end{array} \tag{LeibT2}$$

$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \tau_{[a,c],b}, \tag{LeibT3}$$

$$\psi_{a,[b,c]} = \tau_{[a,b],c} - \psi_{[a,c],b}, \tag{LeibT4}$$

$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \psi_{[a,c],b}, \tag{LeibT5}$$

$$\psi_{a,[b,c]} = \psi_{[a,b],c} - \psi_{[a,c],b}, \qquad a, b, c \in C_0,\ x, y \in C_1. \tag{LeibT6}$$

We will say that $(C_1, C_0, s, t, e, k, (\tau, \psi))$ is a *braided categorical Leibniz K-algebra*.

Definition 2.4.48. Let $(C_1, C_0, s, t, e, k, (\tau, \psi))$ and $(C_1', C_0', s', t', e', k', (\tau', \psi'))$ be two braided categorical Leibniz K-algebras.

An internal functor $(C_1, C_0, s, t, e, k) \xrightarrow{(F_1,F_0)} (C_1', C_0', s', t', e', k')$ is said to be a braided internal functor between two braided categorical Leibniz K-algebras if it satisfies:

$$F_1(\tau_{a,b}) = \tau'_{F_0(a),F_0(b)}, \tag{LeibHT1}$$

$$F_1(\psi_{a,b}) = \psi'_{F_0(a),F_0(b)}, \qquad a, b \in C_0. \tag{LeibHT2}$$

We denote the category of braided categorical Leibniz K-algebras and braided internal functors between them as ***BICat*(*LeibAlg*$_K$)**.

We want to see the braided categorical Lie K-algebras as a particular case of braided categorical Leibniz K-algebras.

Proposition 2.4.49. *Let C_1 and C_0 be Lie K-algebras. Then, $(C_1, C_0, s, t, e, k, \tau)$ is a braided categorical Lie K-algebra if and only if $(C_1, C_0, s, t, e, k, (\tau, \tau^-))$ is a braided categorical Leibniz K-algebra. $\tau^- : C_0 \times C_0 \to C_1$ is defined as $\tau^-_{a,b} = -\tau_{b,a}$.*

Proof. (LeibT1) and (LeibT2) can be rewritten as (LieT1) and LieT2, respectively, using the anticommutativity. Moreover, it is clear that LeibT3 and LieT4 are identical, and that (LeibT6) is equivalent to LieT3.

To see the last equivalences, (LeibT4) with (LieT3), and (LeibT5) with (BXLie4), we must prove $\tau_{[a,b],c} = -\tau_{c,[a,b]}$, for $a, b, c \in C_0$.

- In the Lie case, it is true using Proposition 2.3.5.
- In the Leibniz case it is not true in general, because we need $\tau_{a,b} = -\psi_{b,a}$; but using (LeibT4) and (LeibT5) we can observe that $\tau_{a,[b,c]} = \psi_{a,[b,c]} = \tau^-_{a,[b,c]} = -\tau_{[b,c],a}$.

□

Proposition 2.4.50. *Let $(C_1, C_0, s, t, e, k, (\tau, \psi))$ be a braided categorical Leibniz K-algebra. Then*

$\Bigg(\begin{array}{c} C_1 \\ \big\downarrow{\scriptstyle \pi_{C_1}} \\ \frac{C_1}{[\tau_{C_0,C_0}]} \end{array}, \begin{array}{c} C_0 \\ \big\downarrow{\scriptstyle \pi_{C_0}} \\ \mathrm{Lie}(C_0) \end{array}, (s,\overline{s}), (t,\overline{t}), (e,\overline{e}), (k,\widetilde{k}), \overline{\tau}\Bigg)$ *is a braided categorical Lie object in* $\mathcal{LM}_K$*, where* $\frac{C_1}{[\tau_{C_0,C_0}]}$ *is the Lie* K*-algebra which is a Leibniz quotient of* C_1 *by the ideal generated by elements of the form* $[x,x]$ *and* $\tau_{a,b}+\psi_{b,a}$*,* $x \in C_1$*,* $a,b \in C_0$*; and the maps are the following ones:*

- $\overline{s}: \frac{C_1}{[\tau_{C_0,C_0}]} \to \mathrm{Lie}(C_0)$ *defined as* $\overline{s}(\overline{x}) = \overline{s(x)}$ *for* $\overline{x} \in \frac{C_1}{[\tau_{C_0,C_0}]}$*;*
- $\overline{t}: \frac{C_1}{[\tau_{C_0,C_0}]} \to \mathrm{Lie}(C_0)$ *defined as* $\overline{t}(\overline{x}) = \overline{t(x)}$ *for* $\overline{x} \in \frac{C_1}{[\tau_{C_0,C_0}]}$*;*
- $\overline{e}: \mathrm{Lie}(C_0) \to \frac{C_1}{[\tau_{C_0,C_0}]}$ *defined as* $\overline{e}(\overline{a}) = \overline{e(a)}$ *for* $\overline{a} \in \mathrm{Lie}(C_0)$*;*
- $\widetilde{k}: \frac{C_1}{[\tau_{C_0,C_0}]} \times_{\mathrm{Lie}(C_0)} \frac{C_1}{[\tau_{C_0,C_0}]} \to \frac{C_1}{[\tau_{C_0,C_0}]}$ *defined as* $\overline{k}((\overline{x},\overline{y})) = \overline{\mathring{k}(x,y)}$ *for* $(\overline{x},\overline{y}) \in \frac{C_1}{[\tau_{C_0,C_0}]} \times_{\mathrm{Lie}(C_0)} \frac{C_1}{[\tau_{C_0,C_0}]}$*, where* $\mathring{k}$ *is again the extension to the product* $\mathring{k}(x,y) = x+y-e(s(y))$ *(we can take* $\mathring{k}'(x,y) = x+y-e(t(x))$ *too, because in the quotient it will not change anything);*
- $\overline{\tau}^{C_0,\mathrm{Lie}(C_0)}: C_0 \times \mathrm{Lie}(C_0) \to C_1$ *defined as* $\overline{\tau}^{C_0,\mathrm{Lie}(C_0)}_{a,\overline{b}} = \tau_{a,b}$ *for* $a \in C_0$*,* $\overline{b} \in \mathrm{Lie}(C_0)$*;*
- $\overline{\tau}^{\mathrm{Lie}(C_0),C_0}: \mathrm{Lie}(C_0) \times C_0 \to C_1$ *defined as* $\overline{\tau}^{\mathrm{Lie}(C_0),C_0}_{\overline{a},b} = -\psi_{b,a}$ *for* $\overline{a} \in \mathrm{Lie}(C_0)$*,* $b \in C_0$*;*
- $\overline{\tau}^2: \mathrm{Lie}(C_0) \times \mathrm{Lie}(C_0) \to \frac{C_1}{[\tau_{C_0,C_0}]}$ *defined as* $\tau^2_{\overline{a},\overline{b}} = \overline{\tau_{a,b}} = \overline{-\psi_{b,a}}$ *for* $\overline{a},\overline{b} \in \mathrm{Lie}(C_0)$.

Remark 2.4.51. The bottom part $(\frac{C_1}{[\tau_{C_0,C_0}]}, \mathrm{Lie}(C_0), \overline{s}, \overline{t}, \overline{e}, \widetilde{k}, \overline{\tau}_2)$ will be called Liesation, and it is again functorial.

If we apply this Liesation on a braided categorical Lie K-algebra, thought as a crossed module of Leibniz K-algebras with the action with the braiding (τ,τ^-), the new generator is null

$$\tau_{a,b} + \psi_{b,a} = \tau_{a,b} + \tau^-_{b,a} = \tau_{a,b} - \tau_{a,b} = 0.$$

Proposition 2.4.52. *Let* $\left(\begin{array}{c} C_1 \\ \downarrow{\scriptstyle f_1} \\ D_1 \end{array}, \begin{array}{c} C_0 \\ \downarrow{\scriptstyle f_0} \\ D_0 \end{array}, s, t, e, k, \tau \right)$ *be a braided categorical Lie object in* $\mathcal{LM}_K$.

Then $(C_1, C_0, s_1, t_1, e_1, k_1, (\bar{\tau}^\tau, \bar{\psi}^\tau))$ *is a braided categorical Leibniz* K*-algebra, where* $[x, y]_{C_1} = x *_{D_1}^{C_1} y$ *and* $[a, b]_{C_0} = a *_{D_0}^{C_0} b$ *for* $x, y \in C_1, a, b \in C_0$, *and* $\bar{\tau}^\tau_{a,b} = \tau^{C_0,D_0}_{a,f_0(b)}$, $\bar{\psi}^\tau_{a,b} = -\tau^{D_0,C_0}_{f(b),a}$ *for* $a, b \in C_0$.

We have again the pair of functors $\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K) \underset{BI\Psi}{\overset{BI\Phi}{\rightleftarrows}} \boldsymbol{BICat}(\boldsymbol{Lie}(\mathcal{LM}_K))$ satisfying $BI\Psi \circ BI\Phi = \mathrm{Id}_{\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K)}$, and so, the functor $BI\Phi$ is a full inclusion functor.

2.4.3 The equivalence between the categories of braided crossed modules and braided internal categories in the case of Leibniz algebras

First, we will prove that $\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K)$ and $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)$ are equivalent, as in the case of groups and Lie K-algebras. Moreover, the equivalence must generalize the Lie K-algebras case (i.e. the braidings of the Leibniz K-algebras must satisfy $\{n, n'\} = -\langle n', n\rangle$ and $\tau_{a,b} = -\psi_{b,a}$ and the functors for the Lie case would be recovered) and must be an extension of the one given to the non-braiding case.

Proposition 2.4.53. *Let* $\mathcal{X} = (M \xrightarrow{\partial} N, (\cdot_1, \cdot_2)(\{-,-\}, \langle -,-\rangle))$ *be a braided crossed module of Leibniz* K*-algebras.*

Then $C_{\mathcal{X}} := (M \rtimes N, N, \bar{s}, \bar{t}, \bar{e}, \bar{k}, (\bar{\tau}, \bar{\psi}))$ *is a braided categorical Leibniz* K*-algebra where*

- $\bar{s}: M \rtimes N \to N$, $\bar{s}((m, n)) = n$,
- $\bar{t}: M \rtimes N \to N$, $\bar{t}((m, n)) = \partial m + n$,
- $\bar{e}: N \to M \rtimes N$, $\bar{e}(n) = (0, n)$,
- $\bar{k}: (M \rtimes N) \times_N (M \rtimes N) \to M \rtimes N$, *where the source is the pullback of* $\bar{t}$ *with* $\bar{s}$, *defined as* $k(((m, n), (m', \partial m + n))) = (m + m', n)$,

- $\bar{\tau}: N \times N \to M \rtimes N, \;\; \bar{\tau}_{n,n'} = (-2\{n, n'\}, [n, n'])$,
- $\bar{\psi}: N \times N \to M \rtimes N, \;\; \bar{\psi}_{n,n'} = (-2\langle n, n'\rangle, [n, n'])$.

Proof. We only need to check the braiding axioms, since $(M \rtimes N, N, \bar{s}, \bar{t}, \bar{e}, \bar{k})$ is a categorical Leibniz K-algebra (see [22]).

We will start with (LeibT1). Let $n, n' \in N$.

$$\bar{s}(\bar{\tau}_{n,n'}) = \bar{s}((-2\{n, n'\}, [n, n'])) = [n, n'],$$
$$\bar{t}(\bar{\tau}_{n,n'}) = \bar{t}((-2\{n, n'\}, [n, n'])) = -2\partial\{n, n'\} + [n, n'] = -2[n, n'] + [n, n'] = -[n, n'],$$

where we use (BXLeib1). In the same way we can prove this property of $\bar{\psi}$ by the symmetry of the construction.

We will prove now (LeibT2). Again, we will only check this for $\bar{\tau}$.

Let $x = (m, n), y = (m', n') \in M \rtimes N$.

We need to show that $\tau_{t(x),t(y)} \circ [x, y] = -[x, y] \circ \tau_{s(x),s(y)}$. Now, we will write the equalities in function of the data given by the braided crossed module.

$$\begin{aligned}
&\tau_{t(x),t(y)} \circ [x, y] \\
&= \bar{k}(([(m, n), (m', n')], (-2\{\bar{t}((m, n)), \bar{t}((m', n'))\}, [\bar{t}((m, n)), \bar{t}((m', n'))]))) \\
&= \bar{k}(([(m, n), (m', n')], (-2\{\partial m + n, \partial m' + n'\}, [\partial m + n, \partial m' + n']))) \\
&= \bar{k}((([m, m'] + n \cdot_1 m' + m \cdot_2 n', [n, n']), (-2\{\partial m + n, \partial m' + n'\}, [\partial m + n, \partial m' + n']))) \\
&= ([m, m'] + n \cdot_1 m' + m \cdot_2 n' - 2\{\partial m + n, \partial m' + n'\}, [n, n']) \\
&= ([m, m'] + n \cdot_1 m' + m \cdot_2 n' - 2\{\partial m, \partial m'\} - 2\{\partial m, n'\} - 2\{n, \partial m'\} - 2\{n, n'\}, [n, n']) \\
&= ([m, m'] + n \cdot_1 m' + m \cdot_2 n' - 2[m, m'] - 2(m \cdot_2 n') - 2(n \cdot_1 m') - 2\{n, n'\}, [n, n']) \\
&= (-[m, m'] - n \cdot_1 m' - m \cdot_2 n' - 2\{n, n'\}, [n, n']),
\end{aligned}$$

where we use (BXLeib2), (BXLeib3) and (BXLeib4) in the sixth equality. In the other way,

$$\begin{aligned}
&-[x, y] \circ \tau_{s(x),s(y)} \\
&= \bar{k}(((-2\{\bar{s}((m, n)), \bar{s}((m, n'))\}, [\bar{s}((m, n)), \bar{s}((m', n'))]), -[(m, n), (m', n')]))
\end{aligned}$$

$$
\begin{aligned}
&= \bar{k}(((-2\{n,n'\},[n,n']),-[(m,n),(m',n')]))\\
&= \bar{k}(((-2\{n,n'\},[n,n']),(-[m,m']-n\cdot_1 m'-m\cdot_2 n',-[n,n'])))\\
&= (-2\{n,n'\}-[m,m']-n\cdot_1 m'-m\cdot_2 n',[n,n']).
\end{aligned}
$$

We will verify (LeibT3) below. Let $n, n', n'' \in N$. Then

$$
\begin{aligned}
\bar{\tau}_{n,[n',n'']} &= (-2\{n,[n',n'']\},[n,[n',n'']])\\
&= (-2(\{[n,n'],n''\}-\{[n,n''],n'\}),[[n,n'],n'']-[[n,n''],n'])\\
&= (-2\{[n,n'],n''\},[[n,n'],n''])-(-2\{[n,n''],n'\},[[n,n''],n'])\\
&= \bar{\tau}_{[n,n'],n''}-\bar{\tau}_{[n,n''],n'},
\end{aligned}
$$

where we use (BXLeib5) and the Leibniz identity in the second equality.

The same argument is valid for (LeibT6), using (BXLeib8) and by the symmetry of the properties.

Finally, we will show that (LeibT4) and (LeibT5) are satisfied.

$$
\begin{aligned}
\bar{\psi}_{n,[n',n'']} &= (-2\langle n,[n',n'']\rangle,[n,[n',n'']])\\
&= (-2(\{[n,n'],n''\}-\langle [n,n''],n'\rangle),[[n,n'],n'']-[[n,n''],n'])\\
&= (-2\{[n,n'],n''\},[[n,n'],n''])-(-2\langle [n,n''],n'\rangle,[[n,n''],n'])\\
&= \bar{\tau}_{[n,n'],n''}-\bar{\psi}_{[n,n''],n'}\\
&= (-2(\{[n,n'],n''\}-\langle [n,n''],n'\rangle),[[n,n'],n'']-[[n,n''],n'])\\
&= (-2\{n,[n',n'']\},[n,[n',n'']]) = \bar{\tau}_{n,[n',n'']},
\end{aligned}
$$

where we use (BXLeib6) along with the Leibniz identity in the second equality; and (BXLeib7) with the Leibniz identity in the penultimate equality. □

Remark 2.4.54. Note that if $\mathcal{X}$ is a braided crossed module of Lie K-algebras, then

$$
\bar{\tau}_{n,n'} = (-2\{n,n'\},[n,n']) = -(-2\langle n',n\rangle,[n',n]) = -\bar{\psi}_{n',n}
$$

and we recover the construction for the Lie case (see [24]).

Proposition 2.4.55. *We have a functor* $C:$ $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)\to$ $\boldsymbol{BICat}(\boldsymbol{LeibAlg}_K)$ *defined as*

$$C(\mathcal{X}\xrightarrow{(f_1,f_2)}\mathcal{X}') := C_{\mathcal{X}}\xrightarrow{(f_1\times f_2,f_2)} C_{\mathcal{X}'},$$

,where $C_{\mathcal{X}}$ *is constructed in the previous proposition.*

Proof. It is enough to see that $(f_1\times f_2, f_2)$ is a braided internal functor of braided categorical Leibniz K-algebras, since $(f_1\times f_2, f_2)$ is an internal functor between the respective internal categories (see [22]).

We will verify (LeibHT1). Let $n, n'\in N$.

$$\begin{aligned}(f_1\times f_2)(\overline{\tau}_{n,n'}) &= (f_1\times f_2)((-2\{n,n'\},[n,n'])) = (-2f_1(\{n,n'\}), f_2([n,n']))\\ &= (-2\{f_2(n), f_2(n')\}', [f_2(n), f_2(n')]) = \overline{\tau}'_{f_2(n),f_2(n')},\end{aligned}$$

where we use (BXLeibH1) in the penultimate equality.

Again, the same argument is valid to prove (LeibHT2), using (BXLeibH2) and because of the symmetry of the braiding's properties and the construction. $\square$

Proposition 2.4.56. *Let* $C = (C_1, C_0, s, t, e, k, (\tau,\psi))$ *be a braided categorical Leibniz* K*-algebra.*

Then $\mathcal{X}_C := (\ker(s)\xrightarrow{\partial_t} C_0, ({}^e\cdot, \cdot^e), (\{-,-\}_\tau, \langle -,-\rangle_\psi))$ *is a braided crossed module of Leibniz* K*-algebras where*

- ${}^e\cdot :\ C_0\times\ker(s)\to\ker(s),\ \ a\,{}^e\!\cdot x := [e(a), x],$
- $\cdot^e :\ \ker(s)\times C_0\to\ker(s),\ \ x\cdot^e a := [x, e(a)],$
- $\partial_t := t|_{\ker(s)},$
- $\{-,-\}_\tau :\ C_0\times C_0\to\ker(s),\ \ \{a,b\}_\tau := \frac{e([a,b])-\tau_{a,b}}{2},$
- $\langle -,-\rangle_\psi :\ C_0\times C_0\to\ker(s),\ \ \langle a,b\rangle_\psi := \frac{e([a,b])-\psi_{a,b}}{2}.$

Proof. It is enough to show that $(\{-,-\}_\tau, \langle -,-\rangle_\psi)$ is a braiding on the crossed module of Leibniz K-algebras $(\ker(s)\xrightarrow{\partial_t} C_0, (\cdot^e, {}^e\cdot))$ (see [22]).

First let us see that it is well defined because the image falls in C_1 which is not $\ker(s)$. We will check it only for $\{-,-\}_\tau$, since for $\langle -,-\rangle_\psi$ we will have a completely symmetric argument. Let $a, b \in C_0$, and using (LeibT1), we have

$$s(\{a,b\}_\tau) = s\Big(\frac{e([a,b]) - \tau_{a,b}}{2}\Big) = \frac{[a,b]-[a,b]}{2} = 0.$$

To check (BXLeib1)–(BXLeib4) we will only prove it for $\{-,-\}_\tau$.

First, we will check (BXLeib1). Let $a, b \in C_0$, and using (LeibT1), we get

$$\partial_t\{a,b\}_\tau = t\Big(\frac{e([a,b]) - \tau_{a,b}}{2}\Big) = \frac{[a,b]-(-[a,b])}{2} = \frac{2[a,b]}{2} = [a,b].$$

We will see now (BXLeib2). Let $x, y \in \ker(s)$. Then

$$\{\partial_t x, \partial_t y\}_\tau = \frac{e([\partial_t x, \partial_t y]) - \tau_{\partial_t x, \partial_t y}}{2} = \frac{e([t(x), t(y)]) - \tau_{t(x),t(y)}}{2}.$$

Let us see that $\frac{e([t(x),t(y)]) - \tau_{t(x),t(y)}}{2} = [x,y]$.

Using (LeibT2), we have

$$k(([x,y], \tau_{t(x),t(y)})) = k((\tau_{s(x),s(y)}, -[x,y])).$$

As $x \in \ker(s)$, we have that $s(x) = 0$ (in the same way y), and $\tau_{s(x),s(y)} = 0$ by K-bilinearity. So, we have

$$k((\tau_{s(x),s(y)}, -[x,y])) = k((0, -[x,y])),$$

and therefore

$$k(([x,y], \tau_{t(x),t(y)})) = k((0, -[x,y])).$$

Using now the K-linearity of k in the previous expression, we obtain the equality

$$0 = k(([x,y], \tau_{t(x),t(y)} + [x,y])).$$

Since $t(\tau_{t(x),t(y)} + [x,y]) = -[t(x),t(y)] + [t(x),t(y)] = 0 = s(e(0))$ we can talk about $k((\tau_{t(x),t(y)} + [x,y], e(0)))$. Further $k((\tau_{t(x),t(y)} + [x,y], e(0))) = \tau_{t(x),t(y)} + [x,y]$ by the internal category axioms.

Adding both equalities and by using the K-linearity of k we get

$$k(([x,y]+\tau_{t(x),t(y)}+[x,y],\tau_{t(x),t(y)}+[x,y]))=\tau_{t(x),t(y)}+[x,y].$$

Therefore, by grouping, we have

$$k((2[x,y]+\tau_{t(x),t(y)},\tau_{t(x),t(y)}+[x,y]))=\tau_{t(x),t(y)}+[x,y].$$

By using that $\ker(s)$ is an ideal and the fact that x or y are in $\ker(s)$, we have $s(\tau_{t(x),t(y)}+[x,y])=[t(x),t(y)]-0=[t(x),t(y)]$, and so it makes sense to speak about the composition $k((e([t(x),t(y)]),\tau_{t(x),t(y)}+[x,y]))$, which is equal to $\tau_{t(x),t(y)}+[x,y]$.

Subtracting both equalities and using the K-linearity of k, we obtain

$$k((2[x,y]+\tau_{t(x),t(y)}-e([t(x),t(y)]),0))=0.$$

Again, using the properties for internal categories, we have

$$\begin{aligned}0&=k((2[x,y]+\tau_{t(x),t(y)}-e([t(x),t(y)]),0))\\&=k((2[x,y]+\tau_{t(x),t(y)}-e([t(x),t(y)]),e(0)))\\&=2[x,y]+\tau_{t(x),t(y)}-e([t(x),t(y)]),\end{aligned}$$

which gives us the required equality, since $\mathrm{char}(K)\neq 2$.

As an observation to the above, in the part of the proof where we use that $x,y\in\ker(s)$, it is sufficient that one of the two is in that kernel. Therefore, by repeating the argument, we have the following equalities for $x\in\ker(s)$ and $y\in C_1$:

$$\frac{e([t(x),t(y)])-\tau_{t(x),t(y)}}{2}=[x,y],\qquad \frac{e([t(y),t(x)])-\tau_{t(y),t(x)}}{2}=[y,x].$$

With these equalities, we will check (BXLeib3) and (BXLeib4).

Let $a\in C_0$ and $x\in\ker(s)$. Then

$$\begin{aligned}\{\partial_t x,a\}_\tau&=\frac{e([t(x),t(e(a))])-\tau_{t(x),t(e(a))}}{2}=[x,e(a)]=x\cdot^e a,\\\{a,\partial_t x\}_\tau&=\frac{e([t(e(a)),t(x)])-\tau_{t(e(a)),t(x)}}{2}=[e(a),x]=a\,{}^e\!\cdot x.\end{aligned}$$

We will see now the last conditions, starting with (BXLeib5). Let $a, b, c \in C_0$.

$$\begin{aligned}\{a,[b,c]\}_\tau &= \frac{e([a,[b,c]]) - \tau_{a,[b,c]}}{2} = \frac{e([[a,b],c]) - e([[a,c],b]) - \tau_{[a,b],c} + \tau_{[a,c],b}}{2}\\ &= \frac{e([[a,b],c]) - \tau_{[a,b],c}}{2} - \frac{e([[a,c],b]) - \tau_{[a,c],b}}{2} = \{[a,b],c\}_\tau - \{[a,c],b\}_\tau,\end{aligned}$$

where we use (LeibT3) and the Leibniz identity in the second equality. By symmetry we can prove (BXLeib8), using (LeibT6).

To conclude we will check (BXLeib6) and (BXLeib7).

$$\begin{aligned}\langle a,[b,c]\rangle_\psi &= \frac{e([a,[b,c]]) - \psi_{a,[b,c]}}{2} = \frac{e([[a,b],c]) - e([[a,c],b]) - \tau_{[a,b],c} + \psi_{[a,c],b}}{2}\\ &= \frac{e([[a,b],c]) - \tau_{[a,b],c}}{2} - \frac{e([[a,c],b]) - \psi_{[a,c],b}}{2} = \{[a,b],c\}_\tau - \langle [a,c],b\rangle_\psi\\ &= \frac{e([[a,b],c]) - e([[a,c],b]) - \tau_{[a,b],c} + \psi_{[a,c],b}}{2} = \frac{e([a,[b,c]]) - \tau_{a,[b,c]}}{2}\\ &= \{a,[b,c]\}_\tau,\end{aligned}$$

where we use (LeibT4) in the second equality together with Leibniz identity and (LeibT5) in the penultimate equality with the Leibniz identity. □

Remark 2.4.57. Note that if C is a braided categorical Lie K-algebra, then

$$\{a,b\}_\tau = \frac{e([a,b]) - \tau_{a,b}}{2} = -\frac{e([b,a]) - \psi_{b,a}}{2} = -\langle b,a\rangle_\psi$$

and we recover the construction for the Lie case.

Proposition 2.4.58. *We have a functor $\mathcal{X}$: $\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K) \to \boldsymbol{BX}(\boldsymbol{LeibAlg}_K)$ defined as*

$$\mathcal{X}(C \xrightarrow{(F_1,F_0)} C') = \mathcal{X}_C \xrightarrow{(F_1^s,F_0)} \mathcal{X}_{C'},$$

where $\mathcal{X}_C$ is constructed in the previous proposition and $F_1^s\colon \ker(s) \to \ker(s')$ is defined as $F_1^s(x) = F_1(x)$ for $x \in \ker(s)$.

Proof. $\mathcal{X}$ is a functor between the categories without braiding (see [22]). So, we only must check the axioms of the homomorphisms of braided crossed of Leibniz K-algebras.

We will start with (BXLeibH1). Let $a, b \in C_0$.

$$
\begin{aligned}
F_1^s(\{a,b\}_\tau) &= F_1\Big(\frac{e([a,b]) - \tau_{a,b}}{2}\Big) = \frac{F_1(e([a,b])) - F_1(\tau_{a,b})}{2} \\
&= \frac{e'(F_0([a,b])) - \tau'_{F_0(a),F_0(b)}}{2} = \frac{e'([F_0(a),F_0(b)]) - \tau'_{F_0(a),F_0(b)}}{2} \\
&= \{F_0(a), F_0(b)\}_{\tau'},
\end{aligned}
$$

where we use (LeibHT1) in the third equality.

Again, we can prove (BXLeibH2) using the same argument, (LeibHT2) and the symmetry. □

Remark 2.4.59. Note that, if $(M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,- \rangle))$ then we have that $\ker(\bar{s}) = \{(m,0) \in M \rtimes N \mid m \in M\}$, where $\bar{s}$ is defined for the functor $\mathcal{C}$.

Proposition 2.4.60. *The categories* $\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)$ *and* $\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K)$ *are equivalent categories.*

Further, the functors $\mathcal{C}$ and $\mathcal{X}$ are inverse equivalences, where the natural isomorphisms $\mathrm{Id}_{\boldsymbol{BX}(\boldsymbol{LeibAlg}_K)} \overset{\alpha}{\cong} \mathcal{X} \circ \mathcal{C}$ *and* $\mathrm{Id}_{\boldsymbol{ICat}(\boldsymbol{LeibAlg}_K)} \overset{\beta}{\cong} \mathcal{C} \circ \mathcal{X}$ *are given by:*

- *If* $\mathcal{Z} = (M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,- \rangle))$ *is a braided crossed module of Leibniz K-algebras, then* $\alpha_{\mathcal{Z}} = (\alpha_M, \mathrm{Id}_N)$*, where* $\alpha_M : M \to (M, 0)$ *is defined by* $\alpha_M(m) = (m, 0)$*;*
- *If* $\mathcal{D} = (C_1, C_0, s, t, e, k, (\tau, \psi))$ *is a braided categorical Leibniz K-algebra, then* $\beta_{\mathcal{D}} = (\beta_s, \mathrm{Id}_{C_0})$*, where* $\beta_{C_1} : C_1 \to \ker(s) \rtimes C_0$ *is defined by* $\beta_{C_1}(x) = (x - e(s(x)), s(x))$*.*

Proof. $\mathcal{C}$ and $\mathcal{X}$ are natural isomorphisms in the categories without braiding (see [22]). So, it is enough to show that they are morphisms between braided objects.

Let $\mathcal{Z} = (M \xrightarrow{\partial} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,- \rangle))$ a braided crossed module of Leibniz K-algebras. Let us see that $\alpha_{\mathcal{Z}} = (\alpha_M, \mathrm{Id}_N)$ satisfies (BXLeibH1).

$$
\mathrm{Id}_N(\{n,n'\}_{\bar{\tau}}) = \{n,n'\}_{\bar{\tau}} = \frac{\bar{e}([n,n']) - \bar{\tau}_{n,n'}}{2} = \frac{(0,[n,n']) - (-2\{n,n'\},[n,n'])}{2}
$$

$$= \frac{(2\{n,n'\},0)}{2} = (\{n,n'\},0) = \alpha_M(\{n,n'\}), \qquad n,n' \in N.$$

Analogously (BXLeibH2) is proven, by the similarity of definitions.

Let $\mathcal{D} = (C_1, C_0, s, t, e, k, (\tau, \psi))$ be a braided categorical Leibniz K-algebra. We will check that $\beta_{\mathcal{D}} = (\beta_s, \mathrm{Id}_{C_0})$ satisfies (LeibHT1) and (LeibHT2). We only show the proof for (LeibHT1), since the one for (LeibHT2) is similar.

Let us consider $a, b \in C_0$. We have:

$$\begin{aligned}\mathrm{Id}_{C_0}(\bar{\tau}_{a,b}) = \bar{\tau}_{a,b} = (-2\{a,b\}_\tau, [a,b]) = (-2\frac{e([a,b]) - \tau_{a,b}}{2}, [a,b])\\ = (\tau_{a,b} - e([a,b]), [a,b]) = (\tau_{a,b} - e(s(\tau_{a,b})), s(\tau_{a,b})) = \beta_{C_1}(\tau_{a,b}).\end{aligned}$$

□

2.5 The non-abelian tensor product as example of braiding

If $(M, [-,-])$ is a Leibniz K-algebra, then $([-,-],[-,-])$ is a braiding on $(M \xrightarrow{\mathrm{Id}_M} M, ([-,-],[-,-]))$. This example is the analogous for the case of Leibniz K-algebras to the models of $(G \xrightarrow{\mathrm{Id}_G} G, \mathrm{Conj}, [-,-])$ for groups and $(M \xrightarrow{\mathrm{Id}_M} M, [-,-], [-,-])$ for Lie K-algebras. Further, this example generalizes the Lie example, since $[y,x] = -[x,y]$ in this case.

We will give another symmetric instance in the three frameworks: the non-abelian tensor product.

We will start with the non-abelian tensor product of groups which was introduced by Brown and Loday in [11].

Definition 2.5.1 ([11]). Let G and H be two groups such that G acts on H with $\cdot$ and H acts on G with $*$, both by automorphisms.

The *non-abelian tensor product of G with H*, denoted by $G \otimes H$, is the group generated by the symbols $g \otimes h$, where $g \in G$, $h \in H$, with the relations

$$\begin{aligned}gg' \otimes h &= (gg'g^{-1} \otimes g \cdot h)(g \otimes h),\\ g \otimes hh' &= (g \otimes h)(h * g \otimes hh'h^{-1}).\end{aligned}$$

The following proposition is given for a general case in [10, 11], using actions which are denominated compatible actions for make the tensor product.

Proposition 2.5.2 ([10, 11])**.** *Let G be a group. Then $(G \otimes G \xrightarrow{\partial} G, \cdot)$ is a crossed module of groups where $G \otimes G$ is the non-abelian tensor product of G with itself using the conjugation action. The action $\cdot : G \times (G \otimes G) \to (G \otimes G)$ and the map $\partial : G \otimes G \to G$ are defined on generators as $g \cdot (g_1 \otimes g_2) = gg_1g^{-1} \otimes gg_2g^{-1}$ and $\partial(g_1 \otimes g_2) = [g_1, g_2]$.*

The next example shows that this crossed module can be associated with a natural braiding (see [26]).

Example 2.5.3. Let G be a group. The map $\{-,-\} : G \times G \to G \otimes G$ defined as $\{g_1, g_2\} = g_1 \otimes g_2$ is a braiding on $(G \otimes G \xrightarrow{\partial} G, \cdot)$.

Using the properties of the non-abelian tensor product of groups (see [47, Proposition 1.2.3]) and the definition, the result follows easily.

Once given the example in groups, we look for its analogue in Lie K-algebras. For this we need the concept of non-abelian tensor product of Lie K-algebras, introduced by Ellis in [19].

Definition 2.5.4. Let M and N be two Lie K-algebras such that M acts in N by $\cdot$ and N acts in M with $*$.

The *non-abelian tensor product*, denoted by $M \otimes N$, is the Lie K-algebra generated by the symbols $m \otimes n$, where $m \in M$, $n \in N$, with the relations

$$\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n, \tag{T1}$$

$$(m + m') \otimes n = m \otimes n + m' \otimes n, \tag{T2}$$

$$m \otimes (n + n') = m \otimes n + m \otimes n',$$

$$[m, m'] \otimes n = m \otimes (m' \cdot n) - m' \otimes (m \cdot n), \tag{T3}$$

$$m \otimes [n, n'] = (n' * m) \otimes n - (n * m) \otimes n',$$

$$[(m \otimes n), (m' \otimes n')] = -(n * m) \otimes (m' \cdot n'), \tag{T4}$$

where $m, m' \in M$, $n, n' \in N$, $\lambda \in K$.

The next proposition, following the pattern of the case of groups, was proved more generally in [19], but we restrict ourselves to the case that interests us.

Proposition 2.5.5 ([19]). *Let M be a Lie K-algebra. Then $(M \otimes M \xrightarrow{\partial} M, \cdot)$ is a crossed module of Lie K-algebras, where $M \otimes M$ is the non-abelian tensor product of M with itself using the adjoint action.*

The action $\cdot : M \times (M \otimes M) \to (M \otimes M)$ and the map $\partial : M \otimes M \to M$ are defined on generators as $m \cdot (m_1 \otimes m_2) = [m, m_1] \otimes m_2 + m_1 \otimes [m, m_2]$ and $\partial(m_1 \otimes m_2) = [m_1, m_2]$, where $[-,-]$ is the bracket of M.

Remark 2.5.6. We will rewrite, for clarity, the relations (T3) and (T4) for the case of $M \otimes M$ with the adjoint action of M on itself.

$$\text{(T3)} \qquad [m_1, m_2] \otimes m_3 = m_1 \otimes [m_2, m_3] - m_2 \otimes [m_1, m_3],$$
$$m_1 \otimes [m_2, m_3] = [m_3, m_1] \otimes m_2 - [m_2, m_1] \otimes m_3,$$
$$\text{(T4)}\ [(m_1 \otimes m_2), (m_3 \otimes m_4)] = [m_1, m_2] \otimes [m_3, m_4],$$

where $m_1, m_2, m_3, m_4 \in M$. For the last relation we use the anticommutativity.

Now, we show an example for the case of Lie K-algebras analogous to the case of groups.

Example 2.5.7. Let M be a Lie K-algebra. The K-bilinear map $\{-,-\} : M \times M \to M \otimes M$ defined by $\{m_1, m_2\} = m_1 \otimes m_2$ is a braiding on the crossed module of Lie K-algebras $(M \otimes M \xrightarrow{\partial} M, \cdot)$.

We will check (BXLie1). If $m, m' \in M$, then

$$\partial\{m, m'\} = \partial(m \otimes m') = [m, m'].$$

To check (BXLie2), we will work on generators by the K-linearity and K-bilinearity, since the general case is only a sum of them.

If $m_1 \otimes m_2$ and $m_3 \otimes m_4$ are generators of $M \otimes M$, then

$$\begin{aligned}\{\partial(m_1 \otimes m_2), \partial(m_3 \otimes m_4)\} &= \{[m_1, m_2], [m_3, m_4]\} = [m_1, m_2] \otimes [m_3, m_4] \\ &= [(m_1 \otimes m_2), (m_3 \otimes m_4)],\end{aligned}$$

where the last equality is given by (T4).

For the following properties we need a previous result.

We will use (T3) to prove $m_1 \otimes [m_2, m_3] = -[m_2, m_3] \otimes m_1$.

$$
\begin{aligned}
[m_1, m_2] \otimes m_3 &= m_1 \otimes [m_2, m_3] - m_2 \otimes [m_1, m_3] \\
&= m_1 \otimes [m_2, m_3] - [m_3, m_2] \otimes m_1 + [m_1, m_2] \otimes m_3.
\end{aligned}
$$

Simplifying we have $0 = m_1 \otimes [m_2, m_3] + [m_2, m_3] \otimes m_1$.

Now, we will show (BXLie3). Let $m \in M$ and $m_1 \otimes m_2 \in M \otimes M$.

$$
\begin{aligned}
&\{\partial(m_1 \otimes m_2), m\} = \{[m_1, m_2], m\} = [m_1, m_2] \otimes m \\
&= m_1 \otimes [m_2, m] - m_2 \otimes [m_1, m] = -m_1 \otimes [m, m_2] + m_2 \otimes [m, m_1] \\
&= -m_1 \otimes [m, m_2] - [m, m_1] \otimes m_2 = -m \cdot (m_1 \otimes m_2),
\end{aligned}
$$

where we use (T3) together with the previous result.

Now, we will verify (BXLie4).

$$
\begin{aligned}
\{m, \partial(m_1 \otimes m_2)\} &= m \otimes [m_1, m_2] = -[m_1, m_2] \otimes m = -\{\partial(m_1 \otimes m_2), m\} \\
&= -(-m \cdot (m_1 \otimes m_2)) = m \cdot (m_1 \otimes m_2),
\end{aligned}
$$

where we use (BXLie3) and $m \otimes [m_1, m_2] = -[m_1, m_2] \otimes m$.

Now, we will verify (BXLie5) and (BXLie6). Let $m, m', m'' \in M$.

$$
\begin{aligned}
\{m, [m', m'']\} &= m \otimes [m', m''] = [m'', m] \otimes m' - [m', m] \otimes m'' \\
&= [m, m'] \otimes m'' - [m, m''] \otimes m' = \{[m, m'], m''\} - \{[m, m''], m'\}, \\
\{[m, m'], m''\} &= [m, m'] \otimes m'' = m \otimes [m', m''] - m' \otimes [m, m''] \\
&= \{m, [m', m'']\} - \{m', [m, m'']\}.
\end{aligned}
$$

We use (T3) in the second equality of both chains of equalities.

So, we have shown that $\{m, m'\} = m \otimes m'$ is a braiding.

Remark 2.5.8. Note that the action given in the previous example is actually given by $m \cdot (m_1 \otimes m_2) = m \otimes [m_1, m_2]$.

The non-abelian tensor product of Leibniz K-algebras was introduced by Gnedbaye in [32], where the tensor product is denoted as $M \star N$, and its generators as $m * n$ and $n * m$. In the general case it does not give rise to confusion, but in the case $M = N$ these generators would be denoted in the same way, giving rise to confusion. To avoid this, we change the nomenclature, meaning $m * n$ as $m \otimes n$ and $n * m$ as $n \circledast m$.

Definition 2.5.9. Let M and N two Leibniz K-algebras together with two Leibniz actions $\cdot = (\cdot_1, \cdot_2)$ of M on N and $* = (*_1, *_2)$ of N on M.

The non-abelian tensor product of M and N, denoted by $M \star N$, is the Leibniz K-algebra generated by the symbols $m \otimes n$ and $n \circledast m$ with $m \in M$, $n \in N$, together with the relations:

$$\begin{aligned} \lambda(m \otimes n) &= \lambda m \otimes n = m \otimes \lambda n, \\ \lambda(n \circledast m) &= \lambda n \circledast m = n \circledast \lambda m, \end{aligned} \tag{RTLeib1}$$

$$\begin{aligned} (m + m') \otimes n &= m \otimes n + m' \otimes n, \\ m \otimes (n + n') &= m \otimes n + m \otimes n', \\ (n + n') \circledast m &= n \circledast m + n' \circledast m, \\ n \circledast (m + m') &= n \circledast m + n \circledast m', \end{aligned} \tag{RTLeib2}$$

$$\begin{aligned} m \otimes [n, n'] &= (m *_2 n) \otimes n' - (m *_2 n') \otimes n, \\ n \circledast [m, m'] &= (n \cdot_2 m) \circledast m' - (n \cdot_2 m') \circledast m, \\ [m, m'] \otimes n &= (m \cdot_1 n) \circledast m' - m \otimes (n \cdot_2 m'), \\ [n, n'] \circledast m &= (n *_1 m) \otimes n' - n \circledast (m *_2 n'), \end{aligned} \tag{RTLeib3}$$

$$\begin{aligned} m \otimes (m' \cdot_1 n) &= -m \otimes (n \cdot_2 m'), \\ n \circledast (n' *_1 m) &= -n \circledast (m *_2 n'), \end{aligned} \tag{RTLeib4}$$

$$(m *_2 n) \otimes (m' \cdot_1 n') = [m \otimes n, m' \otimes n'] = (m \cdot_1 n) \circledast (m' *_2 n'), \tag{RTLeib5}$$

$$(m *_2 n) \otimes (n' \cdot_2 m') = [m \otimes n, n' \circledast m'] = (m \cdot_1 n) \circledast (n' *_1 m'),$$
$$(n *_1 m) \otimes (n' \cdot_2 m') = [n \circledast m, n' \circledast m'] = (n \cdot_2 m) \circledast (n' *_1 m'),$$
$$(n *_1 m) \otimes (m' \cdot_1 n') = [n \circledast m, m' \otimes n'] = (n \cdot_2 m) \circledast (m' *_2 n'),$$

$m, m' \in M,\ n, n' \in N$.

Proposition 2.5.10 ([32]). *Let M be a Leibniz K-algebra.*

Then $(M \star M \xrightarrow{\partial} M, (\cdot_1, \cdot_2))$ is a crossed module of Leibniz K-algebras, where $M \star M$ is the non-abelian tensor product of M with itself using the actions given by the Leibniz bracket, where

- *the left action on generators is given by $m \cdot_1 (m_1 \otimes m_2) = [m, m_1] \otimes m_2 - [m, m_2] \circledast m_1$, $m \cdot_1 (m_1 \circledast m_2) = [m, m_1] \circledast m_2 - [m, m_2] \otimes m_1$;*
- *the right action on generators is given by $(m_1 \otimes m_2) \cdot_2 m = [m_1, m] \otimes m_2 + m_1 \otimes [m_2, m]$, $(m_1 \circledast m_2) \cdot_2 m = [m_1, m] \circledast m_2 + m_1 \circledast [m_2, m]$;*
- *the map ∂ is defined on generators as $\partial(m_1 \otimes m_2) = [m_1, m_2] = \partial(m_1 \circledast m_2)$.*

Remark 2.5.11. We will show how are the relations (RTLeib3)–(RTLeib5) for the non-abelian tensor product $M \star M$ with the action $([-,-],[-,-])$ on itself:

$$m_1 \otimes [m_2, m_3] = [m_1, m_2] \otimes m_3 - [m_1, m_3] \otimes m_2, \tag{RTLeib3}$$
$$m_1 \circledast [m_2, m_3] = [m_1, m_2] \circledast m_3 - [m_1, m_3] \circledast m_2,$$
$$[m_1, m_2] \otimes m_3 = [m_1, m_3] \circledast m_2 - m_1 \otimes [m_3, m_2],$$
$$[m_1, m_2] \circledast m_3 = [m_1, m_3] \otimes m_2 - m_1 \circledast [m_3, m_2],$$
$$m_1 \otimes [m_2, m_3] = -m_1 \otimes [m_3, m_2], \tag{RTLeib4}$$
$$m_1 \circledast [m_2, m_3] = -m_1 \circledast [m_3, m_2],$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \otimes m_4] = [m_1, m_2] \circledast [m_3, m_4], \tag{RTLeib5}$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \circledast m_4] = [m_1, m_2] \circledast [m_3, m_4],$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \circledast m_4] = [m_1, m_2] \circledast [m_3, m_4],$$
$$[m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \otimes m_4] = [m_1, m_2] \circledast [m_3, m_4],$$

$m_1, m_2, m_3, m_4 \in M$.

The following example shows the necessity of a pair of braidings for the Leibniz K-algebras case since they will be different.

Example 2.5.12. Let M be a Leibniz K-algebra.

The pair of K-bilinear maps $\{-,-\}, \langle -,- \rangle \colon M \times M \to M \star M$ defined as $\{m_1, m_2\} = m_1 \otimes m_2$ and $\langle m_1, m_2 \rangle = m_1 \circledast m_2$ is a braiding on the crossed module of Leibniz K-algebras $(M \star M \stackrel{\partial}{\longrightarrow} M, (\cdot_1, \cdot_2))$.

First, will check (BXLeib1).

$$\partial\{m_1, m_2\} = \partial(m_1 \otimes m_2) = [m_1, m_2] = \partial(m_1 \circledast m_2) = \partial\langle m_1, m_2 \rangle, \qquad m_1, m_2 \in M.$$

Now, we will prove (BXLeib2).

$$\begin{aligned}
\{\partial(m_1 \otimes m_2), \partial(m_3 \otimes m_4)\} &= [m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \otimes m_4],\\
\{\partial(m_1 \otimes m_2), \partial(m_3 \circledast m_4)\} &= [m_1, m_2] \otimes [m_3, m_4] = [m_1 \otimes m_2, m_3 \circledast m_4],\\
\{\partial(m_1 \circledast m_2), \partial(m_3 \otimes m_4)\} &= [m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \otimes m_4],\\
\{\partial(m_1 \circledast m_2), \partial(m_3 \circledast m_4)\} &= [m_1, m_2] \otimes [m_3, m_4] = [m_1 \circledast m_2, m_3 \circledast m_4],\\
\langle \partial(m_1 \otimes m_2), \partial(m_3 \otimes m_4)\rangle &= [m_1, m_2] \circledast [m_3, m_4] = [m_1 \otimes m_2, m_3 \otimes m_4],\\
\langle \partial(m_1 \otimes m_2), \partial(m_3 \circledast m_4)\rangle &= [m_1, m_2] \circledast [m_3, m_4] = [m_1 \otimes m_2, m_3 \circledast m_4],\\
\langle \partial(m_1 \circledast m_2), \partial(m_3 \otimes m_4)\rangle &= [m_1, m_2] \circledast [m_3, m_4] = [m_1 \circledast m_2, m_3 \otimes m_4],\\
\langle \partial(m_1 \circledast m_2), \partial(m_3 \circledast m_4)\rangle &= [m_1, m_2] \circledast [m_3, m_4] = [m_1 \circledast m_2, m_3 \circledast m_4],
\end{aligned}$$

where in all the cases we used (RTLeib5).

Before to check the following axioms, we need to check a property that can be proven using (RTLeib3) and (RTLeib4).

Using (RTLeib4) in the last equality of relation (RTLeib3) and rewriting that equality and the second one, we get

$$\begin{aligned}
m_1 \circledast [m_2, m_3] &= [m_1, m_2] \circledast m_3 - [m_1, m_3] \circledast m_2,\\
m_1 \circledast [m_2, m_3] &= [m_1, m_2] \circledast m_3 - [m_1, m_3] \otimes m_2.
\end{aligned}$$

Subtracting, we obtain the equality $[m_1, m_3] \otimes m_2 = [m_1, m_3] \circledast m_2$. Using this last equality and the first and second equality of (RTLeib3), we obtain

$$m_1 \otimes [m_2, m_3] = [m_1, m_2] \otimes m_3 - [m_1, m_3] \otimes m_2$$

$$= [m_1, m_2] \circledast m_3 - [m_1, m_3] \circledast m_2 = m_1 \circledast [m_2, m_3].$$

Let us verify now the first equality of (BXLeib3) with $m, m_1, m_2 \in M$,

$$\begin{aligned}\{\partial(m_1 \otimes m_2), m\} &= [m_1, m_2] \otimes m = [m_1, m] \circledast m_2 - m_1 \otimes [m, m_2] \\ &= [m_1, m] \circledast m_2 + m_1 \otimes [m_2, m] = [m_1, m] \otimes m_2 + m_1 \otimes [m_2, m] \\ &= (m_1 \otimes m_2) \cdot_2 m,\end{aligned}$$

where we use (RTLeib3) and (RTLeib4).

The second equality is analogous:

$$\begin{aligned}\{\partial(m_1 \circledast m_2), m\} &= [m_1, m_2] \otimes m = [m_1, m] \circledast m_2 + m_1 \otimes [m_2, m] \\ &= [m_1, m] \circledast m_2 + m_1 \circledast [m_2, m] = (m_1 \circledast m_2) \cdot_2 m.\end{aligned}$$

Using the exchange properties between $\otimes$ and $\circledast$ again, we will see the remaining equalities:

$$\begin{aligned}\langle \partial(m_1 \otimes m_2), m \rangle &= [m_1, m_2] \circledast m = [m_1, m_2] \otimes m = (m_1 \otimes m_2) \cdot_2 m, \\ \langle \partial(m_1 \circledast m_2), m \rangle &= [m_1, m_2] \circledast m = [m_1, m_2] \otimes m = (m_1 \circledast m_2) \cdot_2 m.\end{aligned}$$

Now we will check the next axiom, (BXLeib4), where we will use again that we can exchange the symbols if in one side is the bracket. Starting with the first equality, we have

$$\begin{aligned}\{m, \partial(m_1 \otimes m_2)\} &= m \otimes [m_1, m_2] = [m, m_1] \otimes m_2 - [m, m_2] \otimes m_1 \\ &= [m, m_1] \otimes m_2 - [m, m_2] \circledast m_1 = m \cdot_1 (m_1 \otimes m_2),\end{aligned}$$

where we use (RTLeib3). Analogously we obtain the second equality:

$$\begin{aligned}\{m, \partial(m_1 \circledast m_2)\} &= m \otimes [m_1, m_2] = [m, m_1] \otimes m_2 - [m, m_2] \otimes m_1 \\ &= [m, m_1] \circledast m_2 - [m, m_2] \otimes m_1 = m \cdot_1 (m_1 \circledast m_2).\end{aligned}$$

So, the following properties are immediate:

$$\langle m, \partial(m_1 \otimes m_2) \rangle = m \circledast [m_1, m_2] = m \otimes [m_1, m_2] = m \cdot_1 (m_1 \otimes m_2),$$

$$\langle m, \partial(m_1 \circledast m_2)\rangle = m \circledast [m_1, m_2] = m \otimes [m_1, m_2] = m \cdot_1 (m_1 \circledast m_2).$$

To finalize, we will prove (BXLeib5), because, if it is satisfied, equalities (BXLeib6)–(BXLeib8) will be fulfilled using the following properties:

$$\begin{aligned}
\{m, [m', m'']\} &= m \otimes [m', m''] = m \circledast [m', m''] = \langle m, [m', m'']\rangle, \\
\{[m, m'], m''\} &= [m, m'] \otimes m'' = [m, m'] \circledast m'' = \langle [m, m'], m''\rangle.
\end{aligned}$$

By using (RTLeib3), we have (BXLeib5):

$$\begin{aligned}
\{m, [m', m'']\} = m \otimes [m', m''] &= [m, m'] \otimes m'' - [m, m''] \otimes m' \\
&= \{[m, m'], m''\} - \{[m, m''], m'\},
\end{aligned}$$

Remark 2.5.13. Note that the actions can be written with a simpler notation, given by

$$\begin{aligned}
m \cdot_1 (m_1 \otimes m_2) &= m \cdot_1 (m_1 \circledast m_2) = m \otimes [m_1, m_2] = m \circledast [m_1, m_2], \\
(m_1 \otimes m_2) \cdot_2 m &= (m_1 \circledast m_2) \cdot_2 m = [m_1, m_2] \otimes m = [m_1, m_2] \circledast m.
\end{aligned}$$

Remark 2.5.14. Example 2.5.12 generalizes the Lie example, since if we have that $m_1 \circledast m_2 = -m_2 \otimes m_1$ as a new relation, we obtain the Lie non-abelian tensor product of M with itself using the adjoint action.

CHAPTER 3

Universal central extension of braided crossed modules of Lie algebras

In this chapter, we will study two notions of centre and commutator for the braided crossed modules and Lie algebras. With that, we will define its respective central extensions, and we will show the relationship between them.

3.1 Centre and commutator objects

The category of Lie crossed modules $\boldsymbol{X(LieAlg_K)}$ is a semi-abelian category in the sense of [37].

The notion of the centre of an object was defined in [35], in a category with specific properties. This construction only needs that the category has finite products and zero object.

The category $\boldsymbol{X(LieAlg_K)}$ has centres in the sense of Huq [35], and they were constructed in [13].

Definition 3.1.1. The *centre* of a Lie crossed module $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ is the crossed submodule $Z(\mathcal{M}) = (M^N \xrightarrow{\partial|_{M^N}} \mathrm{st}_N(M) \cap Z(N), \cdot_Z)$, where:

- $M^N = \{m \in M \mid n \cdot m = 0,\ n \in N\}$,
- $Z(N) = \{n \in N \mid [n, n'] = 0,\ n' \in N\}$ is the centre of the Lie K-algebra N,
- $\mathrm{st}_N(M) = \{n \in N \mid n \cdot m = 0,\ m \in M\}$,

and $\cdot_Z$ is the induced action, which means that it is the zero action by the definition of M^N.

The notions of commutator of a Lie crossed module and a perfect Lie crossed module were introduced in [13]. This notion of commutator coincides in the category $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ with the idea of commutator given by Huq in [35] in a category with products, zero objects, kernels and cokernels.

If L is a Lie K-algebra and $S \subset L$, we denote $\langle S \rangle_L$ the Lie subalgebra of L generated by S, that is, the intersection of all subalgebras containing S.

Definition 3.1.2. Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ be a Lie crossed module. The *commutator crossed submodule* is $[\mathcal{M}, \mathcal{M}] = (D_N(M) \xrightarrow{\partial|_{D_N(M)}} [N, N], \cdot_C)$, where $\cdot_C$ is the induced action, and

- $D_N(M) = \langle \{ n \cdot m \mid n \in N,\ m \in M \} \rangle_M$,
- $[N, N] = \langle \{ [n, n'] \mid n, n' \in N \} \rangle_N$ is the commutator of the Lie K-algebra N.

Definition 3.1.3. We will say that a Lie crossed module $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ is *perfect* if it coincides with its commutator crossed submodule $\mathcal{M} = [\mathcal{M}, \mathcal{M}]$, i.e. $M = D_N(M)$ and $N = [N, N]$.

Definition 3.1.4. An *extension* in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ is a regular epimorphism, i.e. a surjective morphism.

Following the theory in [36], we have three kinds of extensions: trivial, normal and central.

An extension $\Phi \colon \mathcal{X} \twoheadrightarrow \mathcal{M}$ is *trivial* if the induced square

$$\begin{tikzcd}
\mathcal{X} \arrow[r, two heads, "\Phi"] \arrow[d, two heads] & \mathcal{M} \arrow[d, two heads] \\
\mathcal{X}_{\text{ab}} \arrow[r, two heads, "\Phi_{ab}"'] & \mathcal{M}_{\text{ab}}
\end{tikzcd}$$

is a pullback in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$, where $\mathcal{M}_{\text{ab}} = \frac{\mathcal{M}}{[\mathcal{M},\mathcal{M}]}$.

An extension is *normal* if one of the projections of the kernel pair is trivial.

An extension $\Phi\colon \mathcal{X} \twoheadrightarrow \mathcal{M}$ is called *central* if there exists another extension $\Psi\colon \mathcal{Y} \twoheadrightarrow \mathcal{M}$ such that π_2 (also denoted $\Psi^*(\Phi)$) in the pullback

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{M}} \mathcal{Y} \arrow[r, two heads, "\pi_1"] \arrow[d, two heads, "\pi_2"] & \mathcal{X} \arrow[d, two heads, "\Phi"] \\ \mathcal{Y} \arrow[r, two heads, "\Psi"'] & \mathcal{M}, \end{tikzcd}$$

is trivial.

In our semi-abelian context, the concepts of normal and central extension are equivalent, and a more practical characterization is the following:

An extension $\mathcal{X} = (X \xrightarrow{\delta} S, *) \xrightarrow{f=(f_1,f_2)} \mathcal{M} = (M \xrightarrow{\partial} N)$ is central if and only if $\ker(f) = (\ker(f_1) \xrightarrow{\delta|_{\ker(f_1)}} \ker(f_2), \cdot_{\ker})$ is a crossed submodule of the centre of $\mathcal{X}$, $Z(\mathcal{X})$.

The central extensions in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ of an object $\mathcal{M}$ constitute another category, whose morphisms are the morphisms of Lie crossed modules $\Theta\colon \mathcal{X} \longrightarrow \mathcal{Y}$ making commutative the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\Theta"] \arrow[dr, two heads, "\Phi"'] & & \mathcal{Y} \arrow[dl, two heads, "\Psi"] \\ & \mathcal{M}. & \end{tikzcd}$$

A central extension $\mathcal{U} \twoheadrightarrow \mathcal{M}$ is said to be *universal* (of $\mathcal{M}$) if it is the initial object in the category of central extensions of $\mathcal{M}$. From the definition, it is clear that the universal central extension is unique up to isomorphisms.

The universal central extension is entirely related to the concept of the non-abelian tensor product. In the case of groups, Brown and Loday in [11] defined the non-abelian tensor product of groups and proved that the universal central extension is the non-abelian tensor product $G \otimes G$ with the epimorphism $G \otimes G \twoheadrightarrow G$ sending $g_1 \otimes g_2$ to its commutator $[g_1, g_2]$. The same happens in Lie algebras' case with the non-abelian tensor product of Lie algebras introduced by Ellis in [19] (see Definition 2.5.4).

In the cases of crossed modules of groups [48] and Lie crossed modules [13], the notions of the non-abelian tensor products are also needed.

When we talk about $M \otimes M$ we will assume that M acts on itself by the adjoint action.

$\mathfrak{U}$ will denote the faithful forgetful functor $\mathfrak{U}: \boldsymbol{BX(LieAlg_K)} \to \boldsymbol{X(LieAlg_K)}$.

In the case of the braiding category $\boldsymbol{BX(LieAlg_K)}$, the idea of braiding changes a little the concepts of centre and commutator from the category of Lie crossed modules $\boldsymbol{X(LieAlg_K)}$, appearing the following subobjects using the definition given by Huq [35] in the general case.

Definition 3.1.5. Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$be a braided Lie crossed module.

The $\boldsymbol{B}$*-centre* of $\mathcal{M}$ is the braided crossed submodule

$$Z_{\boldsymbol{B}}(\mathcal{M}) = (M^N \xrightarrow{\partial|_{M^N}} \mathrm{Z}_B(N), \cdot_Z, \{-,-\}_Z),$$

where

$$\mathrm{Z}_B(N) = \{n \in N \mid \{n, n'\} = 0 = \{n', n\},\ n' \in N\},$$

with $\cdot_Z$ is the induced action and $\{-,-\}_Z$ the induced braided, i.e. the zero action and the zero braiding by the definition of M^N and $\mathrm{Z}_B(N)$.

The $\boldsymbol{B}$-centre is the centre [35] in the category $\boldsymbol{BX(LieAlg_K)}$.

Remark 3.1.6. It is easy to show that the following inclusions of subalgebras are true:

$$M^N \subset Z(M), \qquad Z_B(N) \subset Z(N) \cap \mathrm{st}_N(M).$$

Besides, if we use the properties (BXLie3) and (BXLie4), then we have that $M^N = \{m \in M \mid \partial(m) \in Z_B(N)\}$.

Definition 3.1.7. Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a braided Lie crossed module. The $\boldsymbol{B}$*-commutator braided crossed submodule* is given by

$$[\mathcal{M}, \mathcal{M}]_{\boldsymbol{B}} = (B_N(M) \xrightarrow{\partial|_{B_N(M)}} [N, N], \cdot_C, \{-,-\}_C)$$

where $\cdot_C$ and $\{-,-\}_C$ are the induced operations, and

$$\mathrm{B}_N(M) = \Big\langle \big\{ \{n, n'\} \mid n, n' \in N \big\} \Big\rangle_M.$$

The $\boldsymbol{B}$-commutator is the commutator [35] in the category $\boldsymbol{BX(LieAlg_K)}$.

Remark 3.1.8. $B_N(M)$ is an ideal of M, and we have the following inclusion of subalgebras:

$$[M, M] \subset D_N(M) \subset B_N(M).$$

Definition 3.1.9. A braided Lie crossed module Lie $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is ***B**-perfect* if it coincides with its $\boldsymbol{B}$-commutator braided crossed submodule, i.e. we have the equalities $M = B_N(M)$ and $N = [N, N]$.

Definition 3.1.10. An *extension of braided Lie crossed modules* is given by a morphism $\mathcal{X} \xrightarrow{(f_1,f_2)} \mathcal{Y}$ in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ such that f_1 and f_2 are surjective morphisms.

Besides, we will say that it is ***B**-central* (central in the category $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$) if $\ker(f_1, f_2)$ is a braided crossed submodule of $Z_{\boldsymbol{B}}(\mathcal{X})$, i.e. the kernel is "inside" the $\boldsymbol{B}$-centre.

Definition 3.1.11. We will say that an extension $\mathcal{X} \xrightarrow{(f_1,f_2)} \mathcal{Y}$ of braided Lie crossed modules is a *$\mathfrak{U}$-central extension* if $\mathfrak{U}(\mathcal{X}) \xrightarrow{\mathfrak{U}(f_1,f_2)} \mathfrak{U}(\mathcal{Y})$ is central in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$, i.e. ker $\big(\mathfrak{U}(f_1, f_2)\big)$ is a crossed submodule of the centre of $\mathfrak{U}(\mathcal{X})$, $Z\big(\mathfrak{U}(\mathcal{X})\big)$.

It is immediate that every $\boldsymbol{B}$-central extension in the category $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ is a $\mathfrak{U}$-central extension. The next example shows that not every $\mathfrak{U}$-central extension is a $\boldsymbol{B}$-central extension. Furthermore, it manifests that the concepts of $\boldsymbol{B}$-centre and $\boldsymbol{B}$-commutator of a braided crossed module are different from the notions of centre and commutator.

Example 3.1.12. Let $M \neq 0$ be an abelian K-Lie algebra of finite dimension n, i.e. M is isomorphic as vectorial space to K^n with the Lie bracket $[x, y] = 0$ for $x, y \in M$.

Using Example 2.5.7, we have that $(M \otimes M \xrightarrow{\partial} M, \cdot)$, with $\partial = 0$, $m \cdot (m_1 \otimes m_2) = m \otimes [m_1, m_2] = m \otimes 0 = 0$ and $\{m, m'\} = m \otimes m'$ is a braided Lie crossed module.

Note that, since M is abelian, we have that $M \otimes M$ is isomorphic as vector space to the usual tensor product, and its Lie bracket is 0.

(i) Let $\mathcal{X} = (M \otimes M \xrightarrow{0} M, 0, - \otimes -)$ be the braided Lie crossed module, where the tensor product is the usual one. It is easy to show that the correspondent subalgebras are the following ones:

- $(M \otimes M)^M = M \otimes M$, $\mathrm{st}_M(M \otimes M) = M$, $Z(M) = M$, and $Z_{\boldsymbol{B}}(M) = 0$.
- $D_M(M \otimes M) = 0$, $B_M(M \otimes M) = M \otimes M$, and $[M, M] = 0$.

It is clear that the centre $Z\big(\mathfrak{U}(\mathcal{X})\big) = (M \otimes M \xrightarrow{0} M)$ and the $\boldsymbol{B}$-centre $Z_{\boldsymbol{B}}(\mathcal{X}) = (M \otimes M \xrightarrow{0} 0, - \otimes -)$ are different in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$, i.e. $Z\big(\mathfrak{U}(\mathcal{X})\big) \neq \mathfrak{U}(Z_{\boldsymbol{B}}(\mathcal{X}))$. On the other hand, the commutator $[\mathfrak{U}(\mathcal{X}), \mathfrak{U}(\mathcal{X})] = (0 \xrightarrow{0} 0)$ and the $\boldsymbol{B}$-commutator $[\mathcal{X}, \mathcal{X}]_{\boldsymbol{B}} = (M \otimes M \xrightarrow{0} 0, - \otimes -)$ are also different in the crossed module category, i.e. $[\mathfrak{U}(\mathcal{X}), \mathfrak{U}(\mathcal{X})] \neq \mathfrak{U}([\mathcal{X}, \mathcal{X}]_{\boldsymbol{B}})$.

(ii) Now, we will show a $\mathfrak{U}$-central extension that is not a $\boldsymbol{B}$-central extension.

In particular, we have for $M = K^3$ and $M = K^2$, the braided Lie crossed modules $\mathcal{Y} = (K^3 \otimes K^3 \xrightarrow{0} K^3, 0, - \otimes -)$ and $\mathcal{Z} = (K^2 \otimes K^2 \xrightarrow{0} K^2, 0, - \otimes -)$.

By taking the projection $K^3 \xrightarrow{\pi} K^2$, $(x, y, z) \mapsto (x, y)$, we have that $\pi \otimes \pi : K^3 \otimes K^3 \to K^2 \otimes K^2$ is surjective and $\mathcal{Y} \xrightarrow{(\pi \otimes \pi, \pi)} \mathcal{Z}$ is an extension of braided Lie crossed modules.

It is immediate that $\ker(\pi \otimes \pi) \subset (K^3 \otimes K^3)^{K^3} = K^3 \otimes K^3$ and $\ker(\pi) \subset Z(K^3) \cap \mathrm{st}_{K^3}(K^3 \otimes K^3) = K^3$, i.e. $\ker(\pi \otimes \pi, \pi) \subset Z\big(\mathfrak{U}(\mathcal{Y})\big)$, and so the extension is a $\mathfrak{U}$-central extension.

However $0 \neq \ker(\pi) = \{(x, y, z) \in K^3 \mid x = y = 0\} \nsubseteq Z_{\boldsymbol{B}}(K^3) = 0$, and therefore the extension $\mathcal{Y} \xrightarrow{(\pi \otimes \pi, \pi)} \mathcal{Z}$ is not $\boldsymbol{B}$-central.

3.2 The universal $\boldsymbol{B}$-central extension for $\boldsymbol{B}$-perfect braided Lie crossed modules

Similar to Lie crossed modules we have the following definition of the universal central extension for the case of braided Lie crossed modules.

A $\boldsymbol{B}$-central extension $\mathcal{U} \xrightarrow{u} \mathcal{M}$ of $\mathcal{M}$ in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ is *universal* if it is the initial object in the category of $\boldsymbol{B}$-central extensions of $\mathcal{M}$, i.e. if for any other $\boldsymbol{B}$-central extension $\mathcal{Z} \xrightarrow{f} \mathcal{M}$ in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$, there is a unique morphism $h : \mathcal{U} \to \mathcal{Z}$ such that $u = f \circ h$.

In this section, we will find the expression of this universal initial object when it exists, and we will try to characterize this fact.

Lemma 3.2.1. *If* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a braided Lie crossed module, then* $N \otimes N \xrightarrow{\Phi_1} M$ *defined by* $n \otimes n' \mapsto \{n, n'\}$*, and* $N \otimes N \xrightarrow{\Phi_2} N$ *defined by* $n \otimes n' \mapsto [n, n']$*, are Lie* $\boldsymbol{K}$*-homomorphisms.*

Besides, Φ_1 *and* Φ_2 *are simultaneously surjective if and only if the braided Lie crossed module* $\mathcal{M}$ *is* $\boldsymbol{B}$*-perfect.*

Proof. Since Φ_1 and Φ_2 are determined by generators, we only need to prove that they are well defined to prove that they are Lie K-homomorphisms.

First, we will prove that the two morphisms preserve the relations (T1)–(T4).

(T1) and (T2) are preserved because $[-,-]$ and $\{-,-\}$ are K-bilinear.

Since the two actions on $N \otimes N$ are the Lie bracket of N, $[-,-]$, we can rewrite the relations (T3) and (T4) to obtain the following ones:

$$
\begin{aligned}
[n_1, n_2] \otimes n_3 &= n_1 \otimes [n_2, n_3] - n_2 \otimes [n_1, n_3], && \text{(T3)}\\
n_1 \otimes [n_2, n_3] &= [n_3, n_1] \otimes n_2 - [n_2, n_1] \otimes n_3, &&\\
[(n_1 \otimes n_2), (n_3 \otimes n_4)] &= [n_1, n_2] \otimes [n_3, n_4]. && \text{(T4)}
\end{aligned}
$$

Starting with (T3) we have:

$$
\begin{aligned}
\Phi_1([n_1, n_2] \otimes n_3) &= \{[n_1, n_2], n_3\} = \{n_1, [n_2, n_3]\} - \{n_2, [n_1, n_3]\}\\
&= \Phi_1(n_1 \otimes [n_2, n_3]) - \Phi_1(n_2 \otimes [n_1, n_3])\\
&= \Phi_1(n_1 \otimes [n_2, n_3] - n_2 \otimes [n_1, n_3]),
\end{aligned}
$$

where we have used (BXLie6).

We will see now the second relation in (T3):

$$
\begin{aligned}
\Phi_1(n_1 \otimes [n_2, n_3]) &= \{n_1, [n_2, n_3]\} = \{[n_1, n_2], n_2\} - \{[n_1, n_3], n_2\}\\
&= \Phi_1([n_1, n_2] \otimes n_3 - [n_1, n_3] \otimes n_2)\\
&= \Phi_1(-[n_2, n_1] \otimes n_3 + [n_3, n_1] \otimes n_2)\\
&= \Phi_1([n_3, n_1] \otimes n_2 - [n_2, n_1] \otimes n_3),
\end{aligned}
$$

where we have used (BXLie5).

For Φ_2 is true using a similar argument together with the Jacobi identity in both equalities.

The proof of (T4) for Φ_2 follows since both equalities are $[[n_1, n_2], [n_3, n_4]]$ after applying Φ_2.

For Φ_1 we have the following equalities:

$$\begin{aligned}\Phi_1([n_1 \otimes n_2, n_3 \otimes n_4]) &= [\Phi_1(n_1 \otimes n_2), \Phi_1(n_3 \otimes n_4)] = [\{n_1, n_2\}, \{n_3, n_4\}]\\ &= \{\partial\{n_1, n_2\}, \partial\{n_3, n_4\}\} = \{[n_1, n_2], [n_3, n_4]\}\\ &= \Phi_1([n_1, n_2] \otimes [n_3, n_4]),\end{aligned}$$

where we have used (BXLie2) and (BXLie1).

So, Φ_1 and Φ_2 are well defined and are Lie K-homomorphisms.

For the second part, we have that $\operatorname{Im}\Phi_1 = B_N(M)$ and $\operatorname{Im}\Phi_2 = [N, N]$. Therefore, Φ_1 and Φ_2 are simultaneously surjective if and only if the braided Lie crossed module is $\boldsymbol{B}$-perfect. □

Lemma 3.2.2. *Let* $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *be a braided Lie crossed module, and consider the braided Lie crossed module* $(N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-])$ *(see Example 2.3.8 (1)).*

Then $(\Phi_1, \Phi_2)\colon (N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-]) \longrightarrow (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a morphism in* ***BXLie****, with* Φ_1 *and* Φ_2 *defined in Lemma 3.2.1,.*

Besides, $\ker(\Phi_1) \subset (N \otimes N)^{(N\otimes N)}$ *and* $\ker(\Phi_2) \subset \mathrm{Z}_{\boldsymbol{B}}(N \otimes N)$.

Proof. For the proof, we will denote the action $[-,-]$ of $N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N$ as $*$, and its braiding as $[\![-,-]\!]$.

First, we will show (XLieH1). Let $n \otimes n', n'' \otimes n''' \in N \otimes N$.

$$\begin{aligned}\Phi_1((n \otimes n') * (n'' \otimes n''')) &= \Phi_1([n \otimes n', n'' \otimes n''']) = \Phi_1([n, n'] \otimes [n'', n'''])\\ &= \{[n, n'], [n'', n''']\} = \{[n, n'], \partial\{n'', n'''\}\}\\ &= [n, n'] \cdot \{n'', n'''\} = \Phi_2(n \otimes n') \cdot \Phi_1(n'' \otimes n'''),\end{aligned}$$

where we have used (BXLie1) and (BXLie4).

Now, we will show (XLieH1).

$$\partial \circ \Phi_1(n \otimes n') = \partial\{n, n'\} = [n, n'] = \Phi_2(\mathrm{Id}_{N\otimes N}(n \otimes n')),$$

where we have used (BXLie2).

Now, we will prove (BXLieH3).

$$\begin{aligned}\Phi_1(\llbracket n \otimes n', n'' \otimes n''' \rrbracket) &= \Phi_1([n \otimes n', n'' \otimes n''']) = \Phi_1([n, n'] \otimes [n'', n''']) \\ &= \{[n, n'], [n'', n''']\} = \{\Phi_2(n \otimes n'), \Phi_2(n'' \otimes n''')\}.\end{aligned}$$

So, (Φ_1, Φ_2) is a morphism in ***BXLie***. We will now prove that the inclusions hold.

If $n \otimes n' \in \ker(\Phi_1)$ then $\{n, n'\} = 0$. Using (BXLie1) we have that $0 = \partial\{n, n'\} = [n, n']$.

Since $(N \otimes N)^{(N\otimes N)} = \{x \in N \otimes N \mid (n'' \otimes n''') * x = 0,\ n'' \otimes n''' \in N \otimes N\}$ (it is enough to work on generators), we have

$$(n'' \otimes n''') * (n \otimes n') = [n'' \otimes n''', n \otimes n'] = [n'', n'''] \otimes [n, n'] = [n'', n'''] \otimes 0 = 0.$$

Therefore, we have that $n \otimes n' \in (N \otimes N)^{(N\otimes N)}$ and $\ker(\Phi_1) \subset (N \otimes N)^{(N\otimes N)}$.

For the second inclusion, we take $n \otimes n' \in \ker(\Phi_2)$, i.e. $[n, n'] = 0$.

Since it is enough to work on generators, we have that

$$\mathrm{Z}_B(N\otimes N) = \{x \in N\otimes N \mid \llbracket x, n''\otimes n''' \rrbracket = 0 = \llbracket n''\otimes n''', x \rrbracket,\ n''\otimes n''' \in N\otimes N\}.$$

Taking into account that

$$\begin{aligned}\llbracket n'' \otimes n''', n \otimes n' \rrbracket &= [n'' \otimes n''', n \otimes n'] = [n'', n'''] \otimes [n, n'] = [n'', n'''] \otimes 0 = 0, \\ \llbracket n \otimes n', n'' \otimes n''' \rrbracket &= [n \otimes n', n'' \otimes n'''] = [n, n'] \otimes [n'', n'''] = 0 \otimes [n'', n'''] = 0,\end{aligned}$$

we deduce $n \otimes n' \in Z_B(N \otimes N)$, which proves that $\ker(\Phi_2) \subset Z_B(N \otimes N)$. □

Corollary 3.2.3. *The morphism given in Lemma 3.2.2 is a **B**-central extension if and only if $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is a **B**-perfect braided Lie crossed module.*

Proof. It will be a $\boldsymbol{B}$-central extension if and only if (Φ_1, Φ_2) is an extension, since Lemma 3.2.2 establishes the two inclusions and they have the restricted operations as a braided Lie crossed module.

Moreover, (Φ_1, Φ_2) is an extension if and only if Φ_1 and Φ_2 are simultaneously surjective, and by Lemma 3.2.1 that it happens if and only if the braided Lie crossed module $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect. □

Proposition 3.2.4. *If* $(X_1 \xrightarrow{\delta} X_2, *, (\!(-,-)\!)) \xrightarrow{f=(f_1,f_2)} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a* $\boldsymbol{B}$-*central extension, then we have a morphism in* $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$,

$$h\colon (N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-]) \longrightarrow (X_1 \xrightarrow{\delta} X_2, *, (\!(-,-)\!)),$$

defined by:

- $h_1 : N \otimes N \to X_1$, $n \otimes \eta \mapsto (\!(\overline{n}, \overline{\eta})\!)$, *where* $\overline{n}, \overline{\eta} \in X_2$ *are elements such that* $f_2(\overline{n}) = n$ *and* $f_2(\overline{\eta}) = \eta$*;*
- $h_2 : N \otimes N \to X_2$, $n \otimes \eta \mapsto [\overline{n}, \overline{\eta}]$, *where* $\overline{n}, \overline{\eta} \in X_2$ *are elements such that* $f_2(\overline{n}) = n$ *and* $f_2(\overline{\eta}) = \eta$.

Besides, $f \circ h = \Phi$, *i.e.* h *is a morphism between the extensions.*

Proof. We need to prove that h_1 and h_2 are well defined.

We will start with h_1. We will take $\overline{n}, \widetilde{n}, \overline{\eta}, \widetilde{\eta} \in X_2$ such that $f_2(\overline{n}) = f_2(\widetilde{n}) = n$ and $f_2(\overline{\eta}) = f_2(\widetilde{\eta}) = \eta$ and prove that $(\!(\overline{n}, \overline{\eta})\!) = (\!(\widetilde{n}, \widetilde{\eta})\!)$.

Since $f_2(\overline{n}) = f_2(\widetilde{n})$ and $f = (f_1, f_2)$ is a $\boldsymbol{B}$-central extension, we have that $\overline{n} - \widetilde{n} \in \ker(f_2) \subset \mathrm{Z}_{\boldsymbol{B}}(X_2)$. By the definition of $\mathrm{Z}_{\boldsymbol{B}}(X_2)$ we get that $(\!(\overline{n} - \widetilde{n}, \overline{\eta})\!) = 0$ and so $(\!(\overline{n}, \overline{\eta})\!) = (\!(\widetilde{n}, \overline{\eta})\!)$.

Using an analogue reasoning, we have that $\overline{\eta} - \widetilde{\eta} \in \ker(f_2) \subset \mathrm{Z}_{\boldsymbol{B}}(X_2)$, and, $(\!(\widetilde{n}, \overline{\eta} - \widetilde{\eta})\!) = 0$. So $(\!(\widetilde{n}, \overline{\eta})\!) = (\!(\widetilde{n}, \widetilde{\eta})\!)$.

With both equalities, we have that $(\!(\overline{n}, \overline{\eta})\!) = (\!(\widetilde{n}, \overline{\eta})\!) = (\!(\widetilde{n}, \widetilde{\eta})\!)$, and h_1 is independent of the choice.

Since $\mathrm{Z}_{\boldsymbol{B}}(X_2) \subset \mathrm{Z}(X_2)$ we can change the proof for h_1 taking the equalities for $[-,-]$ instead of $(\!(-,-)\!)$ which proves that h_2 is independent of the choice.

We can use an analogue argument as in Lemma 3.2.1 to prove that h_1 and h_2 are well defined, i.e. they preserve the relations. So, they are Lie K-homomorphisms since they are determined on generators.

To prove that $h = (h_1, h_2)$ is a morphism of braided Lie crossed modules, we also use similar reasoning as the one done in Lemma 3.2.2, since we can make the changes in the choice inside the braidings and brackets.

To finish, if $n \otimes \eta \in N \otimes N$, then

$$f_1 \circ h_1(n \otimes \eta) = f_1(\langle\!\langle \overline{n}, \overline{\eta} \rangle\!\rangle) = \{f_2(\overline{n}), f_2(\overline{\eta})\} = \{n, \eta\} = \Phi_1(n \otimes \eta),$$
$$f_2 \circ h_2(n \otimes \eta) = f_2([\overline{n}, \overline{\eta}]) = [f_2(\overline{n}), f_2(\overline{\eta})] = [n, \eta] = \Phi_2(n \otimes \eta).$$

Therefore, $f \circ h = \Phi$. □

Lemma 3.2.5. *If N is a perfect Lie K-algebra, i.e. $N = [N, N]$, then*

$$(N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-])$$

is a $\boldsymbol{B}$-perfect braided Lie crossed module.

In particular, if $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is a $\boldsymbol{B}$-perfect braided Lie crossed module, then $(N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-])$ is a $\boldsymbol{B}$-perfect braided Lie crossed module.

Proof. Since the braiding in $(N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-])$ is the bracket, we have that $[N \otimes N, N \otimes N] = B_{N\otimes N}(N \otimes N)$, and so it is enough to prove that $[N \otimes N, N \otimes N] = N \otimes N$.

Moreover, it is enough to prove that the generators $[n_1, n_2] \otimes [n_3, n_4]$ are inside $[N \otimes N, N \otimes N]$ since $N = [N, N]$. Using (T4) we have that $[n_1, n_2] \otimes [n_3, n_4] = [n_1 \otimes n_2, n_3 \otimes n_4] \in [N \otimes N, N \otimes N]$.

For the second part, if $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect, then $N = [N, N]$, and we conclude using the first part. □

Proposition 3.2.6. *Let $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \xrightarrow{\Psi} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a morphism of braided Lie crossed modules such that $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect.*

If $(X_1 \xrightarrow{\rho} X_2, *, (\!(-,-)\!)) \overset{f}{\twoheadrightarrow} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a* $\boldsymbol{B}$*-central extension and exists* $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \xrightarrow{h} (X_1 \xrightarrow{\rho} X_2, *, (\!(-,-)\!))$ *such that* $\Psi = f \circ h$*, then* h *is the unique that satisfies the equality.*

Proof. Suppose that there are $g, h\colon (Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \longrightarrow (X_1 \xrightarrow{\rho} X_2, *, (\!(-,-)\!))$ such that $\Psi = f \circ h = f \circ g$, i.e. $\Psi_1 = f_1 \circ h_1 = f_1 \circ g_1$ and $\Psi_2 = f_2 \circ h_2 = f_2 \circ g_2$.

If $y \in Y_2$ then $f_2 \circ h_2(y) = f_2 \circ g_2(y)$, i.e. $h_2(y) - g_2(y) \in \ker(f_2)$. Then there is $k_y \in \ker(f_2)$ such that $h_2(y) = g_2(y) + k_y$. Since f is a $\boldsymbol{B}$-central extension we have that $\ker(f_2) \subset \mathrm{Z}_B(X_2) \subset \mathrm{Z}(X_2)$. If we take $y, z \in Y_2$, and since $k_y, k_z \in \mathrm{Z}(X_2)$, we have

$$[k_y, g_2(z)] = [k_y, k_z] = [g_2(y), k_z] = 0.$$

Using this fact, we have:

$$\begin{aligned} h_2([y,z]) = [h_2(y), h_2(z)] &= [g_2(y) + k_y, g_2(z) + k_z] \\ &= [g_2(y), g_2(z)] + [k_y, g_2(z)] + [k_y, k_z] + [g_2(y), k_z] \\ &= [g_2(y), g_2(z)] = g_2([y,z]). \end{aligned}$$

So, $g_2 = h_2$ since $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect.

Besides, since $\ker(f_2) \subset \mathrm{Z}_B(X_2)$, for $y, z \in Y_2$, we have that:

$$\begin{aligned} h_1([\![y,z]\!]) &= (\!(h_2(y), h_2(z))\!) = (\!(g_2(y) + k_y, g_2(z) + k_z)\!) \\ &= (\!(g_2(y), g_2(z))\!) + (\!(k_y, g_2(z))\!) + (\!(k_y, k_z)\!) + (\!(g_2(y), k_z)\!) \\ &= (\!(g_2(y), g_2(z))\!) = g_1([\![y,z]\!]), \end{aligned}$$

where we have used that $k_y, k_z \in \mathrm{Z}_B(X_2)$.

Therefore, $g_1 = h_1$ because $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect, i.e. $Y_1 = B_{Y_2}(Y_1)$ is generated by the images of the braiding. □

Corollary 3.2.7. *If* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a* $\boldsymbol{B}$*-perfect Lie braided crossed module, then*

$$\mathcal{U} = (N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-]) \overset{\Phi=(\Phi_1,\Phi_2)}{\twoheadrightarrow} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\}) \qquad \text{(UBCE)}$$

is the universal $\boldsymbol{B}$-central extension of $\mathcal{M}$, where Φ_1, Φ_2 were defined in Lemma 3.2.1.

Proof. Since $\mathcal{M}$ is $\boldsymbol{B}$-perfect, Corollary 3.2.3 states that the morphism $\mathcal{U} \overset{\Phi}{\twoheadrightarrow} \mathcal{M}$ is a $\boldsymbol{B}$-central extension.

We need to prove that it is universal.

If we have another $\boldsymbol{B}$-central extension $\mathcal{X} \overset{f}{\twoheadrightarrow} \mathcal{M}$ then by Proposition 3.2.4 there is h such that $\Phi = f \circ h$.

The uniqueness of this morphism is given by Proposition 3.2.6. We can use the previous proposition since $\mathcal{U}$ is $\boldsymbol{B}$-perfect by Lemma 3.2.5 and the fact that $\mathcal{M}$ is $\boldsymbol{B}$-perfect. □

Let us see the converse of Corollary 3.2.7.

Proposition 3.2.8. *Let $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \overset{\Psi}{\twoheadrightarrow} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be an extension in **BXLie** such that $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect. Then $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect.*

Proof. Ψ_1 and Ψ_2 are surjective maps since Ψ is an extension, and $Y_1 = B_{Y_2}(Y_1)$ and $Y_2 = [Y_2, Y_2]$ because $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect.

Since the elements $[\![y,z]\!]$, with $y, z \in Y_2$ are the generators of Y_1, we have that $\Psi_1([\![y,z]\!])$ are the generators of $\operatorname{Im}\Psi_1 = M$. Since $\Psi_1([\![y,z]\!]) = \{\Phi_2(y), \Phi_2(z)\}$, we get that the generators of M are braided elements and $M = B_N(M)$.

We know that the elements $[y,z]$, with $y, z \in Y_2$, are the generators of Y_2. Therefore, $\Phi_2([y,z]) = [\Phi_2(y), \Phi_2(z)]$ are the generators of $\operatorname{Im}\Phi_2 = N$, and then $N = [N, N]$.

So, $(M \xrightarrow{\partial} N \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect. □

Lemma 3.2.9. *Let $\mathcal{Y} = (Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \overset{\Psi}{\twoheadrightarrow} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\boldsymbol{B}$-central extension in **BXLie** such that $\mathcal{Y}$ is not $\boldsymbol{B}$-perfect. Then exists another extension $\mathcal{X} \overset{f}{\twoheadrightarrow} \mathcal{M}$ and two different morphisms $h, g : \mathcal{Y} \to \mathcal{X}$ such that $\Psi = f \circ h = f \circ g$.*

Proof. Let $(B_{Y_2}(Y_1) \xrightarrow{\varrho|_{B_{Y_2}(Y_1)}} [Y_2, Y_2], \star_C, [\![-,-]\!]_C) \xrightarrow{i=(i_1,i_2)} (Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ be the inclusion morphism of the $\boldsymbol{B}$-commutator braided crossed submodule.

Taking the cokernel of i we have the Lie crossed module

$$\overline{\mathcal{Y}} = \left(\frac{Y_1}{B_{Y_2}(Y_1)} \xrightarrow{\overline{\varrho}} \frac{Y_2}{[Y_2,Y_2]}, \overline{\star}, [\![-,-]\!]\right).$$

We will denote in the same way, by abuse of notation, the braidings in $\mathcal{Y}$ and in its quotient $\overline{\mathcal{Y}}$. We will represent the elements in $\frac{Y_1}{B_{Y_2}(Y_1)}$ as $\overline{x}$, $x \in Y_1$, and the ones in $\frac{Y_2}{[Y_2,Y_2]}$ as $\overline{y}$, $y \in Y_2$.

We take now the product in the category $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ and we construct $\mathcal{M}\times\overline{\mathcal{Y}}$. We denote as π^1 the first projection morphism. Since π^1_1 and π^1_2 are surjective maps, we have that $\mathcal{M} \times \overline{\mathcal{Y}} \overset{\pi^1}{\twoheadrightarrow} \mathcal{M}$ is an extension. We will denote the braiding in the product as $\{\!\![-,-]\!\!\}$.

We will prove that it is a $\boldsymbol{B}$-central extension, i.e. we need to prove the inclusions $\ker(\pi^1_1) \subset \left(M \times \frac{Y_1}{B_{Y_2}(Y_2)}\right)^{\left(N\times\frac{Y_2}{[Y_2,Y_2]}\right)}$ and $\ker(\pi^1_2) \subset \mathrm{Z}_B\left(N \times \frac{Y_2}{[Y_2,Y_2]}\right)$.

If $a \in \ker(\pi^1_1)$ then $a = (0,\overline{x})$ with $x \in Y_1$.

If we take $(n,\overline{y}) \in N \times \frac{Y_2}{[Y_2,Y_2]}$ then:

$$(n,\overline{y})(\cdot \times \overline{\star})(0,\overline{x}) = (n\cdot 0, \overline{y}\,\overline{\star}\,\overline{x}) = (0, \overline{y \star x}).$$

But $\overline{y\star x} = \overline{0}$ since $y \star x \in D_{Y_2}(Y_1) \subset B_{Y_2}(Y_1)$.

So $\ker(\pi^1_1) \subset \left(M \times \frac{Y_1}{B_{Y_2}(Y_2)}\right)^{\left(N\times\frac{Y_2}{[Y_2,Y_2]}\right)}$.

If $a \in \ker(\pi^1_2)$ then $a = (0,\overline{y})$ with $y \in Y_2$. If we take $(n,\overline{y_1}) \in N \times \frac{Y_2}{[Y_2,Y_2]}$ then:

$$\{\!\![(0,\overline{y}),(n,\overline{y_1})]\!\!\} = (\{0,n\}, [\![\overline{y},\overline{y_1}]\!]) = (0, \overline{[\![y,y_1]\!]}),$$
$$\{\!\![(n,\overline{y_1}),(0,\overline{y})]\!\!\} = (\{n,0\}, [\![\overline{y_1},\overline{y}]\!]) = (0, \overline{[\![y_1,y]\!]}).$$

Moreover, $\overline{[\![y,y_1]\!]} = \overline{[\![y_1,y]\!]} = \overline{0}$ since $[\![y_1,y]\!], [\![y,y_1]\!] \in B_{Y_2}(Y_1)$.

Therefore $\ker(\pi^1_2) \subset \mathrm{Z}_B\left(N \times \frac{Y_2}{[Y_2,Y_2]}\right)$, and so π^1 is a $\boldsymbol{B}$-central extension.

If $i^c : \mathcal{Y} \twoheadrightarrow \overline{\mathcal{Y}}$ is the cokernel of i, then we have two morphisms, induced by the product, with domain $\mathcal{Y}$ and $\mathcal{M} \times \overline{\mathcal{Y}}$ as codomain. They are $h = (\Psi, 0)$ and

$g = (\Psi, i^c)$. Since they are induced by the universal property of the product, we have that $\Psi = \pi^1 \circ h = \pi^1 \circ g$.

To finish the proof, we only must prove that they are different. Since the braided Lie crossed module $\mathcal{Y}$ is not $\boldsymbol{B}$-perfect and i_1^c and i_2^c are surjective we know that $i_1^c \neq 0$ or $i_2^c \neq 0$ (if both were the zero morphisms, then $\mathcal{Y}$ would be $\boldsymbol{B}$-perfect), and so $h \neq g$. □

Corollary 3.2.10. *If $\mathcal{M}$ is a braided Lie crossed module, then its universal $\boldsymbol{B}$-central extension, if it exists, is $\boldsymbol{B}$-perfect.*

Proof. If the universal extension is not $\boldsymbol{B}$-perfect, then using Lemma 3.2.9 we have another $\boldsymbol{B}$-central extension $\mathcal{X} \twoheadrightarrow \mathcal{M}$ for which there exist two different morphisms from the universal $\boldsymbol{B}$-central extension to $\mathcal{X} \twoheadrightarrow \mathcal{M}$, which contradicts the universality. □

Theorem 3.2.11. *A braided Lie crossed module admits a universal $\boldsymbol{B}$-central extension if and only if it is $\boldsymbol{B}$-perfect.*

Proof. It is a consequence of Corollary 3.2.7, Corollary 3.2.10 and Proposition 3.2.8. □

3.3 Braiding on a universal extension of Lie crossed modules

Universal central extensions of braided crossed modules of groups are not studied in [26]. However, the author constructed a canonical braiding on the universal central extension of a crossed module of groups [48], when the given crossed module is braided as well, and showed that it was universal in a sense that we will explain in this section.

In this part of the paper, we will consider braided Lie crossed modules extensions, but unlike the previous section, we will construct a braiding on the universal central extension of a braided Lie crossed module though as Lie crossed module and with the centre in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$, which we have called $\mathfrak{U}$-central extension. In this sense, we will obtain similar results given by Fukushi in [26] for crossed modules of groups in the category $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$.

Casas and Ladra in [13] proved that the universal central extension of a perfect Lie crossed module $(M \to N, \cdot)$ in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ is given by:

$$(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *) \xrightarrow{c=(c_1,c_2)} (M \xrightarrow{\partial} N, \cdot), \tag{UCE}$$

where $N \otimes M$ is given by the actions $\cdot$ of N on M and $m \star n = [\partial(m), n]$ of M on N; the action of $N \otimes N$ on $N \otimes M$ is given by $(n \otimes n') * (n'' \otimes m) = [[n, n'], n''] \otimes m + n'' \otimes [n, n'] \cdot m$ for $n, n', n'' \in N, m \in M$; and the morphisms are $c_1(n \otimes m) = n \cdot m$ and $c_2(n \otimes n') = [n, n']$.

Proposition 3.3.1. *If* $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a braided Lie crossed module then* $\{\!\!\{-,-\}\!\!\} : (N \otimes N) \times (N \otimes N) \to N \otimes M$, *defined on generators by* $\{\!\!\{n \otimes n', n'' \otimes n'''\}\!\!\} = [n, n'] \otimes \{n'', n'''\}$, *is a braiding for the Lie crossed module* $(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *)$.

Proof. The braiding $\{\!\!\{-,-\}\!\!\}$ is well defined since it preserves the relations (T1) and (T2) using the K-bilinearity of $[-,-]$ and $\{-,-\}$, and (T3) and (T4) are fulfilled too since $\{-,-\}$ and $[-,-]$ satisfy it.

It is sufficient to prove the axioms of braidings. Let $n, n, n', n'' \in N$, $m, m' \in M$. Then

$$(\mathrm{Id}_N \otimes \partial)(\{\!\!\{n \otimes n', n'' \otimes n'''\}\!\!\}) = (\mathrm{Id}_N \otimes \partial)([n, n'] \otimes \{n'', n'''\}) = [n, n'] \otimes \partial\{n'', n'''\}$$
$$= [n, n'] \otimes [n'', n'''] = [n \otimes n', n'' \otimes n'''] \text{ (BXLie1)},$$

$$\begin{aligned} &\{\!\!\{(\mathrm{Id}_N \otimes \partial)(n \otimes m), (\mathrm{Id}_N \otimes \partial)(n' \otimes m')\}\!\!\} \\ &= \{\!\!\{n \otimes \partial(m), n' \otimes \partial(m')\}\!\!\} = [n, \partial(m)] \otimes \{n', \partial(m')\} \\ &= -(m \star n) \otimes (n' \cdot m') = [n \otimes m, n' \otimes m'] \text{ (BXLie2)}, \end{aligned}$$

$$\begin{aligned} &\{\!\!\{(\mathrm{Id}_N \otimes \partial)(n \otimes m), n' \otimes n''\}\!\!\} = \{\!\!\{n \otimes \partial(m), n' \otimes n''\}\!\!\} = [n, \partial(m)] \otimes \{n', n''\} \\ &= -(m \star n) \otimes \{n', n''\} = n \otimes [m, \{n', n''\}] - \{n', n''\} \star n \otimes m \\ &= n \otimes \{\partial(m), \partial(\{n', n''\})\} - [\partial(\{n', n''\}), n] \otimes m \end{aligned}$$

$$= -n \otimes [n', n''] \cdot m - [[n', n''], n] \otimes m = -(n' \otimes n'') * (n \otimes m) \text{ (BXLie3)},$$

where we have used the second relation of (T3) in the third equality.

$$\begin{aligned}
&\{\!\{ n' \otimes n'', (\mathrm{Id}_N \otimes \partial)(n \otimes m) \}\!\} = \{\!\{ n' \otimes n'', n \otimes \partial(m) \}\!\} = [n', n''] \otimes \{n, \partial(m)\} \\
&= [n', n''] \otimes (n \cdot m) = n \otimes [n, n'] \cdot m + [[n', n''], n] \otimes m \\
&= (n' \otimes n'') * (n \otimes m) \text{ (BXLie4)},
\end{aligned}$$

where we have used the first relation of (T3) in the third equality.

$$\begin{aligned}
&\{\!\{ n_1 \otimes n_1', [n_2 \otimes n_2', n_3 \otimes n_3'] \}\!\} = \{\!\{ n_1 \otimes n_1', [n_2 \otimes n_2'] \otimes [n_3 \otimes n_3'] \}\!\} \\
&= [n_1, n_1'] \otimes \{[n_2, n_2'], [n_3, n_3']\} = [n_1, n_1'] \otimes [\{n_2, n_2'\}, \{n_3, n_3'\}] \\
&= (\{n_3, n_3'\} \star [n_1, n_1']) \otimes \{n_2, n_2'\} - (\{n_2, n_2'\} \star [n_1, n_1']) \otimes \{n_3, n_3'\} \\
&= [\partial(\{n_3, n_3'\}), [n_1, n_1']] \otimes \{n_2, n_2'\} - [\partial(\{n_2, n_2'\}), [n_1, n_1']] \otimes \{n_3, n_3'\} \\
&= [[n_3, n_3'], [n_1, n_1']] \otimes \{n_2, n_2'\} - [[n_2, n_2'], [n_1, n_1']] \otimes \{n_3, n_3'\} \\
&= -[[n_1, n_1'], [n_3, n_3']] \otimes \{n_2, n_2'\} + [[n_1, n_1'], [n_2, n_2']] \otimes \{n_3, n_3'\} \\
&= -\{\!\{ [n_1, n_1'] \otimes [n_3, n_3'], n_2 \otimes n_2' \}\!\} + \{\!\{ [n_1, n_1'] \otimes [n_2, n_2'], n_3 \otimes n_3' \}\!\} \\
&= \{\!\{ [n_1 \otimes n_1', n_2 \otimes n_2'], n_3 \otimes n_3' \}\!\} - \{\!\{ [n_1 \otimes n_1', n_3 \otimes n_3'], n_2 \otimes n_2' \}\!\} \text{ (BXLie5)},
\end{aligned}$$

$$\begin{aligned}
&\{\!\{ [n_1 \otimes n_1', n_2 \otimes n_2'], n_3 \otimes n_3' \}\!\} = \{\!\{ [n_1, n_1'] \otimes [n_2, n_2'], n_3 \otimes n_3' \}\!\} \\
&= [[n_1, n_1'], [n_2, n_2']] \otimes \{n_3, n_3'\} \\
&= [n_1, n_1'] \otimes [n_2, n_2'] \cdot \{n_3, n_3'\} - [n_2, n_2'] \otimes [n_1, n_1'] \cdot \{n_3, n_3'\} \\
&= [n_1, n_1'] \otimes \{[n_2, n_2'], \partial(\{n_3, n_3'\})\} - [n_2, n_2'] \otimes \{[n_1, n_1'], \partial(\{n_3, n_3'\})\} \\
&= [n_1, n_1'] \otimes \{[n_2, n_2'], [n_3, n_3']\} - [n_2, n_2'] \otimes \{[n_1, n_1'], [n_3, n_3']\} \\
&= \{\!\{ n_1 \otimes n_1', [n_2, n_2'] \otimes [n_3, n_3'] \}\!\} - \{\!\{ n_2 \otimes n_2', [n_1, n_1'] \otimes [n_3, n_3'] \}\!\} \\
&= \{\!\{ n_1 \otimes n_1', [n_2 \otimes n_2', n_3 \otimes n_3'] \}\!\} - \{\!\{ n_2 \otimes n_2', [n_1 \otimes n_1', n_3 \otimes n_3'] \}\!\} \text{ (BXLie6)}.
\end{aligned}$$

In all equalities, we have used the properties of $\{-,-\}$ and relations of the tensor product. $\square$

Proposition 3.3.2. *If* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a braided Lie crossed module such that* $\mathfrak{U}(\mathcal{M})$ *is perfect, then*

$$(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *, \{\!\!\{-,-\}\!\!\}) \xrightarrow{c=(c_1,c_2)} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$$

is a $\mathfrak{U}$*-central extension, where* $c_1(n \otimes m) = n \cdot m$ *and* $c_2(n \otimes n') = [n, n']$*, and* $\{\!\!\{-,-\}\!\!\}$ *is defined in Proposition 3.3.1.*

Proof. Since $\mathfrak{U}(\mathcal{M}) = (M \to N, \cdot)$ is a perfect Lie crossed module we have the central extension $(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *) \xrightarrow{c=(c_1,c_2)} (M \xrightarrow{\partial} N, \cdot)$ in ***XLie*** (see [13]).

Now, we will prove that c respects the braiding.

$$\begin{aligned} &c_1(\{\!\!\{n_1 \otimes n_1', n_2 \otimes n_2'\}\!\!\}) = c_1([n_1, n_1'] \otimes \{n_2, n_2'\}) = [n_1, n_1'] \cdot \{n_2, n_2'\} \\ &= \{[n_1, n_1'], \partial(\{n_2, n_2'\})\} = \{[n_1, n_1'], [n_2, n_2']\} = \{c_2(n_1 \otimes n_1), c_2(n_2 \otimes n_2')\}. \end{aligned}$$

So, c is a $\mathfrak{U}$-central extension. □

Now, we will provide a similar result to the one Fukushi given in [26] for the case of central extensions of braided crossed modules of groups. A $\mathfrak{U}$-central extension $\mathcal{V} \xrightarrow{v} \mathcal{M}$ of $\mathcal{M}$ in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ is *universal* if it is the initial object in the category of $\mathfrak{U}$-central extensions of $\mathcal{M}$.

Proposition 3.3.3. *If* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a braided Lie crossed module such that* $\mathfrak{U}(\mathcal{M})$ *is perfect, then*

$$\mathcal{V} = (N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *, \{\!\!\{-,-\}\!\!\}) \xrightarrow{c=(c_1,c_2)} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\}), \tag{UUCE}$$

is the universal $\mathfrak{U}$*-central extension of* $\mathcal{M}$*.*

Moreover, the universal initial morphism is the same as in the universality of the non-braiding case.

Proof. Let $(X_1 \xrightarrow{\delta} X_2, \diamond, (\!(-,-)\!)) \xrightarrow{f} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\mathfrak{U}$-central extension of braided Lie crossed modules.

Since $\mathfrak{U}(\mathcal{M})$ is perfect, we have that $\mathfrak{U}(\mathcal{V}) \xrightarrow{c=(c_1,c_2)} \mathfrak{U}(\mathcal{M})$ is the universal central extension in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ (UCE). Therefore, there exists a unique morphism in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$

$$(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *) \xrightarrow{h=(h_1,h_2)} (X_1 \xrightarrow{\delta} X_2, \diamond),$$

defined as $h_1(n \otimes m) = \overline{n} \diamond \overline{m}$ and $h_2(n \otimes n') = [\overline{n}, \overline{n'}]$ where $f_1(\overline{m}) = m$, $f_2(\overline{n}) = n$ and $f_2(\overline{n'}) = n'$, which satisfy $c = f \circ h$.

We check that h is a morphism in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ showing that preserves the braidings $\{\!\{-,-\}\!\}$ and $(\!(-,-)\!)$.

Let $n_1, n_2, \eta_1, \eta_2 \in N$. Then

$$\begin{aligned}
h_1(\{\!\{n_1 \otimes \eta_1, n_2 \otimes \eta_2\}\!\}) = h_1([n_1,\eta_1] \otimes \{n_2,\eta_2\}) &= \overline{[n_1,\eta_1]} \diamond \overline{\{n_2,\eta_2\}} \\
&= [\overline{n_1},\overline{\eta_1}] \diamond (\!(\overline{n_2},\overline{\eta_2})\!) = (\!([\overline{n_1},\overline{\eta_1}], \delta((\!(\overline{n_2},\overline{\eta_2})\!)))\!) \\
&= (\!([\overline{n_1},\overline{\eta_1}], [\overline{n_2},\overline{\eta_2}])\!) = (\!(h_2(n_1 \otimes \eta_1), h_2(n_2 \otimes \eta_2))\!),
\end{aligned}$$

since $f_2(\overline{[n_1,\eta_1]}) = [n_1,\eta_1] = [f_2(\overline{n_1}), f_2(\overline{\eta_1})] = f_2([\overline{n_1},\overline{\eta_1}])$ being f_2 a Lie K-homomorphism, and $f_1(\overline{\{n_2,\eta_2\}}) = \{n_2,\eta_2\} = \{f_2(\overline{n_1}), f_2(\overline{\eta_1})\} = f_1((\!(\overline{n_1},\overline{\eta_1})\!))$ being f a morphism of braided Lie crossed modules.

Then the same pair of homomorphisms as in the non-braiding case satisfies in the braiding case that $c = f \circ h$.

The uniqueness of h in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ is a consequence of that the forgetful functor $\mathfrak{U}$ is faithful. □

Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a braided Lie crossed module. In the next, we will prove that the universal $\mathfrak{U}$-central extension of $\mathcal{M}$ exists if and only if $\mathfrak{U}(\mathcal{M})$ is perfect in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$.

Proposition 3.3.4. *Let $\mathcal{Y} \xrightarrow{\Psi} \mathcal{M}$ be an extension of braided Lie crossed modules such that $\mathfrak{U}(\mathcal{Y})$ is perfect in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$. Then $\mathfrak{U}(\mathcal{M})$ is perfect.*

Proof. Since $\mathfrak{U}(\mathcal{Y}) \xrightarrow{\mathfrak{U}(\Psi)} \mathfrak{U}(\mathcal{M})$ is an extension in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$, by [13, Proposition 2] $\mathfrak{U}(\mathcal{M})$ is perfect. □

Lemma 3.3.5. *Let* $\mathcal{Y} = (Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!]) \xrightarrow{\Psi} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *be a* $\mathfrak{U}$*-central extension of braided Lie crossed modules such that* $\mathfrak{U}(\mathcal{Y})$ *is not perfect. Then exists another* $\mathfrak{U}$*-central extension* $\mathcal{X} \xrightarrow{f} \mathcal{M}$ *in* $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ *and two different morphisms* $h, g : \mathcal{Y} \longrightarrow \mathcal{X}$ *such that* $\Psi = f \circ h = f \circ g$.

Proof. If $(Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!])$ is a braided Lie crossed module, then we know that the Lie crossed module $(D_{Y_2}(Y_1) \xrightarrow{\rho|_{D_{Y_2}(Y_1)}} [Y_2, Y_2], \star_C)$ is a crossed submodule of $(Y_1 \xrightarrow{\rho} Y_2, \star)$. But it is itself a braided crossed submodule of $(Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!])$ since, if we have $[y, y'], [z, z'] \in [Y_2, Y_2]$, then:

$$[\![[y, y'], [z, z']]\!] = [\![[y, y'], \rho([\![z, z']\!])]\!] = [y, y'] \star [\![z, z']\!] \in D_{Y_2}(Y_1).$$

Let us denote $i : (D_{Y_2}(Y_1) \xrightarrow{\rho|_{D_{Y_2}(Y_1)}} [Y_2, Y_2], \star_C, [\![-,-]\!]_C) \longrightarrow (Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!])$.

Let $\mathcal{M} \times \mathrm{coker}(i) \xrightarrow{\pi^1} \mathcal{M}$ be the extension given by the first projection. This extension is a $\mathfrak{U}$-central extension and since $\mathfrak{U}(\mathcal{Y})$ is not perfect, there are two morphisms in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$, $h, g : \mathcal{Y} \longrightarrow \mathcal{M} \times \mathrm{coker}(i)$ such that $\Psi = f \circ h = f \circ g$ (see [13, Lemma 4]). The product in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ is the same as in $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ with induced braiding, so we have that the morphisms are in $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$. □

Corollary 3.3.6. *If the universal* $\mathfrak{U}$*-central extension of a braided Lie crossed module* $\mathcal{M}$ *exists, then* $\mathfrak{U}(\mathcal{M})$ *is perfect in* $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$.

Proof. If the universal extension is not perfect, then using Lemma 3.3.5, we have another $\mathfrak{U}$-central extension and two different morphisms from the universal $\mathfrak{U}$-central extension, which contradicts the universality. □

Corollary 3.3.7. *A braided Lie crossed module admits a universal* $\mathfrak{U}$*-central extension if and only if it is perfect as Lie crossed module.*

Proof. If the braided Lie crossed module is perfect, then using Proposition 3.3.3, we have its universal $\mathfrak{U}$-central extension.

If the braided Lie crossed module has a universal $\mathfrak{U}$-central extension, then using Corollary 3.3.6, we have that the universal $\mathfrak{U}$-central extension is perfect as Lie crossed module. Since it is an extension, we can use Lemma 3.3.4 and conclude that our braided Lie crossed module is perfect as a Lie crossed module. □

3.4 Relationship between the universal $\boldsymbol{B}$-central extension and the universal $\mathfrak{U}$-central extension in the braided case

This section will show the relation between the notions of universal $\boldsymbol{B}$-central extension and universal $\mathfrak{U}$-central extension in the case of braided Lie crossed modules.

Lemma 3.4.1. *Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a braided Lie crossed module. Then, $\mathcal{M}$ is $\boldsymbol{B}$-perfect if and only if $\mathfrak{U}(\mathcal{M})$ is perfect.*
In fact, we have that if $N = [N, N]$ then $B_N(M) = D_N(M)$.

Proof. We have $N = [N, N]$, and we need to check $B_N(M) = D_N(M)$.

If $\mathfrak{U}(\mathcal{M}) = (M \xrightarrow{\partial} N, \cdot)$ is perfect, then $D_N(M) = M$. Since $D_N(M) \subset B_N(M) \subset M$, we have that $B_N(M) = M$ and $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect.

On the other hand, since $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect, we have that $B_N(M) = M$. So, we only need to prove that $B_N(M) \subset D_N(M)$.

If $\{n, n'\}$ is a generator of $B_N(M)$, and since $N = [N, N]$ by being $\boldsymbol{B}$-perfect, we have that on generators $\{[n_1, n_2], [n_1', n_2']\}$,

$$\{[n_1, n_2], [n_1', n_2']\} = \{[n_1, n_2], \partial\{n_1', n_2'\}\} = [n_1, n_2] \cdot \{n_1', n_2'\} \in D_N(M).$$

□

Lemma 3.4.2. *Let $(X_1 \xrightarrow{\delta} X_2, \diamond, (\!(-,-)\!)) \xrightarrow{f} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be an extension of braided Lie crossed modules with $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ $\boldsymbol{B}$-perfect. Then, f is a $\boldsymbol{B}$-central extension if and only if f is a $\mathfrak{U}$-central extension.*
In fact, if $N = [N, N]$ then $\mathrm{Z}_B(N) = Z(N) \cap \mathrm{st}_N(M)$.

Proof. If f is a $\mathfrak{U}$-central extension, then $\ker(f_1) \subset M^N$ and $\ker(f_2) \subset \mathrm{st}_N(M) \cap \mathrm{Z}(N)$. We need to prove that $\ker(f_2) \subset \mathrm{Z}_B(N)$, and it is sufficient to show that $\mathrm{st}_N(M) \cap \mathrm{Z}(N) \subset Z_B(N)$.

Let $n \in \mathrm{st}_N(M) \cap \mathrm{Z}(N)$ and $x = [n_1, n_2] \in N = [N, N]$. We have

$$\{n, x\} = \{n, [n_1, n_2]\} = \{n, \partial\{n_1, n_2\}\} = n \cdot \{n_1, n_2\} = 0,$$
$$\{x, n\} = \{[n_1, n_2], n\} = \{\partial\{n_1, n_2\}, n\} = -n \cdot \{n_1, n_2\} = 0.$$

So, $\ker(f_2) \subset \mathrm{Z}_B(N)$ and f is a $\boldsymbol{B}$-central extension. □

Theorem 3.4.3. *Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\boldsymbol{B}$-perfect braided Lie crossed module. Then its universal $\boldsymbol{B}$-central extension $\mathcal{U} \overset{\Phi}{\twoheadrightarrow} \mathcal{M}$ and its universal $\mathfrak{U}$-central extension $\mathcal{V} \overset{c}{\twoheadrightarrow} \mathcal{M}$ are isomorphic.*

Proof. Since $\mathcal{V} \overset{c}{\twoheadrightarrow} \mathcal{M}$ is a $\mathfrak{U}$-central extension, we know using Lemma 3.4.2 (by hypothesis $\mathcal{M}$ is $\boldsymbol{B}$-perfect) that it is a $\boldsymbol{B}$-central extension, and using the universality of $\mathcal{U}$, there is a unique morphism $\mathcal{U} \xrightarrow{h} \mathcal{V}$ such that $\Phi = c \circ h$.

Since $\mathcal{U} \overset{\Phi}{\twoheadrightarrow} \mathcal{M}$ is a $\boldsymbol{B}$-central extension is also a $\mathfrak{U}$-central extension, and so by the universality of $\mathcal{V}$, there exists a unique morphism $\mathcal{V} \xrightarrow{h'} \mathcal{U}$ such that $c = \Phi \circ h'$.

Using the universality of $\mathcal{U}$, since $\Phi \circ (h' \circ h) = c \circ h = \Phi$, we get that $h' \circ h = \mathrm{Id}_{\mathcal{U}}$.

By the same arguments using the universality of $\mathcal{V}$, we have that $h \circ h' = \mathrm{Id}_{\mathcal{V}}$. □

Corollary 3.4.4.

(i) Let $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\boldsymbol{B}$-perfect braided Lie crossed module. Then $N \otimes M \simeq N \otimes N$.

(ii) If M is a perfect Lie K-algebra, then $M \otimes (M \otimes M) \simeq M \otimes M$.

Proof. (i) By Theorem 3.4.3, (UBCE) and (UUCE) are isomorphic. Therefore we have an isomorphism $N \otimes M \simeq N \otimes N$.

The isomorphism can be described explicitly using Proposition 3.2.4 and Proposition 3.3.3, and it is given by:

$$h_1 : N \otimes N \to N \otimes M, \; n \otimes n' \mapsto n \otimes \{n'_1, n'_2\}, \text{ with } n' = [n'_1, n'_2],$$

and

$$h_1^{-1}: N \otimes M \to N \otimes N,\ n \otimes m \mapsto n \otimes \partial(m).$$

(ii) If M is perfect, then the braiding Lie crossed module $(M \otimes M \xrightarrow{\partial} M, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect (see Example 2.5.7), since $M \otimes M$ is generated by $m_1 \otimes m_2 = \{m_1, m_2\}$. By (i), we have $M \otimes (M \otimes M) \simeq M \otimes M$.

In this case, the isomorphism is described by:

$$h_1: M \otimes M \to M \otimes (M \otimes M),\ m \otimes m' \mapsto m \otimes (m_1' \otimes m_2'),\ \text{with } m' = [m_1', m_2'],$$

and

$$h_1^{-1}: M \otimes (M \otimes M) \to M \otimes M,\ m \otimes (m' \otimes m'') \mapsto m \otimes [m', m''].$$

□

Chapter 4

On the Loday-Pirashvili Category

In this chapter, we generalize the Loday-Pirashvili category to different kinds of tensor categories. Then we use it to prove relationships between internal Lie objects and internal Leibniz objects.

4.1 Tensor Categories

In this section, we will generalize the Loday-Pirashvili category ([44]) to different kinds of tensor categories. The definitions of (braided) semigroupal categories and (braided) monoidal categories are already shown in Section 1.4.

4.1.1 Categories with operations

Definition 4.1.1. Let $\mathbf{C}$ be a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a bifunctor. We will say that the pair $\mathcal{C} = (\mathbf{C}, \otimes)$ is a *category with an operation*.

Definition 4.1.2. Let $\mathbf{C}$ be a category. We will denote as $\mathrm{Hom}(\mathbf{C})$ the category given by:

- $\mathrm{Ob}\left(\mathrm{Hom}(\mathbf{C})\right) = \mathrm{Arw}(\mathbf{C})$

- $\mathrm{Arw}\left(\mathrm{Hom}(\mathbf{C})\right) \subset \mathrm{Arw}(\mathbf{C}) \times \mathrm{Arw}(\mathbf{C})$ with $\begin{array}{c} A \\ \downarrow f \\ B \end{array} \xrightarrow{\alpha=(\alpha^1,\alpha^2)} \begin{array}{c} C \\ \downarrow g \\ D \end{array} \Leftrightarrow \begin{tikzcd} A \arrow[r, "\alpha^1"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "\alpha^2"] & D. \end{tikzcd}$

- $\mathrm{Id}_f = (\mathrm{Id}_A, \mathrm{Id}_B)$ and $\beta \circ \alpha = (\beta^1 \circ \alpha^1, \beta^2 \circ \alpha^2)$.

Since $\mathrm{Hom}(\mathbf{C})$ is defined by pairs of arrows is immediate that if $\mathbf{C}$ has finite coproducts, then

$$\begin{array}{c} A \\ \big\downarrow{\scriptstyle f} \\ B \end{array} \oplus \begin{array}{c} C \\ \big\downarrow{\scriptstyle g} \\ D \end{array} := \begin{array}{c} A \oplus C \\ \big\downarrow{\scriptstyle f \oplus g} \\ B \oplus D, \end{array}$$

and $\mathrm{Hom}(\mathbf{C})$ has finite coproducts. In the same way, if 0 is the zero object for $\mathbf{C}$, then Id_0 is the zero object for $\mathbf{C}$ (same for initial object and final object).

Proposition 4.1.3. *Let $C = (\mathbf{C}, \otimes)$ be a category with an operation such that* $\mathbf{C}$ *has finite coproducts.*

The correspondence $\widehat{\otimes}: \ \mathrm{Hom}(\mathbf{C}) \times \mathrm{Hom}(\mathbf{C}) \to \mathrm{Hom}(\mathbf{C})$ *defined:*

- *in objects as* $\begin{array}{c} A \\ \big\downarrow{\scriptstyle f} \\ B \end{array} \widehat{\otimes} \begin{array}{c} C \\ \big\downarrow{\scriptstyle g} \\ D \end{array} := \begin{array}{c} (A \otimes D) \oplus (B \otimes C) \\ \big\downarrow{\scriptstyle [(f \otimes \mathrm{Id}_D), (\mathrm{Id}_B \otimes g)]} \\ B \otimes D \end{array}$

- *in arrows, if* $f \xrightarrow{\alpha} f'$ *and* $g \xrightarrow{\beta} g'$, $\alpha \widehat{\otimes} \beta = \big((\alpha^1 \otimes \beta^2) \oplus (\alpha^2 \otimes \beta^1), \alpha^2 \otimes \beta^2\big)$

is a functor. Therefore, $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes})$ *is a category with an operation.*

Proof. Since $\otimes$ and $\oplus$ are functors, it is just a matter of checking that $\alpha \widehat{\otimes} \beta$ is a morphism in the category. To do this, we take $A' \xrightarrow{f'} B'$ and $C' \xrightarrow{g'} D'$, and we easily see that the following diagram is commutative:

$$\begin{tikzcd} (A \otimes D) \oplus (B \otimes C) \arrow[r, "(\alpha^1 \otimes \beta^2) \oplus (\alpha^2 \otimes \beta^1)"] \arrow[d, "{[(f \otimes \mathrm{Id}_D), (\mathrm{Id}_B \otimes g)]}"'] & (A' \otimes D') \oplus (B' \otimes C') \arrow[d, "{[(f' \otimes \mathrm{Id}_{D'}), (\mathrm{Id}_{B'} \otimes g')]}"] \\ B \otimes D \arrow[r, "\alpha^2 \otimes \beta^2"] & B' \otimes D' \end{tikzcd}$$

$\square$

Definition 4.1.4. Let $C = (\mathbf{C}, \otimes)$ be a category with an operation such that $\mathbf{C}$ has finite coproducts. The category with an operation $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes})$ defined in Proposition 4.1.3 will be denoted as $\mathrm{LP}(C)$ and it will be called the *Loday-Pirashvili category with an operation* of C.

4.1.2 Semigroupal categories

Definition 4.1.5. Let $\mathcal{C} = (\mathbf{C}, \otimes)$ be a category with an operation and consider three objects A, B, C in $\mathcal{C}$ such that there exists the coproduct of A with B and $A \otimes C$ with $B \otimes C$. We define the *R-$\otimes$-distributor* on A, B, C as the morphism $\varepsilon_{A,B,C}$ given by

$$\begin{tikzcd}
A \otimes C \arrow[r, "\iota_1"] \arrow[dr, "\iota_1 \otimes \mathrm{Id}_C"'] & (A \otimes C) \oplus (B \otimes C) \arrow[d, dashed, "\varepsilon_{A,B,C}"] & (B \otimes C) \arrow[l, "\iota_2"'] \arrow[dl, "\iota_2 \otimes \mathrm{Id}_C"] \\
& (A \oplus B) \otimes C &
\end{tikzcd}$$

Similarly, if there exists the coproduct of A with B and $C \otimes A$ with $C \otimes B$ we define the *L-$\otimes$-distributor* on A, B, C as the morphism $\gamma_{A,B,C}$ given by

$$\begin{tikzcd}
C \otimes A \arrow[r, "\iota_1"] \arrow[dr, "\mathrm{Id}_C \otimes \iota_1"'] & (C \otimes A) \oplus (C \otimes B) \arrow[d, dashed, "\gamma_{A,B,C}"] & (C \otimes B) \arrow[l, "\iota_2"'] \arrow[dl, "\mathrm{Id}_C \otimes \iota_2"] \\
& C \otimes (A \oplus B) &
\end{tikzcd}$$

Proposition 4.1.6. *The correspondences ε and γ defined above are functorial. Furthermore, they are natural transformations.*

Proof. To check that ε is a natural transformation we need to see that the following diagram is commutative,

$$\begin{tikzcd}
(A \otimes C) \oplus (B \otimes C) \arrow[r, "\varepsilon_{A,B,C}"] \arrow[d, "(f \otimes h) \oplus (g \otimes h)"'] & (A \oplus B) \otimes C \arrow[d, "(f \oplus g) \otimes h"] \\
(A' \otimes C') \oplus (B' \otimes C') \arrow[r, "\varepsilon_{A',B',C'}"] & (A' \oplus B') \otimes C'
\end{tikzcd}$$

where $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$, $C \xrightarrow{h} C' \in \mathrm{Arw}(\mathbf{C})$. Since the domain is a coproduct, it easily follows by checking that they coincide when composed with the natural injections. A similar argument works for γ. $\square$

Definition 4.1.7. Let $\mathcal{C} = (\mathbf{C}, \otimes)$ be a category with an operation where $\mathbf{C}$ has finite coproducts. Take ε and γ from Definition 4.1.5.

- We say that **C** is *Right Multiplication distributive with* $\otimes$ or *R-$\otimes$-distributive* if ε is a natural isomorphism. In this case, we will say that $\mathcal{C}$ is R-distributive.
- We say that **C** is *Left Multiplication distributive with* $\otimes$ or *L-$\otimes$-distributive* if γ is a natural isomorphism. In this case, we will say that $\mathcal{C}$ is *L-distributive*.
- We say that **C** is *distributive with* $\otimes$ or *$\otimes$-distributive* if ε and γ are natural isomorphisms. In this case, we will say that $\mathcal{C}$ is *distributive*.

Theorem 4.1.8. *Let* $(\mathbf{C}, \otimes, a)$ *be a semigroupal category such that* $\mathcal{C} = (\mathbf{C}, \otimes)$ *is* $\otimes$-*distributive. For* $\begin{matrix} A \\ \downarrow{\scriptstyle f} \\ B \end{matrix}, \begin{matrix} C \\ \downarrow{\scriptstyle g} \\ D \end{matrix}, \begin{matrix} E \\ \downarrow{\scriptstyle h} \\ F \end{matrix} \in \mathrm{Ob}(\mathrm{Hom}(\mathbf{C}))$ *we define using the universal property of coproducts the morphism*

$$\begin{tikzcd}
((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[r, "\iota_1"] \arrow[d, "\varepsilon^{-1}"'] & (((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[ddd, dashed, "\alpha_{f,g,h}"] & (B \otimes D) \otimes E \arrow[l, "\iota_2"'] \arrow[d, "a"] \\
((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[d, "a \oplus a"'] & & B \otimes (D \otimes E) \arrow[d, "\mathrm{Id} \otimes \iota_2"] \\
(A \otimes (D \otimes F)) \oplus (B \otimes (C \otimes F)) \arrow[dr, "\mathrm{Id} \oplus (\mathrm{Id} \otimes \iota_1)"'] & & B \otimes ((C \otimes F) \oplus (D \otimes E)) \arrow[dl, "\iota_2"] \\
& (A \otimes (D \otimes F)) \oplus (B \otimes ((C \otimes F) \oplus (D \otimes E))) &
\end{tikzcd}$$

Then $\widehat{a}_{f,g,h} = (\alpha_{f,g,h}, a_{B,D,F})$ *gives an associator for* $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes})$. *The inverse of this natural isomorphism is given by* $(\omega_{f,g,h}, a^{-1}_{B,D,F})$ *where* $\omega_{f,g,h}$ *is defined by the diagram*

$$\begin{tikzcd}
A \otimes (D \otimes F) \arrow[r, "\iota_1"] \arrow[d, "a^{-1}"'] & (A \otimes (D \otimes F)) \oplus (B \otimes ((C \otimes F) \oplus (D \otimes E))) \arrow[ddd, dashed, "\omega_{f,g,h}"] & B \otimes ((C \otimes F) \oplus (D \otimes E)) \arrow[l, "\iota_2"'] \arrow[d, "\gamma^{-1}"] \\
(A \otimes D) \otimes F \arrow[d, "\iota_1 \otimes \mathrm{Id}_F"'] & & (B \otimes (C \otimes F)) \oplus (B \otimes (D \otimes E)) \arrow[d, "a^{-1} \oplus a^{-1}"] \\
((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[dr, "\iota_1"'] & & ((B \otimes C) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[dl, "(\iota_2 \otimes \mathrm{Id}) \oplus \mathrm{Id}"] \\
& (((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) &
\end{tikzcd}$$

Proof. First, we will prove that $\widehat{a}_{f,g,h} = (\alpha_{f,g,h}, a_{B,D,F})\colon (f\widehat{\otimes}g)\widehat{\otimes}h \to f\widehat{\otimes}(g\widehat{\otimes}h)$, that means, we have the diagram

$$\begin{tikzcd}
(((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[r, "\alpha_{f,g,h}"] \arrow[d, "{[([(f\otimes \mathrm{Id}),(\mathrm{Id}\otimes g)]\otimes \mathrm{Id}),(\mathrm{Id}\otimes h)]}"'] & (A \otimes (D \otimes F)) \oplus (B \otimes ((C \otimes F) \oplus (D \otimes E))) \arrow[d, "{[(f\otimes \mathrm{Id}),(\mathrm{Id}\otimes [(g\otimes \mathrm{Id}),(\mathrm{Id}\otimes h)])]}"] \\
(B \otimes D) \otimes F \arrow[r, "a"'] & B \otimes (D \otimes F).
\end{tikzcd}$$

Note that we use the notation $[-,-]$ for the natural morphism to the coproduct.

The domain is a coproduct so that we can study each part separately. The first one is $((A \otimes C) \oplus (B \otimes D)) \otimes F$ and the first morphism going down is ε^{-1}, so we will prove that the diagram is commutative when precomposed with ε, since it is an isomorphism. Then the domain is again a coproduct so that we can analyse both parts separately again. To see the diagram's commutativity is just a matter of resolving the coproduct injections, as we can see in the following diagrams. The second part and the other ones can be checked similarly. The outer diagram is commutative since the inner diagrams are easily commutative.

$$\begin{tikzcd}
(((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[dd, "{[([(f\otimes \mathrm{Id}),(\mathrm{Id}\otimes g)]\otimes \mathrm{Id}),(\mathrm{Id}\otimes h)]}"'] & ((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[l, "\iota_1"'] \arrow[dddl, "{[(f\otimes \mathrm{Id}),(\mathrm{Id}\otimes g)]\otimes \mathrm{Id}}"] & ((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[l, "\varepsilon"'] \\
 & & (A \otimes D) \otimes F \arrow[ul, "\iota_1 \otimes \mathrm{Id}"] \arrow[u, "\iota_1"'] \arrow[dd, "a"] \arrow[ddll, bend left, "(f\otimes \mathrm{Id})\otimes \mathrm{Id}"] \\
(B \otimes D) \otimes F \arrow[d, "a"'] & & \\
B \otimes (D \otimes F) & & A \otimes (D \otimes F) \arrow[ll, "f\otimes \mathrm{Id}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[ddd, "\alpha"] & ((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"'] & ((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[l, "\varepsilon"'] \\
 & ((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[d, "a \oplus a"'] & (A \otimes D) \otimes F \arrow[u, "\iota_1"'] \arrow[d, "a"] \\
 & (A \otimes (D \otimes F)) \oplus (B \otimes (C \otimes F)) \arrow[dl, "\mathrm{Id} \oplus (\mathrm{Id} \otimes \iota_1)"'] & A \otimes (D \otimes F) \arrow[l, "\iota_1"'] \arrow[dll, bend left, "\iota_1"] \arrow[ddll, bend left, "f \otimes \mathrm{Id}"] \\
(A \otimes (D \otimes F)) \oplus (B \otimes ((C \otimes F) \oplus (D \otimes E))) \arrow[d, "{[(f \otimes \mathrm{Id}), (\mathrm{Id} \otimes [(g \otimes \mathrm{Id}), (\mathrm{Id} \otimes h)])]}"] & & \\
B \otimes (D \otimes F) & &
\end{tikzcd}$$

Note that the rightmost part is the same for these two diagrams, so we conclude that the leftmost part is the same when precomposed with the injections and ε. Therefore, we have that $\widehat{a}_{f,g,h}$ is a morphism.

Let us now see that it is a natural transformation. Let us consider the following objects

$$\begin{tikzcd}
A \arrow[d, "f"] \arrow[r, "\lambda"] & A' \arrow[d, "f'"] \\
B & B'
\end{tikzcd}, \quad
\begin{tikzcd}
C \arrow[d, "g"] \arrow[r, "\beta"] & C' \arrow[d, "g'"] \\
D & D'
\end{tikzcd}, \quad
\begin{tikzcd}
E \arrow[d, "h"] \arrow[r, "\sigma"] & E' \arrow[d, "h'"] \\
F & F'
\end{tikzcd}.$$

The lower part of the naturalness is satisfied, since there the operation is the standard one. We will focus in the upper part.

$$\begin{tikzcd}
(((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[r, "\alpha_{f,g,h}"] \arrow[d, "(((\lambda^1 \otimes \beta^2) \oplus (\lambda^2 \otimes \beta^1)) \otimes \sigma^2) \oplus ((\lambda^2 \otimes \beta^2) \otimes \sigma^1)"] & (A \otimes (D \otimes F)) \oplus (B \otimes ((C \otimes F) \oplus (D \otimes E))) \arrow[d, "(\lambda^1 \otimes (\beta^2 \otimes \sigma^2)) \oplus (\lambda^2 \otimes ((\beta^1 \otimes \sigma^2) \oplus (\beta^2 \otimes \sigma^1)))"] \\
(((A' \otimes D') \oplus (B' \otimes C')) \otimes F') \oplus ((B' \otimes D') \otimes E') \arrow[r, "\alpha_{f',g',h'}"] & (A' \otimes (D' \otimes F')) \oplus (B' \otimes ((C' \otimes F') \oplus (D' \otimes E'))).
\end{tikzcd}$$

Again, the domain is a coproduct so that we can study each part separately. It is the same coproduct as the proof of well defined, so we need to use the same steps to prove it, using the injections and ε, as shown in the following diagrams. The other cases can be checked similarly.

$$
\begin{tikzcd}
(((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[ddd, "\alpha"'] & ((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"'] & ((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[l, "\varepsilon"'] \\
 & ((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[d, "a \oplus a"'] & (A \otimes D) \otimes F \arrow[u, "\iota_1"'] \arrow[d, "a"] \arrow[ddd, bend left, "(\lambda^1 \otimes \beta^2) \otimes \sigma^2"] \\
 & (A \otimes (D \otimes F)) \oplus (B \otimes (C \otimes F)) \arrow[dl, "\mathrm{Id} \oplus (\mathrm{Id} \otimes \iota_1)"'] & A \otimes (D \otimes F) \arrow[l, "\iota_1"'] \arrow[dll, bend left, "\iota_1"] \arrow[ddl, "\lambda^1 \otimes (\beta^2 \otimes \lambda^2)"] \\
(A \otimes (D \otimes F)) \oplus (B \otimes ((C \otimes F) \oplus (D \otimes E))) \arrow[d, "(\lambda^1 \otimes (\beta^2 \otimes \sigma^2)) \oplus (\lambda^2 \otimes ((\beta^1 \otimes \sigma^2) \oplus (\beta^2 \otimes \sigma^1)))"] & & \\
(A' \otimes (D' \otimes F')) \oplus (B' \otimes ((C' \otimes F') \oplus (D' \otimes E'))) & A' \otimes (D' \otimes F') \arrow[l, "\iota_1"] & (A' \otimes D') \otimes F', \arrow[l, "a"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[ddd, "(((\lambda^1 \otimes \beta^2) \oplus (\lambda^2 \otimes \beta^1)) \otimes \sigma^2) \oplus ((\lambda^2 \otimes \beta^2) \otimes \sigma^1)"'] & ((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[l, "\iota_1"'] \arrow[d, "((\lambda^1 \otimes \beta^2) \oplus (\lambda^2 \otimes \beta^2)) \otimes \sigma^2"'] & ((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[l, "\varepsilon"'] \\
 & ((A' \otimes D') \oplus (B' \otimes C')) \otimes F' \arrow[d, "\varepsilon^{-1}"] \arrow[ddl, bend right, "\iota_1"'] & (A \otimes D) \otimes F \arrow[u, "\iota_1"] \arrow[ul, "\iota_1 \otimes \mathrm{Id}"'] \arrow[d, "(\lambda^1 \otimes \beta^1) \otimes \sigma^2"] \\
 & ((A' \otimes D') \otimes F') \oplus ((B' \otimes C') \otimes F') \arrow[d, "a \oplus a"] & (A' \otimes D') \otimes F' \arrow[ul, "\iota_1 \otimes \mathrm{Id}"'] \arrow[l, "\iota_1"] \arrow[ddl, bend left, "a"] \\
(((A' \otimes D') \oplus (B' \otimes C')) \otimes F') \oplus ((B' \otimes D') \otimes E') \arrow[d, "\alpha"'] & (A' \otimes (D' \otimes F')) \oplus (B' \otimes (C' \otimes F')) \arrow[dl, "\mathrm{Id} \oplus (\mathrm{Id} \otimes \iota_1)"] & \\
(A' \otimes (D' \otimes F')) \oplus (B' \otimes ((C' \otimes F') \oplus (D' \otimes E'))) & A' \otimes (D' \otimes F'). \arrow[l, "\iota_1"] \arrow[u, "\iota_1"'] &
\end{tikzcd}
$$

Therefore, $\widehat{a}\colon\ \widehat{\otimes}\circ(\widehat{\otimes}\times \mathrm{Id}_{\mathrm{Hom}(\mathbf{C})}) \Rightarrow \widehat{\otimes}\circ(\mathrm{Id}_{\mathrm{Hom}(\mathbf{C})} \times\widehat{\otimes})\circ A^{\mathrm{Hom}(\mathbf{C}),\mathrm{Hom}(\mathbf{C}),\mathrm{Hom}(\mathbf{C})}$ is a natural transformation.

We will show now that it is a natural isomorphism. We need to show that for $f, g, h \in \mathrm{Hom}(\mathbf{C})$, the morphism $\widehat{a}_{f,g,h}$ is an isomorphism with inverse $(\omega_{f,g,h}, a^{-1}_{B,D,F})$. Since $a_{B,D,F}$ is an isomorphism with inverse $a^{-1}_{B,D,F}$ we only need to prove that $\alpha_{f,g,h}$ is an isomorphism with inverse $\omega_{f,g,h}$.

We will prove $\omega_{f,g,h}\circ\alpha_{f,g,h} = \mathrm{Id}$, the other composition is analogous. As in the other proofs, the domain is a coproduct. In fact, it is the same coproduct as in the other proofs, so we need to use the same steps to prove it, using the injections and ε, as can see in the following diagrams. The other cases are checked similarly.

$$
\begin{tikzcd}
(((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E) \arrow[ddd, "\alpha"] & ((A\otimes D)\oplus(B\otimes C))\otimes F \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"'] & ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[l, "\varepsilon"'] \\
& ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[d, "a\oplus a"'] & (A\otimes D)\otimes F \arrow[u, "\iota_1"'] \arrow[d, "a"] \\
& (A\otimes(D\otimes F))\oplus(B\otimes(C\otimes F)) \arrow[dl, "\mathrm{Id}\oplus(\mathrm{Id}\otimes\iota_1)"'] & A\otimes(D\otimes F) \arrow[l, "\iota_1"'] \arrow[d, "a^{-1}"] \arrow[dll, bend left, "\iota_1"] \\
(A\otimes(D\otimes F))\oplus(B\otimes((C\otimes F)\oplus(D\otimes E))) \arrow[d, "\omega"] & & (A\otimes D)\otimes F \arrow[d, "\iota_1"] \arrow[dl, "\iota_1\otimes\mathrm{Id}"] \\
(((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E) & ((A\otimes D)\oplus(B\otimes C))\otimes F \arrow[l, "\iota_1"] & ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[l, "\varepsilon"]
\end{tikzcd}
$$

For completion we will show another diagram, composing first with ι_1 and then with ι_2, to show when the morphism γ from the L-$\otimes$-distributivity appears.

$$
\begin{tikzcd}
(((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E) \arrow[ddd, "\alpha"] & ((A\otimes D)\oplus(B\otimes C))\otimes F \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"'] & ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[l, "\varepsilon"'] \\
& ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[d, "a\oplus a"'] & (B\otimes C)\otimes F \arrow[u, "\iota_2"'] \arrow[d, "a"] \\
& (A\otimes(D\otimes F))\oplus(B\otimes(C\otimes F)) \arrow[dl, "\mathrm{Id}\oplus(\mathrm{Id}\otimes\iota_1)"'] & B\otimes(C\otimes F) \arrow[l, "\iota_2"'] \arrow[dl, "\mathrm{Id}\otimes\iota_1"] \arrow[ddl, bend left, "\iota_1"] \arrow[ddd, "a^{-1}"] \\
(A\otimes(D\otimes F))\oplus(B\otimes((C\otimes F)\oplus(D\otimes E))) \arrow[ddd, "\omega"] & B\otimes((C\otimes F)\oplus(D\otimes E)) \arrow[l, "\iota_2"] \arrow[d, "\gamma^{-1}"] & \\
& (B\otimes(C\otimes F))\oplus(B\otimes(D\otimes E)) \arrow[d, "a^{-1}\oplus a^{-1}"] & \\
& ((B\otimes C)\otimes F)\oplus((B\otimes D)\otimes E) \arrow[dl, "(\iota_2\otimes\mathrm{Id})\oplus\mathrm{Id}"'] & (B\otimes C)\otimes F \arrow[l, "\iota_1"] \arrow[d, "\iota_2"] \arrow[dl, "\iota_2\otimes\mathrm{Id}"] \\
(((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E) & ((A\otimes D)\oplus(B\otimes C))\otimes F \arrow[l, "\iota_1"] & ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[l, "\varepsilon"]
\end{tikzcd}
$$

To prove that the natural isomorphism $\widehat{a}$ is an associator, we need to prove the associativity coherence diagram.

$$\begin{tikzcd}
((f\widehat{\otimes}g)\widehat{\otimes}h)\widehat{\otimes}k \arrow[r, "\widehat{a}_{f\widehat{\otimes}g,h,k}"] \arrow[d, "\widehat{a}_{f,g,h}\widehat{\otimes}\,\mathrm{Id}_k"'] & (f\widehat{\otimes}g)\widehat{\otimes}(h\widehat{\otimes}k) \arrow[dd, "\widehat{a}_{f,g,h\widehat{\otimes}k}"] \\
(f\widehat{\otimes}(g\widehat{\otimes}h))\widehat{\otimes}k \arrow[d, "\widehat{a}_{f,g\widehat{\otimes}h,k}"'] & \\
f\widehat{\otimes}((g\widehat{\otimes}h)\widehat{\otimes}k) \arrow[r, "\mathrm{Id}_f\,\widehat{\otimes}\widehat{a}_{g,h,k}"] & f\widehat{\otimes}(g\widehat{\otimes}(h\widehat{\otimes}k)).
\end{tikzcd}$$

That means that we need to obtain two commutative diagrams given by the domain and codomain of the tensor product $\widehat{\otimes}$. The lower part is immediate since it is the coherence diagram for the associator a.

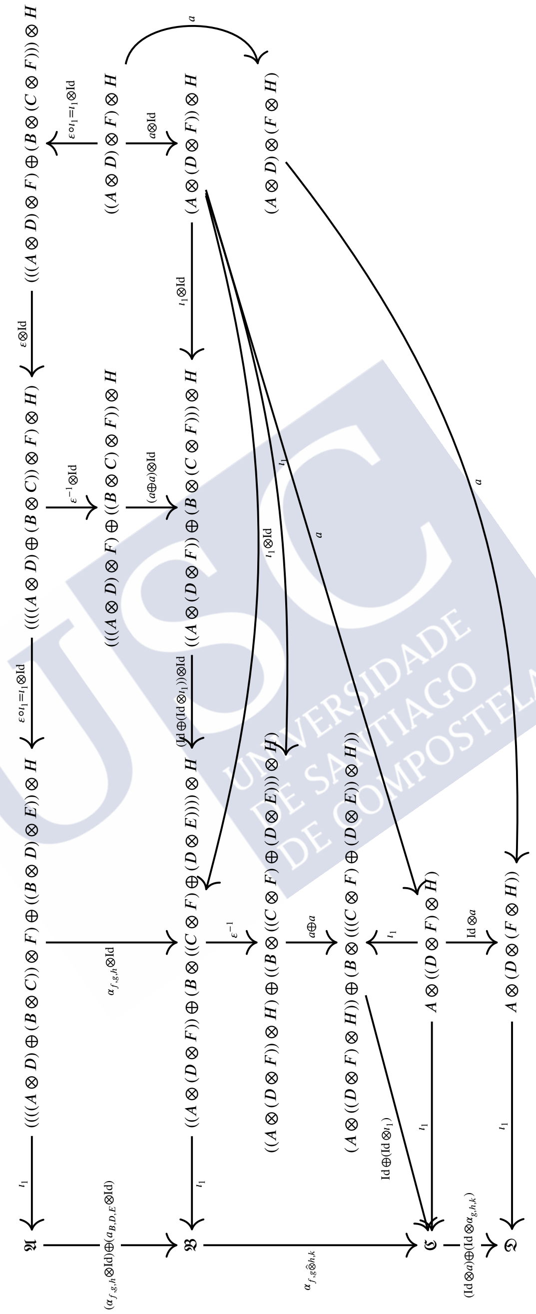
The upper part is the following one:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\alpha_{f\widehat{\otimes}g,h,k}"] \arrow[d, "(\alpha_{f,g,h}\otimes\mathrm{Id})\oplus(a_{B,D,E}\otimes\mathrm{Id})"'] & \mathfrak{E} \arrow[dd, "\alpha_{f,g,h\widehat{\otimes}k}"] \\
\mathfrak{B} \arrow[d, "\alpha_{f,g\widehat{\otimes}h,k}"'] & \\
\mathfrak{C} \arrow[r, "(\mathrm{Id}\otimes a_{D,F,H})\oplus(\mathrm{Id}\otimes\alpha_{g,h,k})"] & \mathfrak{D},
\end{tikzcd}$$

Where, for clarity, we will denote:

$$\begin{aligned}
\mathfrak{A} &:= (((((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E))\otimes H)\oplus(((B\otimes D)\otimes F)\otimes G),\\
\mathfrak{B} &:= (((A\otimes(D\otimes F))\oplus(B\otimes((C\otimes F)\oplus(D\otimes E))))\otimes H)\oplus((B\otimes(D\otimes F))\otimes G),\\
\mathfrak{C} &:= (A\otimes((D\otimes F)\otimes H))\oplus(B\otimes((((C\otimes F)\oplus(D\otimes E))\otimes H)\oplus((D\otimes F)\otimes G))),\\
\mathfrak{D} &:= (A\otimes(D\otimes(F\otimes H)))\oplus(B\otimes((C\otimes(F\otimes H))\oplus(D\otimes((E\otimes H)\oplus(F\otimes G))))),\\
\mathfrak{E} &:= (((A\otimes D)\oplus(B\otimes C))\otimes(F\otimes H))\oplus((B\otimes D)\otimes((E\otimes H)\oplus(F\otimes G))).
\end{aligned}$$

Following other cases, this can be proved by resolving injections to the coproduct, as we can see in the following diagrams taking the first injection.



$\mathfrak{A}$
$(((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E))\otimes H$
$(((A\otimes D)\oplus(B\otimes C))\otimes F)\otimes H)$
$(((A\otimes D)\otimes F)\oplus(B\otimes(C\otimes F)))\otimes H$
$((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F))\otimes H$
$((A\otimes D)\otimes F)\otimes H$
$\mathfrak{B}$
$((A\otimes(D\otimes F))\oplus(B\otimes((C\otimes F)\oplus(D\otimes E))))\otimes H$
$((A\otimes(D\otimes F))\oplus(B\otimes(C\otimes F)))\otimes H$
$(A\otimes(D\otimes F))\otimes H$
$((A\otimes(D\otimes F))\otimes H)\oplus((B\otimes((C\otimes F)\oplus(D\otimes E)))\otimes H)$
$(A\otimes D)\otimes(F\otimes H)$
$(A\otimes((D\otimes F)\otimes H))\oplus(B\otimes(((C\otimes F)\oplus(D\otimes E))\otimes H))$
$\mathfrak{C}$
$A\otimes((D\otimes F)\otimes H)$
$\mathfrak{D}$
$A\otimes(D\otimes(F\otimes H))$

$$
\begin{tikzcd}
\mathfrak{A} \arrow[ddd, "\alpha_{f\hat{\otimes}g,h,k}"'] & ((((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E))\otimes H \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"] & ((((A\otimes D)\oplus(B\otimes C))\otimes F)\otimes H) \arrow[l, "\varepsilon\circ\iota_1=\iota_1\otimes\mathrm{Id}"'] \arrow[dl, "\iota_1"] \arrow[ddd, bend right, "a"'] & (((A\otimes D)\otimes F)\oplus(B\otimes(C\otimes F)))\otimes H \arrow[l, "\varepsilon\otimes\mathrm{Id}"'] \\
 & ((((A\otimes D)\oplus(B\otimes C))\otimes F)\otimes H)\oplus(((B\otimes D)\otimes E)\otimes H) \arrow[d, "a\oplus a"] & & ((A\otimes D)\otimes F)\otimes H \arrow[u, "\varepsilon\circ\iota_1=\iota_1\otimes\mathrm{Id}"'] \arrow[ul, bend left, "(\iota_1\otimes\mathrm{Id})\otimes\mathrm{Id}"] \arrow[dd, "a"] \\
 & (((A\otimes D)\oplus(B\otimes C))\otimes(F\otimes H))\oplus((B\otimes D)\otimes(E\otimes H)) \arrow[dl, "\mathrm{Id}\oplus(\mathrm{Id}\otimes\iota_1)"'] & & \\
\mathfrak{E} \arrow[ddd, "\alpha_{f,g,h\hat{\otimes}k}"'] & ((A\otimes D)\oplus(B\otimes C))\otimes(F\otimes H) \arrow[l, "\iota_1"'] \arrow[u, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"] & & (A\otimes D)\otimes(F\otimes H) \arrow[ll, "\iota_1\otimes\mathrm{Id}"'] \arrow[dll, bend left, "\iota_1"] \arrow[ddd, "a"] \\
 & ((A\otimes D)\otimes(F\otimes H))\oplus((B\otimes D)\otimes(F\otimes H)) \arrow[d, "a\oplus a"] & & \\
 & (A\otimes(D\otimes(F\otimes H)))\oplus(B\otimes(D\otimes(F\otimes H))) \arrow[dl, "\mathrm{Id}\oplus(\mathrm{Id}\otimes\iota_1)"'] & & \\
\mathfrak{D} & & & A\otimes(D\otimes(F\otimes H)) \arrow[lll, "\iota_1"'] \arrow[ull, "\iota_1"']
\end{tikzcd}
$$

Therefore, $\widehat{a}$ is an associator for $\mathrm{LP}(\mathcal{C})$. $\square$

Definition 4.1.9. Let $\mathcal{C} = (\mathbf{C}, \otimes, a)$ be a semigroupal category where $\mathbf{C}$ is $\otimes$-distributive. The semigroupal category $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a})$ defined in Theorem 4.1.8 will be denoted as $\mathrm{LP}(\mathcal{C})$ and will be called the *Loday-Pirashvili semigroupal category* of $\mathcal{C}$.

4.1.3 Braided semigroupal categories

Theorem 4.1.10. *Let $(\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal category such that $(\mathbf{C}, \otimes)$ is distributive. Let $\mathcal{C} = (\mathbf{C}, \otimes, a)$ and consider its Loday-Pirashvili semigroupal category $\mathrm{LP}(\mathcal{C}) = (\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a})$.*

Given $\begin{array}{c} A \\ \downarrow{\scriptstyle f} \\ B \end{array}, \begin{array}{c} C \\ \downarrow{\scriptstyle g} \\ D \end{array} \in \mathrm{Ob}\left(\mathrm{Hom}(\mathbf{C})\right)$, *we define* $\widehat{\tau}^1_{f,g} := \tau^{\oplus}_{D\otimes A, C\otimes B} \circ (\tau_{A,D} \oplus \tau_{B,C})$ *where $\tau^{\oplus}$ is the interchange transformation of the coproduct, i.e. the natural braiding for the category with an operation $(\mathbf{C}, \oplus)$.*

Then $\widehat{\tau}_{f,g} = (\widehat{\tau}^1_{f,g}, \tau^{\oplus}_{B,D})$ defines a braiding for $\mathrm{LP}(\mathcal{C})$.

Proof. First we will show that $\widehat{\tau}_{f,g}$ is well defined, so we just need to prove that the diagram

$$\begin{tikzcd}
(A \otimes D) \oplus (B \otimes C) \arrow[r, "\widehat{\tau}^1_{f,g}"] \arrow[d, "{[(f\otimes \mathrm{Id}_D),(\mathrm{Id}_B \otimes g)]}"'] & (C \otimes B) \oplus (D \otimes A) \arrow[d, "{[(g\otimes \mathrm{Id}_B),(\mathrm{Id}_D \otimes f)]}"] \\
B \otimes D \arrow[r, "\tau_{B,D}"] & D \otimes B
\end{tikzcd}$$

is commutative

Since the domain is a coproduct, it follows by the commutative diagram and its analogue:

$$\begin{tikzcd}
(A\otimes D)\oplus(B\otimes C) \arrow[rrr, "\widehat{\tau}^1_{f,g}"] \arrow[drr, "\tau_{A,D}\oplus\tau_{B,C}"] \arrow[ddd, "{[(f\otimes\mathrm{Id}_D),(\mathrm{Id}_B\otimes g)]}"'] & & & (C\otimes B)\oplus(D\otimes A) \arrow[ddd, "{[(g\otimes\mathrm{Id}_B),(\mathrm{Id}_D\otimes f)]}"] \\
 & & (D\otimes A)\oplus(C\otimes B) \arrow[ur, "\tau^{\oplus}"] & \\
 & A\otimes D \arrow[uul, "\iota_1"] \arrow[r, "\tau_{A,D}"'] \arrow[dl, "f\otimes\mathrm{Id}_D"] & D\otimes A \arrow[u, "\iota_1"'] \arrow[dr, "\mathrm{Id}_D\otimes f"] & \\
B\otimes C \arrow[rrr, "\tau_{B,D}"'] & & & D\otimes B
\end{tikzcd}$$

The fact that $\widehat{\tau}$ is a natural isomorphism is a consequence of being a composition of natural isomorphisms.

Let us see now that the first hexagonal diagram is satisfied. The second one will be analogous. Let $\begin{array}{c} E \\ \downarrow h \\ F \end{array}$. We need to see that the following diagram is commutative:

$$\begin{tikzcd}
 & (f\widehat{\otimes}g)\widehat{\otimes}h \arrow[r, "\tau_{f\widehat{\otimes}g,h}"] \arrow[dl, "a_{f,g,h}"'] & h\widehat{\otimes}(f\widehat{\otimes}g) & \\
f\widehat{\otimes}(g\widehat{\otimes}h) \arrow[dr, "\mathrm{Id}_f\widehat{\otimes}\tau_{g,h}"'] & & & (h\widehat{\otimes}f)\widehat{\otimes}g \arrow[ul, "a_{h,f,g}"'] \\
 & f\widehat{\otimes}(h\widehat{\otimes}g) \arrow[r, "a^{-1}_{f,h,g}"'] & (f\widehat{\otimes}h)\widehat{\otimes}g \arrow[ur, "\tau_{f,h}\widehat{\otimes}\mathrm{Id}_g"'] &
\end{tikzcd}$$

As usual, the lower part is satisfied since the operation is the standard one. Let us focus on the upper part. The domain is a coproduct so that we can study each part separately. The first one is $((A\otimes D)\oplus(B\otimes C))\otimes F$ and the first morphism going down is ϵ^{-1}, so we will prove that the diagram is commutative when precomposed with the natural transformation ϵ. Then the domain is again a coproduct so that we can analyse both parts separately again. To see the commutativity of the diagram is just a matter of resolving the coproduct injections as can be seen in the following two diagrams. The second part and the other can be checked similarly. □

$$
\begin{tikzcd}
(((A\otimes D)\oplus(B\otimes C))\otimes F)\oplus((B\otimes D)\otimes E) \arrow[ddd, "\alpha"] & ((A\otimes D)\oplus(B\otimes C))\otimes F \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"] & ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[l, "\varepsilon"'] \\
 & ((A\otimes D)\otimes F)\oplus((B\otimes C)\otimes F) \arrow[d, "a\oplus a"] & (A\otimes D)\otimes F \arrow[u, "\iota_1"'] \arrow[d, "a"] \arrow[ddddddd, bend left, "\tau"] \\
 & (A\otimes(D\otimes F))\oplus(B\otimes(C\otimes F)) \arrow[dl, "\mathrm{Id}\oplus(\mathrm{Id}\otimes\iota_1)"'] & A\otimes(D\otimes F) \arrow[l, "\iota_1"'] \arrow[dll, "\iota_1"] \arrow[dd, "\mathrm{Id}\otimes\tau"] \\
(A\otimes(D\otimes F))\oplus(B\otimes((C\otimes F)\oplus(D\otimes E))) \arrow[d, "(\mathrm{Id}\otimes\tau)\oplus(\mathrm{Id}\otimes\widehat{\tau}^1_{g,h})"'] & & \\
(A\otimes(F\otimes D))\oplus(B\otimes((E\otimes D)\oplus(F\otimes C))) \arrow[d, "\omega"'] & & A\otimes(F\otimes D) \arrow[ll, "\iota_1"'] \arrow[d, "a^{-1}"] \\
(((A\otimes F)\oplus(B\otimes E))\otimes D)\oplus((B\otimes F)\otimes C) \arrow[d, "(\widehat{\tau}^1_{f,h}\otimes Id)\oplus(\tau\otimes\mathrm{Id})"'] & ((A\otimes F)\oplus(B\otimes E))\otimes D \arrow[l, "\iota_1"'] \arrow[d, "\widehat{\tau}^1_{f,h}\otimes\mathrm{Id}"] & (A\otimes F)\otimes D \arrow[l, "\iota_1\otimes\mathrm{Id}"'] \arrow[d, "\tau\otimes\mathrm{Id}"] \\
(((E\otimes B)\oplus(F\otimes A))\otimes D)\otimes((F\otimes B)\otimes C) \arrow[ddd, "\alpha"'] & ((E\otimes B)\oplus(F\otimes A))\otimes D \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"] & (F\otimes A)\otimes D \arrow[l, "\iota_2\otimes\mathrm{Id}"'] \arrow[dl, "\iota_2"] \arrow[dd, "a"] \\
 & ((E\otimes B)\otimes D)\oplus((F\otimes A)\otimes D) \arrow[d, "a\oplus a"] & \\
 & (E\otimes(B\otimes D))\oplus(F\otimes(A\otimes D)) \arrow[dl, "\mathrm{Id}\oplus(\mathrm{Id}\otimes\iota_1)"'] & F\otimes(A\otimes D) \arrow[l, "\iota_2"'] \arrow[dl, "\mathrm{Id}\otimes\iota_1"] \\
(E\otimes(B\otimes D))\oplus(F\otimes((A\otimes D)\oplus(B\otimes C))) & F\otimes((A\otimes D)\oplus(B\otimes C)) \arrow[l, "\iota_2"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[ddd, "\widehat{\tau}^1_{f \widehat{\otimes} g, h}"'] \arrow[ddr, "\tau \oplus \tau"] & ((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[l, "\iota_1"'] \arrow[d, "\tau"] & ((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[l, "\varepsilon"'] \\
 & F \otimes ((A \otimes D) \oplus (B \otimes C)) \arrow[d, "\iota_1"] & (A \otimes D) \otimes F \arrow[u, "\iota_1"] \arrow[d, "\tau"] \arrow[ul, "\iota_1 \otimes \mathrm{Id}"'] \\
 & (F \otimes ((A \otimes D) \oplus (B \otimes C))) \oplus (E \otimes (B \otimes D)) \arrow[dl, "\tau^{\oplus}"'] & F \otimes (A \otimes D) \arrow[ul, "\mathrm{Id} \otimes \iota_1"'] \arrow[dl, "\mathrm{Id} \otimes \iota_1"] \\
(E \otimes (B \otimes D)) \oplus (F \otimes ((A \otimes D) \oplus (B \otimes C))) & F \otimes ((A \otimes D) \oplus (B \otimes C)) \arrow[l, "\iota_2"] \arrow[u, "\iota_1"] &
\end{tikzcd}
$$

Definition 4.1.11. Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal category such that $\mathbf{C}$ is $\otimes$-distributive.

The braided semigroupal category $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a}, \widehat{\tau})$ defined in Theorem 4.1.10, will be denoted as $\mathrm{LP}(\mathcal{C})$ and will be called the *Loday-Pirashvili braided semigroupal category* of $\mathcal{C}$.

Definition 4.1.12. Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal category. We say that $\mathcal{C}$ is *symmetric* if for all objects A, B of $\mathbf{C}$ is satisfied that

$$\tau_{A,B}^{-1} = \tau_{B,A}.$$

Proposition 4.1.13. *Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal category such that $\mathbf{C}$ is $\otimes$-distributive. Then $\mathrm{LP}(\mathcal{C})$ is symmetric if and only if $\mathcal{C}$ is symmetric.*

Proof. Let us assume first that $\mathrm{LP}(\mathcal{C})$ is symmetric. For any $A, B \in \mathrm{Ob}(\mathbf{C})$ we know that $(\widehat{\tau}_{\mathrm{Id}_A,\mathrm{Id}_B})^{-1} = \widehat{\tau}_{\mathrm{Id}_B,\mathrm{Id}_A}$. Then, by taking the second component of the morphisms we have $\tau_{A,B}^{-1} = \tau_{B,A}$

Assume now that $\mathbf{C}$ is symmetric. For any $\begin{array}{c} A \\ \downarrow{\scriptstyle f} \\ B \end{array}, \begin{array}{c} C \\ \downarrow{\scriptstyle g} \\ D \end{array} \in \mathrm{Ob}\left(\mathrm{Hom}(\mathbf{C})\right)$ we have to prove that $(\widehat{\tau}_{f,g})^{-1} = \widehat{\tau}_{g,f}$. That means that we need to prove that $(\widehat{\tau}^1_{f,g}, \tau_{A,B})^{-1} = (\widehat{\tau}^1_{g,f}, \tau_{B,A})$, i.e. $(\widehat{\tau}^1_{f,g})^{-1} = \widehat{\tau}^1_{g,f}$ and $\tau^{-1}_{A,B} \circ \tau_{B,A}$. The second one is true by hypothesis.

We know that $\widehat{\tau}^1_{f,g}$ is an isomorphism in $\mathbf{C}$, since it is the component of an isomorphism, so it is just enough to prove $\widehat{\tau}^1_{g,f} \circ \widehat{\tau}^1_{f,g} = \mathrm{Id}_{(A\otimes D)\oplus(B\otimes C)}$.

$$(A \otimes D) \oplus (B \otimes C) \xrightarrow{\widehat{\tau}^1_{f,g}} (C \otimes B) \oplus (D \otimes A) \xrightarrow{\widehat{\tau}^1_{g,f}} (A \otimes D) \oplus (B \otimes C)$$

But this is obviously true, since what $\widehat{\tau}^1_{f,g}$ does is insert each part of the coproduct into its correspondent one and then twist it. But these twists are inverse to each other by assumption. □

4.1.4 Monoidal categories

Definition 4.1.14. Let $(\mathbf{C}, \otimes)$ be a category with an operation such that $\mathbf{C}$ has an initial object Λ. Then:

- $\mathbf{C}$ is said to be *left-$\otimes$-annihilated* (*right-$\otimes$-annihilated*) if the unique morphism $\Lambda_{A\otimes\Lambda} : \Lambda \to A \otimes \Lambda$ (respectively, $\Lambda_{\Lambda\otimes A} : \Lambda \to \Lambda \otimes A$) is an isomorphism. That means $A \otimes \Lambda$ ($\Lambda \otimes A$) is an initial object.

- $\mathbf{C}$ is *$\otimes$-annihilated* if it is both left-$\otimes$-annihilated and right-$\otimes$-annihilated.

Remark 4.1.15. The definition of left-$\otimes$-annihilated does not depend on the initial object. Assume that Λ' is also another initial object, we have $\Lambda_{\Lambda'}$ and Λ'_{Λ} are inverse to each other, so

$$\Lambda'_{A\otimes\Lambda'} = (\mathrm{Id}_A \otimes \Lambda_{\Lambda'}) \circ \Lambda_{A\otimes\Lambda} \circ \Lambda'_{\Lambda}$$

and $\Lambda'_{A\otimes\Lambda'}$ is an isomorphism by composition, since $\otimes$ preserves isomorphisms.

Theorem 4.1.16. *Let* $(\mathbf{C}, \otimes, a, I, l, r)$ *be a monoidal category such that* $(\mathbf{C}, \otimes)$ *is distributive and annihilated. Let* $\mathcal{C} = (\mathbf{C}, \otimes, a)$ *and consider its Loday-Pirashvili semigroupal category* $\mathrm{LP}(\mathcal{C}) = (\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a})$.

We will denote $\widehat{I} := \begin{array}{c}\Lambda\\ \downarrow{\scriptstyle \Lambda_I}\\ I\end{array}$, *and take* $\begin{array}{c}A\\ \downarrow{\scriptstyle f}\\ B\end{array} \in \mathrm{Ob}(\mathrm{Hom}(\mathbf{C}))$.

Let $\widehat{l}_f := (\widehat{l}^1_f, l_B)$ *and* $\widehat{r}_f := (\widehat{r}^1_f, r_B)$, *where* $\widehat{l}^1_f$ *is the following composition:*

$$(\Lambda \otimes B) \oplus (I \otimes A) \xrightarrow{\Lambda^{-1}_{\Lambda\otimes B} \oplus \mathrm{Id}_{I\otimes A}} \Lambda \oplus (I \otimes A) \xrightarrow{(\iota_2)^{-1}} I \otimes A \xrightarrow{l_A} A\,,$$

and $\widehat{r}^1_f$ *is the following composition:*

$$(A \otimes I) \oplus (B \otimes \Lambda) \xrightarrow{\mathrm{Id}_{A\otimes I} \oplus \Lambda^{-1}_{B\otimes\Lambda}} (A \otimes I) \oplus \Lambda \xrightarrow{(\iota_1)^{-1}} A \otimes I \xrightarrow{r_A} A\,.$$

Then $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a}, \widehat{I}, \widehat{l}, \widehat{r})$ *is a monoidal category.*

Proof. We will first prove that $\widehat{r}_f$ is well defined. This is the same as showing that the following diagram is commutative:

$$\begin{tikzcd}
(A \otimes I) \oplus (B \otimes \Lambda) \arrow[r, "\widehat{r}^1_f"] \arrow[d, "{[(f\otimes \mathrm{Id}_I),(\mathrm{Id}_B \otimes \Lambda_I)]}"'] & A \arrow[d, "f"] \\
B \otimes I \arrow[r, "r_B"'] & B
\end{tikzcd}$$

The composition with the second coproduct injection is trivial since $B \otimes \Lambda$ is an initial object. The first one follows by the commutative diagram:

$$\begin{tikzcd}
(A \otimes I) \oplus (B \otimes \Lambda) \arrow[rrrr, "\widehat{r}^{1}_{f}"] \arrow[drr, "\mathrm{Id} \otimes \Lambda^{-1}"] \arrow[ddd, "{[(f \otimes \mathrm{Id}),(\mathrm{Id} \otimes \Lambda)]}"'] & & & & A \arrow[ddd, "f"] \\
 & & (A \otimes I) \oplus \Lambda \arrow[r, "(\iota_1)^{-1}"] & A \otimes I \arrow[ur, "r"] & \\
 & & A \otimes I \arrow[uull, "\iota_1"] \arrow[ur, "\mathrm{Id}"'] \arrow[dll, "f \otimes \mathrm{Id}"'] & & \\
B \otimes I \arrow[rrrr, "r"'] & & & & B
\end{tikzcd}$$

The fact that $\widehat{r}$ is a natural isomorphism is immediate by being composition of natural isomorphisms. The same arguments work for $\widehat{l}$.

So we have to prove the triangle equation. For that, we will take $\begin{tikzcd} C \arrow[d, "g"] \\ D \end{tikzcd}$. The lower part is immediate since it is the triangle equation for the original monoidal category. We have to prove the triangle equation for the upper part, i.e. we need to prove the following diagram:

$$\begin{tikzcd}
(((A \otimes I) \oplus (B \otimes \Lambda)) \otimes D) \oplus ((B \otimes I) \otimes C) \arrow[rr, "\alpha_{f,\widehat{1},g}"] \arrow[dr, "(\widehat{r}^{1}_{f} \otimes \mathrm{Id}_D) \oplus (r_B \otimes \mathrm{Id}_C)"'] & & (A \otimes (I \otimes D)) \oplus (B \otimes ((\Lambda \otimes D) \oplus (I \otimes C))) \arrow[dl, "(\mathrm{Id}_A \otimes l_D) \oplus (\mathrm{Id}_B \otimes \widehat{l}^{1}_{g})"] \\
 & (A \otimes D) \oplus (B \otimes C) &
\end{tikzcd}$$

As in the previous theorem, we will use that the domain is a coproduct to study each part separately. The first one is $((A \otimes I) \oplus (B \otimes \Lambda)) \otimes D$ and the first morphism from there is ε, so we can precompose with that isomorphism. Now we have again a coproduct that we can study separately, but the second part is immediate since its domain, $(B \oplus \Lambda) \oplus D$, is an initial object. This is true because, since $(\mathbf{C}, \otimes)$ is annihilated. Left-annihilation say that $(B \otimes \Lambda)$ is an initial object, and since annihilation does not depend on the initial object selected, right-annihilation say that $(B \otimes \Lambda) \otimes C$

is an initial object. So, we will prove the first part. The second part of the original coproduct is established similarly.

$$
\begin{tikzcd}
(((A \otimes I) \oplus (B \otimes \Lambda)) \otimes D) \oplus ((B \otimes I) \otimes C) \arrow[dd, "\alpha_{f,\hat{I},g}"'] & ((A \otimes I) \oplus (B \otimes \Lambda)) \otimes D \arrow[l, "\iota_1"'] \arrow[d, "\varepsilon^{-1}"] & ((A \otimes I) \otimes D) \oplus ((B \otimes \Lambda) \otimes D) \arrow[l, "\varepsilon"'] \\
& ((A \otimes I) \otimes D) \oplus ((B \otimes \Lambda) \otimes D) \arrow[d, "a \oplus a"] & (A \otimes I) \otimes D \arrow[u, "\iota_1"'] \arrow[d, "a"] \arrow[dd, bend left, "r \otimes \mathrm{Id}"] \\
(A \otimes (I \otimes D)) \oplus (B \otimes ((\Lambda \otimes D) \oplus (I \otimes C))) \arrow[d, "(\mathrm{Id} \otimes l) \oplus (\mathrm{Id} \otimes \hat{l}^1_g)"'] & (A \otimes (I \otimes D)) \oplus (B \otimes (\Lambda \otimes D)) \arrow[l, "\mathrm{Id} \oplus (\mathrm{Id} \otimes \iota_1)"'] & A \otimes (I \otimes D) \arrow[l, "\iota_1"'] \arrow[ll, bend left, "\iota_1"] \arrow[d, "\mathrm{Id} \otimes l"] \\
(A \otimes D) \oplus (B \otimes C) & & A \otimes D \arrow[ll, "\iota_1"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(((A \otimes I) \oplus (B \otimes \Lambda)) \otimes D) \oplus ((B \otimes I) \otimes C) \arrow[d, "(\hat{r}^1_f \otimes \mathrm{Id}) \oplus (r \otimes \mathrm{Id})"'] & ((A \otimes I) \oplus (B \otimes \Lambda)) \otimes D \arrow[l, "\iota_1"'] \arrow[d, "\hat{r}^1_f \otimes \mathrm{Id}"'] & ((A \otimes I) \otimes D) \oplus ((B \otimes \Lambda) \otimes D) \arrow[l, "\varepsilon"'] \\
(A \otimes D) \oplus (B \otimes C) & A \otimes D \arrow[l, "\iota_1"] & (A \otimes I) \otimes D \arrow[l, "r \otimes \mathrm{Id}"] \arrow[u, "\iota_1"'] \arrow[ul, "\iota_1 \otimes \mathrm{Id}"]
\end{tikzcd}
$$

With this we know that the triangular equation holds and $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a}, \widehat{I}, \widehat{l}, \widehat{r})$ is a monoidal category. $\square$

Definition 4.1.17. Let $C = (\mathbf{C}, \otimes, a, I, l, r)$ be a monoidal category such that $\mathbf{C}$ is $\otimes$-distributive and $\otimes$-annihilated. The monoidal category $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a}, \widehat{I}, \widehat{l}, \widehat{r})$ defined in Theorem 4.1.16 will be denoted as $\mathrm{LP}(C)$ and will be called the *Loday-Pirashvili monoidal category* of C.

4.1.5 Braided monoidal categories

The notion of braided monoidal category was introduced by Joyal and Street in [38].

Definition 4.1.18. A *braided symmetric monoidal category* is a braided monoidal category $C = (\mathbf{C}, \otimes, a, I, l, r, \tau)$ where $(\mathbf{C}, \otimes, a, \tau)$ is a braided symmetric semigroupal category.

Definition 4.1.19. Let $C = (\mathbf{C}, \otimes, a, I, l, r, \tau)$ be a braided (symmetric) monoidal category such that $\mathbf{C}$ is $\otimes$-distributive and $\otimes$-annihilated. The braided (symmetric) monoidal category $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes}, \widehat{a}, \widehat{I}, \widehat{l}, \widehat{r}, \widehat{\tau})$ defined between Theorem 4.1.10 and Theorem 4.1.16, will be denoted as $\mathrm{LP}(C)$ and will be called the *Loday-Pirashvili braided monoidal category* of C.

Lemma 4.1.20. *Let $C = (\mathbf{C}, \otimes, a, I, l, r, \tau)$ be a braided monoidal category such that $\mathbf{C}$ is $\otimes$-distributive and $\otimes$-annihilated. Then $\mathrm{LP}(C)$ is symmetric if and only if C is symmetric.* □

4.2 Additive Categories with operations

In this section, we will try to give properties to the sum of two morphisms in an additive category with an operation in that category.

Lemma 4.2.1. *Let $(\mathbf{C}, \otimes)$ be a category with an operation, where $\mathbf{C}$ has finite coproducts. Then the following diagrams are commutative:*

$$
\begin{tikzcd}
(A \otimes B) \oplus (A \otimes B) \arrow[dr, "\bigtriangledown_{A\otimes B}"] \arrow[dd, "\gamma_{A,B,B}"'] & \\
& A \otimes B \\
A \otimes (B \oplus B) \arrow[ur, "\mathrm{Id}_A \otimes \bigtriangledown_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \oplus (A \otimes B) \arrow[dr, "\bigtriangledown_{A\otimes B}"] \arrow[dd, "\varepsilon_{A,A,B}"'] & \\
& A \otimes A \\
(A \oplus A) \otimes B \arrow[ur, "\bigtriangledown_A \otimes \mathrm{Id}_B"'] &
\end{tikzcd}
$$

where ε and γ are defined in Definition 4.1.5.

Proof. Since the domain of both are coproducts we can use the universal property. Take $k \in \{1, 2\}$.

$$
\begin{aligned}
(\mathrm{Id}_A \otimes \bigtriangledown_B) \circ \gamma_{A,B,B} \circ \iota_k &= (\mathrm{Id}_A \otimes \bigtriangledown_B) \circ (\mathrm{Id}_A \otimes \iota_k) = \mathrm{Id}_A \otimes (\bigtriangledown_B \circ \iota_k) \\
&= \mathrm{Id}_A \otimes \mathrm{Id}_B = \mathrm{Id}_{A\otimes B} = \bigtriangledown_{A\otimes B} \circ \iota_k
\end{aligned}
$$

The second one follows analogously. □

Theorem 4.2.2. *Let* $(\mathbf{C}, \otimes)$ *a category with an operation, distributive and annihilated, where* $\mathbf{C}$ *has biproducts. Then,*

(i) $(f+g)\otimes h = (f\otimes h) + (g\otimes h)$ *for* $A \xrightarrow{f,g} B$, $C \xrightarrow{h} D$,

(ii) $f\otimes(g+h) = (f\otimes g) + (f\otimes h)$ *for* $A \xrightarrow{f} B$, $C \xrightarrow{g,h} D$.

Proof. We will show part (i) since the second one is completely analogue. We need to prove that the following diagram is commutative:

$$\begin{tikzcd}
 & (A\oplus A)\otimes C \arrow[rr, "(f\oplus g)\otimes h"] & & (B\oplus B)\otimes D \arrow[dr, "\nabla\otimes \mathrm{Id}"] & \\
A\otimes C \arrow[ur, "\triangle\otimes \mathrm{Id}"] \arrow[dr, "\triangle"'] & & & & B\otimes D \\
 & (A\otimes C)\oplus(A\otimes C) \arrow[rr, "(f\otimes h)\oplus(g\otimes h)"'] & & (B\otimes D)\oplus(B\otimes D) \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

Besides, to check that the previous diagram is commutative is equivalent to prove that the following subdiagrams are commutative:

$$\begin{tikzcd}
 & (A\oplus A)\otimes C \arrow[rr, "(f\oplus g)\otimes h"] & & (B\oplus B)\otimes D \arrow[dr, "\nabla\otimes \mathrm{Id}"] & \\
A\otimes C \arrow[ur, "\triangle\otimes \mathrm{Id}_C"] \arrow[dr, "\triangle"'] & & & & B\otimes D \\
 & (A\otimes C)\oplus(A\otimes C) \arrow[rr, "(f\otimes h)\oplus(g\otimes h)"'] \arrow[uu, "\varepsilon"] & & (B\otimes D)\oplus(B\otimes D) \arrow[ur, "\nabla"'] \arrow[uu, "\varepsilon"] &
\end{tikzcd}$$

The rightmost subdiagram has already appeared in Lemma 4.2.1, whereas the middle subdiagram is naturalness from Proposition 4.1.6. Let us prove that the left subdiagram is commutative.

For doing this, we will define $\delta_{i,j}^{X\oplus Y} := \iota_i^{X\oplus Y} \circ \pi_j^{X\oplus Y}$, i.e. $\delta_{i,j} = \mathrm{Id}$ if $i=j$ and $\delta_{i,j} = 0$ if $i \neq j$.

Given the following diagram

$$\begin{tikzcd}
& (A \oplus A) \otimes C \arrow[dd, shift right, "\varepsilon^{-1}"'] \arrow[rrdd, bend left, "\pi_1 \otimes \mathrm{Id}"] & & \\
A \otimes C \arrow[ru, "\triangle \otimes \mathrm{Id}"] \arrow[rd, "\triangle"'] & & A \otimes C \arrow[lu, "\iota_i \otimes \mathrm{Id}"'] \arrow[ld, "\iota_i"] \arrow[rd, "\iota_i \circ \pi_j"] & \\
& (A \otimes C) \oplus (A \otimes C) \arrow[uu, shift right, "\varepsilon"'] \arrow[rr, "\pi_j"'] & & A \otimes C
\end{tikzcd}$$

for $i, j \in \{1, 2\}$. Note that $\iota_i \circ \pi_j = \mathrm{Id}_{X \otimes Y}$ if $i = j$, and $\iota_i \circ \pi_j = 0$, if $i \neq j$. Then, it is straightforward that each subdiagram of the right is commutative: if $i = j$, it is immediate, and if $i \neq j$, the topmost triangle comes from left-$\otimes$-annihilation ($0 \otimes \mathrm{Id}_C = 0$). Therefore, using that the object in the bottom is a coproduct, the outer right triangle is commutative:

$$\begin{tikzcd}
& (A \oplus A) \otimes C \arrow[dd, shift right, "\varepsilon^{-1}"'] \arrow[rdd, bend left, "\pi_1 \otimes \mathrm{Id}"] & \\
A \otimes C \arrow[ru, "\triangle \otimes \mathrm{Id}"] \arrow[rd, "\triangle"'] & & \\
& (A \otimes C) \oplus (A \otimes C) \arrow[uu, shift right, "\varepsilon"'] \arrow[r, "\pi_j"'] & A \otimes C
\end{tikzcd}$$

Since the object of the bottom is also a product and we have the out square is commutative, we conclude that the left triangle is also commutative. □

Corollary 4.2.3. *Let* $(\mathbf{C}, \otimes)$ *a category with an operation distributive and annihilated, where* $\mathbf{C}$ *is additive. Then, for* $A \xrightarrow{f} B$, $C \xrightarrow{g} D$ *morphisms and* $E \xrightarrow{0} F$, *we have:*

(i) $f \otimes 0 \colon A \otimes E \to B \otimes F$ *is the zero morphism.*

(ii) $0 \otimes f \colon E \otimes A \to F \otimes B$ *is the zero morphism*

(iii) $f \otimes (-g) = (-f) \otimes g = -(f \otimes g)$

Proof. It follows using the previous theorem together with the Abelian group properties of the homomorphisms. □

Remark 4.2.4. Since the definition of additive category resides in natural morphisms, it is immediate that if **C** has biproducts, then the category Hom(**C**) also has biproducts. In fact, it can be easily proved that the operation in this category is $(f, g) + (h, k) = (f + h, g + k)$. Thus, if **C** is additive, then Hom(**C**) is additive and $-(f, g) = (-f, -g)$.

4.3 Lie and Leibniz objects in LP

4.3.1 Lie objects and Leibniz Objects

In this subsection, we will study the Lie objects and the Leibniz objects. The definitions of that concepts can be seen in Definition 2.4.3 and Definition 2.4.5.

Lemma 4.3.1. *Let $C = (\mathbf{C}, \otimes, a, \tau)$ be a symmetric semigroupal category where* **C** *is an additive category. Let (L, μ) be a Leibniz object. Then,*

$$\mu\circ(\mathrm{Id}_L \otimes \mu) = -\mu\circ(\mathrm{Id}_L \otimes \mu)\circ(\mathrm{Id}_L \otimes \tau_{L,L}).$$

Proof. By symmetry, if we compose the Leibniz identity with $a^{-1}\circ(\mathrm{Id}_L \otimes \mu)\circ a$, we get

$$\mu\circ(\mu \otimes \mathrm{Id}_L)\circ a_L^{-1}\circ(\mathrm{Id}_L \otimes \tau_{L,L})\circ a_L = \mu\circ(\mu \otimes \mathrm{Id}_L) + \mu\circ(\mathrm{Id}_L \otimes \mu)\circ(\mathrm{Id}_L \otimes \tau_{L,L})\circ a_L.$$

Substituting it in the Leibniz identity, we obtain

$$\begin{aligned}\mu\circ(\mu \otimes \mathrm{Id}_L) = \mu\circ(\mathrm{Id}_L \otimes \mu)\circ a_L &+ \mu\circ(\mu \otimes \mathrm{Id}_L)\\ &+ \mu\circ(\mathrm{Id}_L \otimes \mu)\circ(\mathrm{Id}_L \otimes \tau_{L,L})\circ a_L\end{aligned}$$

Finally, subtracting $\mu\circ(\mu \otimes \mathrm{Id}_L) + \mu\circ(\mathrm{Id}_L \otimes \mu)\circ(\mathrm{Id}_L \otimes \tau_{L,L})\circ a_L$ in both sides of the identity, we get

$$-\mu\circ(\mathrm{Id}_L \otimes \mu)\circ(\mathrm{Id}_L \otimes \tau_{L,L})\circ a_L = \mu\circ(\mathrm{Id}_L \otimes \mu)\circ a_L,$$

which composed with a_L^{-1} gives us the desired identity. □

Proposition 4.3.2. *Let $C = (\mathbf{C}, \otimes, a, \tau)$ be a braided symmetrical semigroupal category where* $\mathbf{C}$ *is an additive $\otimes$-distributive $\otimes$-annihilated category. The following identity is called the* Jacobi identity*:*

$$\mu \circ (\mathrm{Id}_L \otimes \mu) \circ (\mathrm{Id}_{L\otimes(L\otimes L)} + a_L \circ \tau_{L,L\otimes L} + \tau_{L\otimes L,L} \circ a_L^{-1}) = 0. \tag{Jac}$$

Then (L, μ) is a Lie object if and only if it satisfies (AC) *and* (Jac).

Proof. Let (L, μ) be a Lie object and let us prove that (Jac) holds. We can rewrite the Leibniz identity as

$$0 = \mu \circ (\mathrm{Id}_L \otimes \mu) + \mu \circ (\mu \otimes \mathrm{Id}_L) \circ a_L^{-1} \circ (\mathrm{Id}_L \otimes \tau_{L,L}) - \mu \circ (\mu \otimes \mathrm{Id}_L) \circ a_L^{-1}.$$

The first and third summands are equal to the first and third summands of (Jac), respectively, since

$$\tau_{L,L} \circ (\mu \otimes \mathrm{Id}_L) = (\mathrm{Id}_L \otimes \mu) \circ \tau_{L\otimes L,L}.$$

To see the second one, we will use (AC), the naturalness and symmetry of τ, the second hexagon equation and Lemma 4.3.1:

$$\begin{aligned}
\mu \circ (\mu \otimes \mathrm{Id}_L) \circ a_L^{-1} \circ (\mathrm{Id}_L \otimes \tau_{L,L}) &= -\mu \circ \tau_{L,L} \circ (\mu \otimes \mathrm{Id}_L) \circ a_L^{-1} \circ (\mathrm{Id}_L \otimes \tau_{L,L}) \\
&= -\mu \circ (\mathrm{Id}_L \otimes \mu) \circ \tau_{L\otimes L,L} \circ a_L^{-1} \circ (\mathrm{Id}_L \otimes \tau_{L,L}) \\
&= -\mu \circ (\mathrm{Id}_L \otimes \mu) \circ \tau_{L,L\otimes L}^{-1} \circ a_L^{-1} \circ (\mathrm{Id}_L \otimes \tau_{L,L}) \\
&= -\mu \circ (\mathrm{Id}_L \otimes \mu) \circ a_L \circ (\tau_{L,L} \otimes \mathrm{Id}_L)^{-1} \circ a_L^{-1} \\
&= -\mu \circ (\mathrm{Id}_L \otimes \mu) \circ a_L \circ (\tau_{L,L} \otimes \mathrm{Id}_L) \circ a_L^{-1} \\
&= \mu \circ (\mathrm{Id}_L \otimes \mu) \circ (\mathrm{Id}_L \otimes \tau_{L,L}) \circ a_L \circ (\tau_{L,L} \otimes \mathrm{Id}_L) \circ a_L^{-1} \\
&= \mu \circ (\mathrm{Id}_L \otimes \mu) \circ a_L \circ \tau_{L,L\otimes L}.
\end{aligned}$$

Conversely, using Theorem 4.2.2 and Corollary 4.2.3, the identity (AC) gives us the same equality given in Lemma 4.3.1.

$$\begin{aligned}
-\mu \circ \mathrm{Id}_L \otimes \mu \circ \mathrm{Id}_L \otimes \tau_{L,L} &= \mu \circ \mathrm{Id}_L \otimes (-\mu) \circ \mathrm{Id}_L \otimes \tau_{L,L} \\
&= \mu \circ \mathrm{Id}_L \otimes (-\mu \circ \tau_{L,L}) = \mu \circ \mathrm{Id}_L \otimes \mu.
\end{aligned}$$

Then, the identity (Lb) automatically follows. □

Remark 4.3.3. Note that in the last proof, we do not use the anticommutativity inside μ to show that a Lie object satisfies (Jac), so Theorem 4.2.2 and Corollary 4.2.3 are not needed. Therefore, for that implication, the category does not need to be $\otimes$-distributive nor $\otimes$-annihilated.

4.3.2 Liesation

In this subsection, we will construct the Lieisation functor from the category of Leibniz objects to the category of Lie objects.

Definition 4.3.4. Let $(\mathbf{C}, \otimes, a, \tau)$ to be a braiding semigroupal where $\mathbf{C}$ is an additive category with coequalisers. The *Liesation coequaliser* of a Leibniz object (L, μ) is the following coequaliser

$$L \otimes L \underset{\mu \circ \tau_{L,L}}{\overset{-\mu}{\rightrightarrows}} L \xrightarrow{\pi^L} \overline{L}$$

Definition 4.3.5. Let $\mathcal{C} = (\mathbf{C}, \otimes)$ be a category with an operation. We will say that $\mathcal{C}$ is a *closed category* (or that $\mathbf{C}$ is $\otimes$-*closed*) if the functor $- \otimes L$ is a left adjoint for each $L \in \mathrm{Ob}(\mathbf{C})$.

Remark 4.3.6. Let $\mathcal{C} = (\mathbf{C}, \otimes)$ be a closed category. Since left adjoints preserve colimits we have that:

- If $\mathbf{C}$ has finite coproducts, then $\mathbf{C}$ is R-$\otimes$-distributive.
- If $\mathbf{C}$ has zero object, then $\mathbf{C}$ is R-$\otimes$-annihilated.

In addition, if we have a braiding, then $L \otimes -$ is also a left adjoint. Therefore, if we are in a braided category we have that:

- If $\mathbf{C}$ has finite coproducts then it is $\otimes$-distributive.
- If $\mathbf{C}$ has zero object then it is $\otimes$-annihilated.

We will state the main theorem for Liesation.

Theorem 4.3.7. *Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a semigroupal symmetric category such that $\mathbf{C}$ is an additive $\otimes$-clossed finitely cocomplete category. Let (L, μ) be a Leibniz object and consider the Liesation coequaliser $L \otimes L \underset{\mu\circ\tau_{L,L}}{\overset{-\mu}{\rightrightarrows}} L \xrightarrow{\pi^L} \overline{L}$. There exists a unique $\overline{\mu}$ making the following diagram commutative*

$$\begin{tikzcd} L \otimes L \arrow[r, "\pi^L \otimes \pi^L"] \arrow[d, "\mu"'] & \overline{L} \otimes \overline{L} \arrow[d, "\overline{\mu}"] \\ L \arrow[r, "\pi^L"] & \overline{L} \end{tikzcd}$$

Moreover, it satisfies the following properties:

(i) $(\overline{L}, \overline{\mu})$ *is a Lie object.*

(ii) $(L, \mu) \xrightarrow{\pi^L} (\overline{L}, \overline{\mu})$ *is a morphism between Leibniz objects.*

(iii) There is a functor $\overline{(-)}:\ \mathrm{Leib}(\mathcal{C}) \to \mathrm{Lie}(\mathcal{C})$ *defined in arrows as*

$$\overline{(-)}\Big((L, \mu) \xrightarrow{f} (M, \xi)\Big) = (\overline{L}, \overline{\mu}) \xrightarrow{\overline{f}} (\overline{M}, \overline{\xi}),$$

where $\overline{f}$ is the unique morphism induced by the coequaliser

$$\begin{tikzcd} L \otimes L \arrow[r, shift left, "-\mu"] \arrow[r, shift right, "\mu\circ\tau_{L,L}"'] & L \arrow[r, "\pi^L"] \arrow[d, "f"'] & \overline{L} \arrow[d, dashed, "\overline{f}"] \\ & M \arrow[r, "\pi^M"] & \overline{M} \end{tikzcd}$$

(iv) The functor $\overline{(-)}$ is left adjoint to the forgetful functor U. Moreover, we have the identity $\overline{(-)} \circ \mathrm{U} \cong \mathrm{Id}_{\mathrm{Lie}(\mathcal{C})}$.

Proof. To define $\overline{\mu}$ we will begin defining an auxiliary $\widehat{\mu}$:

$$\begin{tikzcd} L \otimes L \arrow[r, "\mathrm{Id}_L \otimes \pi^L"] \arrow[dr, "\mu"'] & L \otimes \overline{L} \arrow[d, dashed, "\widehat{\mu}"] \\ & L \end{tikzcd}$$

Since **C** is $\otimes$-closed and we are in braided category, we know that the functor $(L \otimes -)$ preserves colimits. Then, applying it to the Liesation diagram we know that the following diagram is a coequaliser.

$$L \otimes (L \otimes L) \underset{(\mathrm{Id}_L \otimes \mu)\circ(\mathrm{Id}_L \otimes \tau_{L,L})}{\overset{-\mathrm{Id}_L \otimes \mu}{\rightrightarrows}} L \otimes L \xrightarrow{\mathrm{Id}_L \otimes \pi^L} L \otimes \overline{L}.$$

By Lemma 4.3.1 it is clear that both arrows composed with μ are equal, so $\widehat{\mu}$ is defined by the universal properties of the coequaliser.

Now, since $(-\otimes \overline{L})$ also preserves colimits, the following diagram is a coequaliser.

$$(L \otimes L) \otimes \overline{L} \underset{(\mu\otimes\mathrm{Id}_{\overline{L}})\circ(\tau_{L,L}\otimes\mathrm{Id}_{\overline{L}})}{\overset{-\mu\otimes\mathrm{Id}_{\overline{L}}}{\rightrightarrows}} L \otimes \overline{L} \xrightarrow{\pi^L\otimes\mathrm{Id}_{\overline{L}}} \overline{L} \otimes \overline{L}.$$

Let us see that both arrows composed with $\pi^L\circ\widehat{\mu}$ are equal, using the fact that $\mathrm{Id}_{L\otimes L} \otimes\pi^L$ is an epimorphism:

$$\begin{aligned}
&\pi^L\circ\widehat{\mu}\circ(-\mu \otimes \mathrm{Id}_{\overline{L}})\circ(\mathrm{Id}_{L\otimes L} \otimes\pi^L) = -\pi^L\circ\widehat{\mu}\circ(\mu \otimes \mathrm{Id}_{\overline{L}})\circ(\mathrm{Id}_{L\otimes L} \otimes\pi^L)\\
&= -\pi^L\circ\widehat{\mu}\circ(\mathrm{Id}_L \otimes\pi^L)\circ(\mu \otimes \mathrm{Id}_L) = -\pi^L\circ\mu\circ(\mu \otimes \mathrm{Id}_L)\\
&= \pi^L\circ\mu\circ\tau_{L,L}\circ(\mu \otimes \mathrm{Id}_L) = \pi^L\circ\mu\circ(\mathrm{Id}_L \otimes\mu)\circ\tau_{L\otimes L,L}\\
&= -\pi^L\circ\mu\circ(\mathrm{Id}_L \otimes\mu)\circ(\mathrm{Id}_L \otimes\tau_{L,L})\circ\tau_{L\otimes L,L}\\
&= -\pi^L\circ\mu\circ\tau_{L,L}\circ(\mu \otimes \mathrm{Id}_L)\circ(\tau_{L,L} \otimes \mathrm{Id}_L)\\
&= \pi^L\circ\mu\circ(\mu \otimes \mathrm{Id}_L)\circ(\tau_{L,L} \otimes \mathrm{Id}_L)\\
&= \pi^L\circ\widehat{\mu}\circ(\mathrm{Id}_L \otimes\pi^L)\circ(\mu \otimes \mathrm{Id}_L)\circ(\tau_{L,L} \otimes \mathrm{Id}_L)\\
&= \pi^L\circ\widehat{\mu}\circ(\mu \otimes \mathrm{Id}_{\overline{L}})\circ(\tau_{L,L} \otimes \mathrm{Id}_{\overline{L}})\circ \mathrm{Id}_{L\otimes L}\,.
\end{aligned}$$

Then, we define $\overline{\mu}$ as the unique arrow

$$\begin{tikzcd}
L \otimes \overline{L} \arrow[r, "\pi^L\otimes\mathrm{Id}_{\overline{L}}"] \arrow[d, "\widehat{\mu}"'] & \overline{L} \otimes \overline{L} \arrow[d, dashed, "\overline{\mu}"] \\
L \arrow[r, "\pi^L"] & \overline{L}
\end{tikzcd}$$

Let us see that $\overline{\mu}$ is the unique morphism that makes the following diagram commutative:

$$\begin{tikzcd}
L \otimes L \arrow[r, "\pi^L \otimes \pi^L"] \arrow[d, "\mu"'] & \overline{L} \otimes \overline{L} \arrow[d, "\overline{\mu}"] \\
L \arrow[r, "\pi^L"] & \overline{L}.
\end{tikzcd}$$

It is commutative since

$$\overline{\mu}\circ(\pi^L \otimes \pi^L) = \overline{\mu}\circ(\pi^L \circ \mathrm{Id}_{\overline{L}})\circ(\mathrm{Id}_L \otimes \pi^L) = \pi^L \circ \widehat{\mu}\circ(\mathrm{Id}_L \otimes \pi^L) = \pi^L \circ \mu,$$

whereas uniqueness follows by the fact that $\pi^L \otimes \pi^L$ is an epimorphism, by being composition of coequalisers.

Now we want to see that $(\overline{L}, \overline{\mu})$ is a Lie object. First we will prove (AC). We know that

$$\begin{aligned}
-\mu\circ(\pi^L \otimes \pi^L) &= -\pi^L \circ \mu = \pi^L \circ \mu \circ \tau_{L,L} = \overline{\mu}\circ(\pi^L \otimes \pi^L)\circ\tau_{L,L} \\
&= \overline{\mu}\circ\tau_{\overline{L},\overline{L}}\circ(\pi^L \otimes \pi^L).
\end{aligned}$$

Since they are equal right-composed with an epimorphism, we conclude that

$$-\overline{\mu} = \overline{\mu}\circ\tau_{\overline{L},\overline{L}}.$$

To prove (Lb), we will use a similar argument, since $(\pi^L \otimes \pi^L) \otimes \pi^L$ is also an epimorphism.

$$\begin{aligned}
&\overline{\mu}\circ(\overline{\mu} \otimes \mathrm{Id}_{\overline{L}})\circ\big((\pi^L \otimes \pi^L) \otimes \pi^L\big) \\
&= \overline{\mu}\circ\big((\overline{\mu}\circ(\pi^L \otimes \pi^L)) \otimes \pi^L\big) = \mu\circ\big((\pi^L \circ \mu) \otimes \pi^L\big) \\
&= \overline{\mu}\circ(\pi^L \otimes \pi^L)\circ(\mu \otimes \mathrm{Id}_L) = \pi^L \circ \mu\circ(\mu \otimes \mathrm{Id}_L) \\
&= \pi^L \circ \mu \circ \mu \otimes \mathrm{Id}_L \circ a_L^{-1} \circ \mathrm{Id}_L \otimes \tau_{L,L} \circ a_L + \pi^L \circ \mu \circ \mathrm{Id}_L \otimes \mu \circ a_L \\
&= \overline{\mu}\circ(\pi^L \otimes \pi^L)\circ(\mu \otimes \mathrm{Id}_L)\circ a_L^{-1}\circ(\mathrm{Id}_L \otimes \tau_{L,L})\circ a_L \\
&\quad + \overline{\mu}\circ(\pi^L \otimes \pi^L)\circ(\mathrm{Id}_L \otimes \mu)\circ a_L \\
&= \overline{\mu}\circ(\overline{\mu} \otimes \mathrm{Id}_L)\circ\big((\pi^L \otimes \pi) \otimes \pi^L\big)\circ a_L^{-1}\circ(\mathrm{Id}_L \otimes \tau_{L,L})\circ a_L
\end{aligned}$$

$$
\begin{aligned}
&+\overline{\mu}\circ(\mathrm{Id}_L\otimes\overline{\mu})\circ\big(\pi^L\otimes(\pi^L\otimes\pi^L)\big)\circ a_L\\
&=(\overline{\mu}\circ(\overline{\mu}\otimes\mathrm{Id}_{\overline{L}})\circ a^{-1}_{\overline{L}}\circ(\mathrm{Id}_{\overline{L}}\otimes\tau_{\overline{L},\overline{L}})\circ a_{\overline{L}}+\overline{\mu}\circ(\mathrm{Id}_{\overline{L}}\otimes\overline{\mu})\circ a_{\overline{L}})\circ\big((\pi^L\otimes\pi^L)\circ\pi^L\big).
\end{aligned}
$$

Now that we know that $(\overline{L},\overline{\mu})$ satisfies (Lb), the fact that $\pi^L:\ (L,\mu)\to(\overline{L},\overline{\mu})$ is a Leibniz morphism is a direct consequence of the following diagram being commutative:

$$
\begin{tikzcd}
L\otimes L \arrow[r, "\pi^L\otimes\pi^L"] \arrow[d, "\mu"'] & \overline{L}\otimes\overline{L} \arrow[d, "\overline{\mu}"] \\
L \arrow[r, "\pi^L"] & \overline{L}
\end{tikzcd}
$$

Let $(L,\mu)\xrightarrow{f}(M,\xi)$ a morphism in Leib$(\mathcal{C})$. Then,

$$
\begin{aligned}
\pi^M\circ f\circ(-\mu)&=-\pi^M\circ f\circ\mu=-\pi^M\circ\xi\circ(f\otimes f)=\pi^M\circ\xi\circ\tau_{M,M}\circ(f\otimes f)\\
&=\pi^M\circ\xi\circ(f\otimes f)\circ\tau_{L,L}=\pi^M\circ f\circ\mu\circ\tau_{L.L},
\end{aligned}
$$

shows that $\overline{f}$ is well defined. To see that it is a Lie morphism we need to check that the following diagram is commutative:

$$
\begin{tikzcd}
\overline{L}\otimes\overline{L} \arrow[r, "\overline{f}\otimes\overline{f}"] \arrow[d, "\overline{\mu}"'] & \overline{M}\otimes\overline{M} \arrow[d, "\overline{\xi}"] \\
\overline{L} \arrow[r, "\overline{f}"] & \overline{M}
\end{tikzcd}
$$

It is true, since

$$
\begin{aligned}
\overline{\xi}\circ(\overline{f}\otimes\overline{f})\circ(\pi^L\otimes\pi^L)&=\overline{\xi}\circ\big((\overline{f}\circ\pi^L)\otimes(\overline{f}\otimes\pi^L)\big)\\
&=\overline{\xi}\circ\big((\pi^M\circ f)\otimes(\pi^M\circ f)\big)=\overline{\xi}\circ(\pi^M\otimes\pi^M)\circ(f\otimes f)\\
&=\pi^M\circ\xi\circ(f\otimes f)=\pi^M\circ f\circ\mu=\overline{f}\circ\pi^L\circ\mu\\
&=\overline{f}\circ\overline{\mu}\circ(\pi^L\otimes\pi^L),
\end{aligned}
$$

so it follows using the fact that $\pi^L\otimes\pi^L$ is an epimorphism.

To finish the proof we will show that $\overline{(-)}\dashv$ U. Let $g:\ (L,\mu)\to(M,\xi)$ a morphism of Leibniz objects, where (M,ζ) is also a Lie object. Then, we define $\widetilde{g}$ as the unique arrow such that

$$\begin{tikzcd} L \arrow[r, "\pi^L"] \arrow[dr, "g"'] & \overline{L} \arrow[d, dashed, "\widetilde{g}"] \\ & M \end{tikzcd}$$

It is well defined since

$$\begin{aligned} g\circ(-\mu) &= -g\circ\mu = -\xi\circ(g\otimes g) = \xi\circ\tau_{M,M}\circ(g\otimes g) \\ &= \xi\circ(g\otimes g)\circ\tau_{L,L} = g\circ\mu\circ\tau_{L,L}, \end{aligned}$$

and it is a Leibniz morphism since

$$\begin{aligned} \widetilde{g}\circ\overline{\mu}\circ(\pi^L\otimes\pi^L) &= \widetilde{g}\circ\pi^L\circ\mu = g\circ\mu = \xi\circ(g\otimes g) \\ &= \xi\circ\big((\widetilde{g}\circ\pi^L)\otimes(\widetilde{g}\circ\pi^L)\big) = \xi\circ(\widetilde{g}\otimes\widetilde{g})\circ(\pi^L\otimes\pi^L). \end{aligned}$$

This assignment induces a natural isomorphism between $\mathrm{Hom}_{\mathrm{Lie}(\mathcal{C})}\big((\overline{L},\overline{\mu}),(M,\xi)\big)$ and $\mathrm{Hom}_{\mathrm{Leib}(\mathcal{C})}\big((L,\mu),(M,\xi)\big)$, proving the adjunction. $\square$

We will now internalise the definition of a right-module of a Lie algebra (Definition 2.4.7).

Definition 4.3.8. Let $(\mathbf{C},\otimes,a,\tau)$ be a braided semigroupal category such that $\mathbf{C}$ is additive and consider a Lie object (L,μ). An (L,μ)-*right module* is a pair (S,ν) where S is an object of $\mathbf{C}$ and ν is a morphism from $S\otimes L$ to S satisfying:

$$\nu\circ(\nu\otimes\mathrm{Id}_L) = \nu\circ(\nu\otimes\mathrm{Id}_L)\circ a^{-1}_{S,L,L}\circ(\mathrm{Id}_S\otimes\tau_{L,L})\circ a_{S,L,L} + \nu\circ(\mathrm{Id}_S\otimes\mu)\circ a_{S,L,L}.$$

Note that if (L,μ) is a Lie object, then (L,μ) is itself a (L,μ)-right module.

Definition 4.3.9. Let $\mathcal{C}=(\mathbf{C},\otimes,a,\tau)$ be a braided semigroupal category such that $\mathbf{C}$ is additive. Let (S,ν) be an (L,μ)-right module, (S',ν') an (L',μ')-right module, and $\gamma\colon (L,\mu)\to(L',\mu')$ a Lie morphism in $\mathcal{C}$. Then, a morphism $f\colon S\to S'$ is called (γ,ν,ν')-*equivariant* if the following diagram is commutative:

$$\begin{tikzcd} S\otimes L \arrow[r, "\nu"] \arrow[d, "f\otimes\gamma"'] & S \arrow[d, "f"] \\ S'\otimes L' \arrow[r, "\nu'"] & S' \end{tikzcd}$$

If $\gamma = \mathrm{Id}_L$ then we will say that f is $\big((L,\mu),\nu,\nu'\big)$-*equivariant*.

Theorem 4.3.10. *Let $\mathcal{C} = (C,\otimes,a,\tau)$ be a semigroupal symmetric category such that $\mathbf{C}$ is an additive clossed finitely cocomplete category and let (L,μ) be a Leibniz object. There exists a unique $\widehat{\mu}$ such that the following diagram is commutative*

$$\begin{tikzcd}
L \otimes L \arrow[r, "\mathrm{Id}_L \otimes \pi^L"] \arrow[dr, "\mu"'] & L \otimes \overline{L} \arrow[d, dashed, "\widehat{\mu}"] \\
& L
\end{tikzcd}$$

and it satisfies that:

(i) $(L,\widehat{\mu})$ *is a* $(\overline{L},\overline{\mu})$*-right module.*

(ii) π^L *is* $\big((\overline{L},\overline{\mu}),\widehat{\mu},\overline{\mu}\big)$*-equivariant.*

Proof. That $\widehat{\mu}$ is well defined and it is unique is already obtained in the first part of the proof of Theorem 4.3.7. Let us see that $(L,\widehat{\mu})$ is an $(\overline{L},\overline{\mu})$. Since,

$$\begin{aligned}
&\widehat{\mu}\circ(\widehat{\mu}\otimes \mathrm{Id}_{\overline{L}})\circ\big((\mathrm{Id}_L\otimes\pi^L)\otimes\pi^L\big) = \widehat{\mu}\circ\big((\widehat{\mu}\circ \mathrm{Id}_L\otimes\pi^L)\otimes\pi^L\big) = \widehat{\mu}\circ(\mu\otimes\pi^L)\\
&= \widehat{\mu}\circ(\mathrm{Id}_L\otimes\pi^L)\circ(\mu\otimes \mathrm{Id}_L) = \mu\circ(\mu\otimes \mathrm{Id}_L)\\
&= \mu\circ(\mu\otimes \mathrm{Id}_L)\circ a_L^{-1}\circ(\mathrm{Id}_L\otimes\tau_{L,L})\circ a_L + \mu\circ(\mathrm{Id}_L\otimes\mu)\circ a_L\\
&= \widehat{\mu}\circ(\mathrm{Id}_L\otimes\pi^L)\circ(\mu\otimes \mathrm{Id}_L)\circ a_L^{-1}\circ(\mathrm{Id}_L\otimes\tau_{L,L})\circ a_L\\
&\qquad + \widehat{\mu}\circ(\mathrm{Id}_L\otimes\pi^L)\circ(\mathrm{Id}_L\otimes\mu)\circ a_L\\
&= \widehat{\mu}\circ\big((\widehat{\mu}\circ \mathrm{Id}_L\otimes\pi^L)\otimes \mathrm{Id}_L\big)\circ(\mathrm{Id}_{L\otimes L}\otimes\pi^L)\circ a_L^{-1}\circ(\mathrm{Id}_L\otimes\tau_{L,L})\circ a_L\\
&\qquad + \widehat{\mu}\circ\big(\mathrm{Id}_L\otimes(\pi^L\circ\mu)\big)\circ a_L\\
&= \widehat{\mu}\circ(\widehat{\mu}\otimes \mathrm{Id}_L)\circ\big((\mathrm{Id}_L\otimes\pi^L)\otimes \mathrm{Id}_L\big)\circ(\mathrm{Id}_{L\otimes L}\otimes\pi^L)\circ a_L^{-1}\circ(\mathrm{Id}_L\otimes\tau_{L,L})\circ a_L\\
&\qquad + \widehat{\mu}\circ\big(\mathrm{Id}_L\otimes(\overline{\mu}\circ\pi^L\otimes\pi^L)\big)\circ a_L\\
&= \widehat{\mu}\circ(\widehat{\mu}\otimes \mathrm{Id}_L)\circ\big((\mathrm{Id}_L\otimes\pi^L)\otimes\pi^L\big)\circ a_L^{-1}\circ(\mathrm{Id}_L\otimes\tau_{L,L})\circ a_L\\
&\qquad + \widehat{\mu}\circ(\mathrm{Id}_L\otimes\overline{\mu})\circ\big(\mathrm{Id}_L\otimes(\pi^L\otimes\pi^L)\big)\circ a_L\\
&= \widehat{\mu}\circ(\widehat{\mu}\otimes \mathrm{Id}_L)\circ a^{-1}_{L,\overline{L},\overline{L}}\circ(\mathrm{Id}_L\otimes\tau_{\overline{L},\overline{L}})\circ a_{L,\overline{L},\overline{L}}\circ\big((\mathrm{Id}_L\otimes\pi^L)\otimes\pi^L\big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \widehat{\mu}\circ(\mathrm{Id}_L \otimes\overline{\mu})\circ a_{L,\overline{L},\overline{L}}\circ\big((\mathrm{Id}_L \otimes\pi^L) \otimes \pi^L\big)\\
&= (\widehat{\mu}\circ(\widehat{\mu} \otimes \mathrm{Id}_L)\circ a^{-1}_{L,\overline{L},\overline{L}}\circ(\mathrm{Id}_L \otimes\tau_{\overline{L},\overline{L}})\circ a_{L,\overline{L},\overline{L}}\\
&\qquad\qquad + \widehat{\mu}\circ(\mathrm{Id}_L \otimes\overline{\mu})\circ a_{L,\overline{L},\overline{L}})\circ\big((\mathrm{Id}_L \otimes\pi^L) \otimes \pi^L\big),
\end{aligned}
$$

using the fact that $(\mathrm{Id}_L \otimes\pi^L) \otimes \pi^L$ is an epimorphism, we obtain the result. Equivariance follows again by the diagram that define $\overline{\mu}$ in the proof of Theorem 4.3.7. □

4.3.3 Lie Objects in LP

In this subsection, we show that the Leibniz objects are a full coreflective subcategory of the Lie objects in the correspondent LP category.

Proposition 4.3.11. *Let $C = (\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal symmetric category where $\mathbf{C}$ is an additive, $\otimes$-distributive and $\otimes$-annihilated category. Then, a Lie Object in* $\mathrm{LP}(C)$ *is equivalent to a triple* $(\begin{smallmatrix}M\\ \downarrow f\\ N\end{smallmatrix}, *^M_N, \mu_N)$ *where:*

(i) (N, μ_N) *is a Lie object in* $\mathbf{C}$*,*

(ii) $*^M_N :\ M \otimes N\to M$ *is a morphism in* $\mathbf{C}$*.*

(iii) $(M, *^M_N)$ *is a* (N, μ_N)*-right module.*

(iv) f *is* $\big((N, \mu_N), *^M_N, \mu_N\big)$*-equivariant.*

Moreover, a morphism between Lie Objects $\begin{smallmatrix}M\\ \downarrow f\\ N\end{smallmatrix} \xrightarrow{(h,k)} \begin{smallmatrix}O\\ \downarrow g\\ P\end{smallmatrix}$ *can be seen as a morphism* (h, k) *in* $\mathrm{Hom}(\mathbf{C})$ *such that* $k :\ (N, \mu_N) \to\ (P, \mu_P)$ *is a Lie morphism and* $h :\ (M, *^M_N)\to (O, *^O_P)$ *is* $(k, *^M_N, *^O_P)$*-equivariant.*

Proof. Let $(\begin{smallmatrix}M\\ \downarrow f\\ N\end{smallmatrix}, \mu_f)$ be a Lie object in $\mathrm{LP}(C)$. Then, $\mu^1_f :\ (M \otimes N) \oplus (N \otimes M)\to M$ and $\mu^2_f :\ N \otimes N\to\ N$ are the two components of μ_f. Taking μ_N as μ^2_f, since the bottom part of the tensor product in the LP category works in the same way as the tensor product of the original category, we can conclude that (N, μ_N) is a Lie object and that the bottom parts of the morphisms between Lie objects in $\mathrm{LP}(\mathcal{C})$ are in fact Lie morphisms in $\mathcal{C}$.

Let us focus on the top part. We can define $*^M_N$ as the composition

$$\begin{tikzcd}
M \otimes N \arrow[r, "\iota_1"] \arrow[dr, "*^M_N"'] & (M \otimes N) \oplus (N \otimes M) \arrow[d, "\mu^1_f"] \\
& M
\end{tikzcd}$$

On the other hand, from $*^M_N$ we can define μ^1_f as

$$\begin{tikzcd}
M \otimes N \arrow[r, "\iota_1"] \arrow[dr, "*^M_N"'] & (M \otimes N) \oplus (N \otimes M) \arrow[d, dashed, "\mu^1_f"] & N \otimes M \arrow[l, "\iota_2"'] \arrow[dl, "-(*^M_N \circ \tau_{N,M})"] \\
& M &
\end{tikzcd}$$

With the aid of the following diagram we will show the equivalence between both constructions:

$$\begin{tikzcd}
(M \otimes N) \oplus (N \otimes M) \arrow[ddd, "\mu^1_f"'] \arrow[dr, "(\tau_{M,N} \oplus \tau_{N,M})"] \arrow[ddr, bend right, "\hat{\tau}^1_{f,f}"'] & & N \otimes M \arrow[ll, "\iota_2"'] \arrow[dd, "\tau_{N,M}"] \\
& (N \otimes M) \oplus (M \otimes N) \arrow[d, "\tau^{\oplus}_{N\otimes M, M\otimes N}"'] & \\
& (M \otimes N) \oplus (N \otimes M) \arrow[dl, "-\mu^1_f"'] & M \otimes N \arrow[ul, "\iota_2"'] \arrow[l, "\iota_1"'] \arrow[d, "\iota_1"] \arrow[dll, "-(*^M_N)"] \\
M & & (M \otimes N) \oplus (N \otimes M) \arrow[ll, "-\mu^1_f"]
\end{tikzcd}$$

If we start with μ^1_f, then all subdiagrams are commutative so that the outer diagram proves that we recover μ^1_f. On the other hand, if we start with $*^M_N$, it is obvious that we recover the same $*^M_N$ again. Moreover, the multiplication that it defines is anticommutative, since the leftmost subdiagram commutes by the commutativity of all others together with the outer one.

We want to work with $*_N^M$, so we need to know what means, in terms of $*_N^M$, that μ_f satisfies (Lb). We need to know what means that the first morphism of the pair satisfies the property (Lb) and we are familiar with that the lower part of that property is satisfied if (N, μ_N) is a Leibniz object in $\mathcal{C}$. Since we have a Lie object, the top part gives us the following equation.

$$\mu_f^1\circ(\mu_f \widehat{\otimes} \mathrm{Id}_f)^1 = \mu_f^1\circ(\mu_f \widehat{\otimes} \mathrm{Id}_f)^1\circ\alpha_f^{-1}\circ(\mathrm{Id}_f \widehat{\otimes}\widehat{\tau}_{f,f})^1\circ\alpha_f + \mu_f^1\circ(\mathrm{Id}_f \widehat{\otimes}\mu_f)^1\circ\alpha_f.$$

Where we denote $\alpha_f := \alpha_{f,f,f}$.

Expanding, we have:

$$\begin{aligned}
&\mu_f^1\circ(\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M)\\
&= \mu_f^1\circ(\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M)\circ\alpha_f^{-1}\circ(\mathrm{Id}_M \otimes\tau_{N,N}) \oplus (\mathrm{Id}_N \otimes\widehat{\tau}^1_{f,f})\circ\alpha_f\\
&\qquad\qquad + \mu_f^1\circ(\mathrm{Id}_M \otimes\mu_N) \oplus (\mathrm{Id}_N \otimes\mu_f^1)\circ\alpha_f.
\end{aligned}$$

That equality holds when we compose with the isomorphism $\omega_f = \alpha_f^{-1}$, obtaining

$$\begin{aligned}
&\mu_f^1\circ(\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M)\circ\omega_f\\
&= \mu_f^1\circ(\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M)\circ\omega_f\circ(\mathrm{Id}_M \otimes\tau_{N,N}) \oplus (\mathrm{Id}_N \otimes\widehat{\tau}^1_{f,f})\\
&\qquad\qquad + \mu_f^1\circ(\mathrm{Id}_M \otimes\mu_N) \oplus (\mathrm{Id}_N \otimes\mu_f^1).
\end{aligned}$$

Now, if we compose with the first natural injection of the coproduct domain, and using that $\mu_f^1 = [*_N^M, (- *_N^M \circ\tau_{N,M})]$ we obtain:

$$\begin{aligned}
&*_N^M \circ(*_N^M \otimes \mathrm{Id}_N)\circ a^{-1}_{M,N,N}\\
&= \mu_f^1\circ\iota_1\circ(\mu_f^1 \otimes \mathrm{Id}_N)\circ(\iota_1 \otimes \mathrm{Id}_N)\circ a^{-1}_{M,N,N}\\
&= \mu_f^1\circ((\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M))\circ\iota_1\circ(\iota_1 \otimes \mathrm{Id}_N)\circ a^{-1}_{M,N,N}\\
&= \mu_f^1\circ((\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M))\circ\omega_f\circ\iota_1\\
&= \mu_f^1\circ((\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M))\circ\omega_f\circ((\mathrm{Id}_M \otimes\tau_{N,N}) \oplus (\mathrm{Id}_N \otimes\widehat{\tau}^1_{f,f}))\circ\iota_1\\
&\qquad\qquad + \mu_f^1\circ((\mathrm{Id}_M \otimes\mu_N) \oplus (\mathrm{Id}_N \otimes\mu_f^1))\circ\iota_1\\
&= \mu_f^1\circ((\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M))\circ\omega_f\circ\iota_1\circ(\mathrm{Id}_M \otimes\tau_{N,N}) + \mu_f^1\circ(\iota_1\circ \mathrm{Id}_M \otimes\mu_N)
\end{aligned}$$

$$
\begin{aligned}
&= \mu_f^1 \circ ((\mu_f^1 \otimes \mathrm{Id}_N) \oplus (\mu_N \otimes \mathrm{Id}_M)) \circ \iota_1 \circ (\iota_1 \otimes \mathrm{Id}_N) \circ a^{-1}_{M,N,N} \circ (\mathrm{Id}_M \otimes \tau_{N,N}) \\
&\qquad\qquad + *_N^M \circ (\mathrm{Id}_M \otimes \mu_N) \\
&= \mu_f^1 \circ \iota_1 \circ (\mu_f^1 \otimes \mathrm{Id}_N) \circ (\iota_1 \otimes \mathrm{Id}_N) \circ a^{-1}_{M,N,N} \circ (\mathrm{Id}_M \otimes \tau_{N,N}) + *_N^M \circ (\mathrm{Id}_M \otimes \mu_N) \\
&= *_N^M \circ (*_N^M \otimes \mathrm{Id}_N) \circ a^{-1}_{M,N,N} \circ (\mathrm{Id}_M \otimes \tau_{N,N}) + *_N^M \circ (\mathrm{Id}_M \otimes \mu_N).
\end{aligned}
$$

In the previous equation, composing with the isomorphism $a_{M,N,N}$ we obtain:

$$
\begin{aligned}
*_N^M \circ (*_N^M \otimes \mathrm{Id}_N) = &*_N^M \circ (*_N^M \otimes \mathrm{Id}_N) \circ a^{-1}_{M,N,N} \circ (\mathrm{Id}_M \otimes \tau_{N,N}) \circ a_{M,N,N} \\
&+ *_N^M \circ (\mathrm{Id}_M \otimes \mu_N) \circ a_{M,N,N},
\end{aligned}
$$

which means that $M, *_N^M$ is an (N, μ_N)-right module. Note that we abuse the language naming all the different injections as ι_1.

In the same way we can prove that, if $(M, *_N^M)$ is an (N, μ_N)-right module, then $[*_N^M, (- *_N^M \circ \tau_{N,M})]$ satisfies (Lb). For this is necessary Theorem 4.2.2 to move the "$-$" inside the tensor product.

We need to show that f is $(N\mu_N, *_N^M, \mu_N)$-equivariant. That comes from the fact that (μ_f^1, μ_f^2) is a morphism.

$$
\begin{tikzcd}
M \otimes N \arrow[r, "\iota_1"'] \arrow[d, "f \otimes \mathrm{Id}"'] \arrow[rr, bend left, "*_N^M"] & (M \otimes N) \oplus (N \otimes M) \arrow[r, "\mu_f^1"'] \arrow[d, dashed, "{[f \otimes \mathrm{Id}_N, \mathrm{Id}_N \otimes f]}"] & M \arrow[d, "f"] \\
N \otimes N \arrow[r, "\mathrm{Id}"'] & N \otimes N \arrow[r, "\mu_N = \mu_f^2"'] & N
\end{tikzcd}
$$

The outer diagram is commutative since all the internal diagrams are commutative.

Using the injections, one can easily see that f is $(N\mu_N, *_N^M, \mu_N)$-equivariant implies that (μ_f^1, μ_f^2) is a morphism.

Now we will prove that this also works with morphisms. The second component for morphisms is immediate, since for k we have the original tensor product, and for that, the diagram of the lower part gives that it will be a Lie morphism.

We will prove that h is $(k, *_N^M, *_P^O)$-equivariant. To do this we will use an analogous diagram as the previous part.

$$\begin{tikzcd}
M \otimes N \arrow[r, "\iota_1"'] \arrow[d, "h\otimes k"'] \arrow[rr, bend left, "*^M_N"] & (M \otimes N) \oplus (N \otimes M) \arrow[r, "\mu^1_f"'] \arrow[d, dashed, "(h\otimes k)\oplus(k\otimes h)"] & M \arrow[d, "h"] \\
O \otimes P \arrow[r, "\iota_1"] \arrow[rr, bend right, "*^O_P"'] & (O \otimes P) \oplus (P \otimes O) \arrow[r, "\mu^1_g"] & O
\end{tikzcd}$$

The commutativity of the outer diagram is the $(k, *^M_N, *^O_P)$-equivariance of h.

Again we have an equivalence, since we can recover that (h, k) is a morphism using the $(k, *^M_N, *^O_P)$-equivariance of h, since we have $[*^M_N, (- *^M_N \circ\tau_{N,M})]$, only by using the natural injections.

This shows that arrows can also be simplified in diagrams that are to commute.

□

Lemma 4.3.12. *Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a braided semigroupal symmetric category where $\mathbf{C}$ is an additive, $\otimes$-distributive and $\otimes$-annihilated category.*

Let $\left(\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}, *^M_N, \mu_N\right)$ be a Lie object in $\mathrm{LP}(\mathcal{C})$. If we define $\mathring{\mu}_M := *^M_N \circ(\mathrm{Id}_M \otimes f)$, then, $(M, \mathring{\mu}_M)$ is a Leibniz object in $\mathcal{C}$ and f is a Leibniz morphism.

Proof. Let us see that it satisfies (Lb). Since $(M, *^M_N)$ is a right-(N, μ_N)-module and $\mathrm{Id}_M \otimes f$ is $((N, \mu_N), *^M_N, \mu_N)$-equivariant,

$$\begin{aligned}
&\mathring{\mu}_M\circ(\mathring{\mu}_M \otimes \mathrm{Id}_M) = *^M_N \circ(\mathrm{Id}_M \otimes f)\circ((*^M_N \circ \mathrm{Id}_M \otimes f) \otimes \mathrm{Id}_M)\\
&= *^M_N \circ(\mathrm{Id}_M \otimes f)\circ(*^M_N \otimes \mathrm{Id}_M)\circ((\mathrm{Id}_M \otimes f) \otimes \mathrm{Id}_M)\\
&= *^M_N \circ(*^M_N \otimes \mathrm{Id}_N)\circ(\mathrm{Id}_{M\otimes N} \otimes f)\circ((\mathrm{Id}_M \otimes f) \otimes \mathrm{Id}_M)\\
&= *^M_N \circ(*^M_N \otimes \mathrm{Id}_N)\circ((\mathrm{Id}_M \otimes f) \otimes f)\\
&= *^M_N \circ(*^M_N \otimes \mathrm{Id}_N)\circ a^{-1}_{M,N,N}\circ(\mathrm{Id}_M \otimes\tau_{N,N})\circ a_{M,N,N}\circ((\mathrm{Id}_M \otimes f) \otimes f)\\
&\qquad\qquad + *^M_N \circ(\mathrm{Id}_M \otimes\mu_N)\circ a_{M,N,N}\circ((\mathrm{Id}_M \otimes f) \otimes f)\\
&= *^M_N \circ(*^M_N \otimes \mathrm{Id}_N)\circ((\mathrm{Id}_M \otimes f) \otimes f)\circ a^{-1}_{M,M,M}\circ(\mathrm{Id}_M \otimes\tau_{M,M})\circ a_{M,M,M}\\
&\qquad\qquad + *^M_N \circ(\mathrm{Id}_M \otimes\mu_N)\circ(\mathrm{Id}_M \otimes(f \otimes f))\circ a_{M,M,M}
\end{aligned}$$

$$
\begin{aligned}
&= *^M_N \circ (\mathrm{Id}_M \otimes f) \circ (*^M_N \otimes \mathrm{Id}_M) \circ ((\mathrm{Id}_M \otimes f) \otimes \mathrm{Id}_M) \circ a^{-1}_{M,M,M} \circ (\mathrm{Id}_M \otimes \tau_{M,M}) \circ a_{M,M,M} \\
&\qquad\qquad + *^M_N \circ (\mathrm{Id}_M \otimes \mu_N) \circ (\mathrm{Id}_M \otimes (f \otimes \mathrm{Id}_N)) \circ (\mathrm{Id}_M \otimes (\mathrm{Id}_M \otimes f)) \circ a_{M,M,M} \\
&= \mathring{\mu}_M \circ (\mathring{\mu}_M \otimes \mathrm{Id}_M) \circ a^{-1}_{M,M,M} \circ (\mathrm{Id}_M \otimes \tau_{M,M}) \circ a_{M,M,M} \\
&\qquad\qquad + *^M_N \circ (\mathrm{Id}_M \otimes f) \circ (\mathrm{Id}_M \otimes *^M_N) \circ (\mathrm{Id}_M \otimes (\mathrm{Id}_M \otimes f)) \circ a_{M,M,M} \\
&= \mathring{\mu}_M \circ (\mathring{\mu}_M \otimes \mathrm{Id}_M) \circ a^{-1}_M \circ (\mathrm{Id}_M \otimes \tau_{M,M}) \circ a_M + \mathring{\mu}_M \circ (\mathrm{Id}_M \otimes \mathring{\mu}_M) \circ a_M .
\end{aligned}
$$

The fact that f is a Leibniz morphism in $\mathcal{C}$ follows by

$$
f \circ \mathring{\mu}_M = f \circ *^M_N \circ (\mathrm{Id}_M \otimes f) = \mu_N \circ (f \otimes \mathrm{Id}_N) \circ (\mathrm{Id}_M \otimes f) = \mu_N \circ (f \otimes f).
$$

□

Lemma 4.3.13. *Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a symmetric semigroupal category where $\mathbf{C}$ is an additive, $\otimes$-distributive and $\otimes$-annihilated category. There are two functors* $\mathrm{T}\colon \mathrm{Lie}\left(\mathrm{LP}(\mathcal{C})\right) \to \mathrm{Leib}(\mathcal{C})$ *and* $\mathrm{B}\colon \mathrm{Lie}\left(\mathrm{LP}(\mathcal{C})\right) \to \mathrm{Lie}(\mathcal{C})$ *defined as:*

$$
(i)\ \ \mathrm{T}\left(\left(\begin{array}{c} M \\ \downarrow_{f} \\ N \end{array}, *^M_N, \mu_N\right) \xrightarrow{h} \left(\begin{array}{c} O \\ \downarrow_{f} \\ P \end{array}, *^O_P, \mu_P\right) \right) := (M, \mathring{\mu}_M) \xrightarrow{h^1} (O, \mathring{\mu}_O).
$$

$$
(ii)\ \ \mathrm{B}\left(\left(\begin{array}{c} M \\ \downarrow_{f} \\ N \end{array}, *^M_N, \mu_N\right) \xrightarrow{h} \left(\begin{array}{c} O \\ \downarrow_{f} \\ P \end{array}, *^O_P, \mu_P\right) \right) := (N, \mu_N) \xrightarrow{h^2} (P, \mu_P).
$$

Proof. By Lemma 4.3.12 we know that $(M, \mathring{\mu}_M)$ and $(O, \mathring{\mu}_O)$ are Leibniz objects, so we just need to check that h_1 is $(h_2, *^M_N, *^O_P)$-equivariant. It follows from

$$
\begin{aligned}
h_1 \circ \mathring{\mu}_M &= h_1 \circ *^M_N \circ (\mathrm{Id}_M \otimes f) = *^O_P \circ (h_1 \otimes h_2) \circ (\mathrm{Id}_M \otimes h_1) \\
&= *^O_P \circ (\mathrm{Id}_O \otimes h_2) \circ (h_1 \otimes h_1) = \mathring{\mu}_O \circ (h_1 \otimes h_1).
\end{aligned}
$$

By Proposition 4.3.11 we know that B is also a functor. □

Theorem 4.3.14. *Let $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ be a semigroupal symmetric category such that $\mathbf{C}$ is an additive $\otimes$-closed finitely cocomplete category. Let (L, μ) be a Leibniz object. Then,* $\left(\begin{array}{c} L \\ \downarrow_{\pi_L} \\ \overline{L} \end{array}, \widehat{\mu}, \overline{\mu}\right)$ *is a Lie object in* $\mathrm{LP}(\mathbf{C})$*, where $\overline{\mu}$ and $\widehat{\mu}$ are defined in Theorem 4.3.7. Therefore, we obtain a functor* $\mathrm{I}\colon \mathrm{Leib}(\mathcal{C}) \to \mathrm{Lie}\left(\mathrm{LP}(\mathcal{C})\right)$*. Moreover,* *(i)* $\mathrm{T} \circ \mathrm{I} = \mathrm{Id}_{\mathrm{Leib}(\mathcal{C})}$. *This implies* I *is a section functor.*

(ii) $\mathrm{B} \circ \mathrm{I} = \overline{(-)}$.

Proof. Using the characterisation of Proposition 4.3.11, Theorem 4.3.7 together with Theorem 4.3.10 we know that it is a Lie object in $\mathrm{LP}(\mathcal{C})$.

Given a morphism $(L_1, \mu_1) \xrightarrow{f} (L_2, \mu_2)$, we have to see that $\mathrm{I}(f) = (f, \overline{f})$ is a Lie morphism in $\mathrm{LP}(\mathcal{C})$. In Theorem 4.3.7 it is already proved that $\widehat{f}$ is a Lie morphism in $\mathcal{C}$, so we just need to prove that $f : (L_1, \widehat{\mu}_1) \rightarrow f : (L_2, \widehat{\mu}_2)$ is $(\overline{f}, \widehat{\mu}_1, \widehat{\mu}_2)$-equivariant. Using that $\mathrm{Id}_{L_1} \otimes \pi^{L_1}$ is an epimorphism.

$$\begin{aligned} f \circ \widehat{\mu}_1 \circ (\mathrm{Id}_{L_1} \otimes \pi^{L_1}) &= f \circ \mu_1 = \mu_{L_2} \circ (f \otimes f) = \widehat{\mu}_2 \circ (\mathrm{Id}_{L_2} \otimes \pi^{L_2}) \circ (f \otimes f) \\ &= \widehat{\mu}_2 \circ (f \otimes (\pi^{L_2} \circ f)) = \widehat{\mu}_2 \circ (f \otimes (\overline{f} \circ \pi^{L_1})) = \widehat{\mu}_2 \circ (f \otimes \overline{f}) \circ (\mathrm{Id}_{L_1} \otimes \pi^{L_1}). \end{aligned}$$

It is immediate to see that $\mathrm{B} \circ \mathrm{I} = \overline{(-)}$.

Moreover, $\mathrm{T} \circ \mathrm{I} = \mathrm{Id}_{\mathrm{Leib}(\mathcal{C})}$ is immediate in morphisms, so we just need to prove the equality in objects. That means that for the Leibniz object (L, μ), we have $\mu = \overset{\circ}{\overline{\mu}}_M$, where $\overset{\circ}{\overline{\mu}}_M := \widehat{\mu} \circ (\mathrm{Id}_L \otimes \pi^L)$ is defined in Lemma 4.3.12. But that is exactly the definition of $\widehat{\mu}$ in Theorems 4.3.7 and 4.3.10. □

Theorem 4.3.15. *Let* $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ *be a semigroupal symmetric category such that* $\mathbf{C}$ *is an additive* $\otimes$*-closed finitely cocomplete category. Then we have that* $\mathrm{Leib}(\mathcal{C})$ *is a full coreflective subcategory of* $\mathrm{Lie}\left(\mathrm{LP}(\mathcal{C})\right)$.

Proof. Let $\mathrm{I} : \mathrm{Leib}(\mathcal{C}) \rightarrow \mathrm{Lie}\left(\mathrm{LP}(\mathcal{C})\right)$ be the inclusion functor from Theorem 4.3.14. It is full since all morphisms $\mathrm{I}(L_1, \mu_1) \xrightarrow{f_1, f_2} \mathrm{I}(L_2, \mu_2)$ must satisfy the following diagram:

$$\begin{tikzcd} L_1 \arrow[r, "f_1"] \arrow[d, "\pi^{L_1}"'] & L_2 \arrow[d, "\pi^{L_2}"] \\ \overline{L_1} \arrow[r, "f_2"] & \overline{L_2} \end{tikzcd}$$

We know that $f_1 : (L_1, \mu_1) \rightarrow (L_2, \mu_2)$, as we prove it in Lemma 4.3.13, by uniqueness of $\overline{f_1}$ (Theorem 4.3.7) we conclude that $f_2 = \overline{f_1}$.

Now, we will prove that I is a coreflective functor with the coreflection given by $\mathrm{T}: \ \mathrm{Lie}\big(\mathrm{LP}(\mathcal{C})\big) \to \mathrm{Leib}(\mathcal{C})$, i.e. $\mathrm{I} \dashv \mathrm{T}$.

Let (L, μ) be an object in $\mathrm{Leib}(\mathcal{C})$ and $\left(\begin{array}{c}M\\ \downarrow f\\ N\end{array}, *_N^M, \mu_N\right)$ a Lie object in $\mathrm{LP}(\mathcal{C})$. Let $(L, \mu) \xrightarrow{h} (M, \mathring{\mu}_M)$ be a Leibniz morphism $(M, \mathring{\mu}_M) = \mathrm{T}\left(\left(\begin{array}{c}M\\ \downarrow f\\ N\end{array}, *_N^M, \mu_N\right)\right)$.

We define $\check{h}$ using the coequalizer:

$$\begin{tikzcd} L \otimes L \arrow[r, shift left, "-\mu"] \arrow[r, shift right, "\mu\circ\tau_{L,L}"'] & L \arrow[r, "\pi^L"] \arrow[d, "h"'] & \overline{L} \arrow[d, dashed, "\check{h}"] \\ & M \arrow[r, "f"] & N. \end{tikzcd}$$

If the pair $(h, \check{h})$ is a Lie morphism in $\mathrm{LP}(\mathcal{C})$, then $h \mapsto (h, \check{h})$ induces a natural isomorphism between

$$\mathrm{Hom}_{\mathrm{Leib}(\mathcal{C})}\big((L, \mu), (M, \mathring{\mu}_M)\big) \text{ and } \mathrm{Hom}_{\mathrm{Lie}(\mathrm{LP}(\mathcal{C}))}\left(\left(\begin{array}{c}L\\ \downarrow \pi^L\\ \overline{L}\end{array}, \hat{\mu}, \overline{\mu}\right), \left(\begin{array}{c}M\\ \downarrow f\\ N\end{array}, *_N^M, \mu_N\right)\right),$$

proving the adjunction.

First, we need to prove that $\check{h}$ is well defined, i.e. we can use the coequalizer.

$$\begin{aligned} f\circ h\circ(-\mu) &= -f\circ h\circ\mu = -\mu_N\circ(f \otimes f)\circ(h \otimes h) = \mu_N\circ\tau_{N,N}\circ(f \otimes f)\circ(h \otimes h) \\ &= \mu_N\circ(f \otimes f)\circ(h \otimes h)\circ\tau_{L,L} = f\circ h\circ\mu\circ\tau_{L,L}. \end{aligned}$$

Since $\check{h}$ is a well-defined morphism in $\boldsymbol{C}$, we need to show that $(h, \check{h})$ is a Lie morphism in $\mathrm{LP}(\mathcal{C})$, i.e. $\check{h}: \ (\overline{L}, \overline{\mu}) \to (N, \mu_N)$ is a Lie morphism and h is $(\check{h}, \widehat{\mu}, *_N^M)$-equivariant.

To show that $\check{h}$ is a Lie morphism we will see its composition with the epimorphism $\pi^L \otimes \pi^L$.

$$\begin{aligned} \check{h}\circ\overline{\mu}\circ(\pi^L \otimes \pi^L) &= \check{h}\circ\pi^L\circ\mu = f\circ h\circ\mu = \mu\circ(f \otimes f)\circ(h \otimes h) \\ &= \mu\circ\big((f\circ h) \otimes (f\circ h)\big) = \mu\circ(\check{h}\circ\pi^L) \otimes (\check{h}\circ\pi^L) \\ &= \mu\circ\check{h} \otimes \check{h}\circ\pi^L \otimes \pi^L. \end{aligned}$$

Finally, we prove that h is $(\check{h}, \widehat{\mu}, *_N^M)$-equivariant, that means that the following diagram is commutative:

$$\begin{tikzcd} L \otimes \overline{L} \arrow[r, "\widehat{\mu}"] \arrow[d, "h\otimes\check{h}"'] & L \arrow[d, "h"] \\ M \otimes N \arrow[r, "*_N^M"] & M \end{tikzcd}$$

To do this, we will compose with the epimorphism $\mathrm{Id}_L \otimes \pi^L$:

$$\begin{aligned} *_N^M \circ (h \otimes \check{h}) \circ (\mathrm{Id}_L \otimes \pi^L) &= *_N^M \circ \big(h \otimes (\check{h} \circ \pi^L)\big) = *_N^M \circ \big(h \otimes (f \circ h)\big) \\ &= *_N^M \circ (\mathrm{Id}_M \otimes f) \circ (h \otimes h) = \mathring{\mu}_M \circ (h \otimes h) = h \circ \mu \\ &= h \circ \widehat{\mu} \circ (\mathrm{Id}_L \otimes \pi^L). \end{aligned}$$

$\square$

Conclusions

As stated in our objectives, we proved:

In Chapter 2, we introduce the notion of braiding for the associative and Leibniz case, showing the relationship in both cases with the Lie case. We also show a new definition for the Lie case, which gives an example using the non-abelian tensor product, as it already exists for the group case. We show the same kind of example for the Leibniz case. We also show in that chapter that the braided categories of the associative and Leibniz case are equivalent.

In Chapter 3, we define the $\mathfrak{U}$-central extensions and $\boldsymbol{B}$-central extensions of crossed modules of Lie algebras. In this chapter, we prove that we have $\mathfrak{U}$-perfect if and only if there is a universal $\mathfrak{U}$-central extension. We also prove that we have $\boldsymbol{B}$-perfect if and only if we have a universal $\boldsymbol{B}$-central extension. In fact, we prove that despite $\mathfrak{U}$-central extensions and $\boldsymbol{B}$-central extensions are different, $\mathfrak{U}$-perfect and $\boldsymbol{B}$-perfect are the same, and, in fact, both universal central extensions coincide when they exist.

In Chapter 4, we define the Loday-Pirashvili category for categories with operations, (braided) semigroupal categories and (braided) monoidal categories, adding, only when they are needed, the properties of $\otimes$-distributivity and $\otimes$-annihilation. In this chapter, we also construct the Liesation functor for Lie objects in braided semigroupal additive categories, adding the properties that were just said or one stronger, the $\otimes$-closedness, when it is needed. We show that this Liesation functor is a left adjoint to the forgetful functor. Then using those properties we show which is a Lie object in the Loday-Pirashvili category, and we show that Leibniz algebras in the base category are a particular case for this Lie objects, i.e. we have a full coreflective subcategory.

Results

The results presented in this thesis appear in the following four research articles:

- **"Braiding for categorical algebras and crossed modules of algebras I: Associative and Lie algebras"**, *Journal of Algebra and Its Applications*, Vol. 19, No. 9 (2020) 2050176.

  doi: `https://doi.org/10.1142/S0219498820501765`

- **"Braiding for categorical algebras and crossed modules of algebras II: Leibniz Algebras"**, *Filomat*, Vol. 34, No. 5 (2020) 1443–1469.

  doi: `https://doi.org/10.2298/FIL2005443F`

- **"Universal central extensions of braided Lie crossed modules"**. Submitted for publication.

- **"On the Loday-Pirashvili category"**. Submitted for publication.

Resumo da Tese de Doutoramento (in Galician)

Braided Crossed Modules and Loday-Pirashvili category

Resumo abreviado: Módulos cruzados trenzados e a categoría de Loday-Pirashvili

Esta tese está dedicada ao estudo das trenzas en diferentes contextos matemáticos, así como nun estudo máis profundo da categoría de Loday e Pirashvili.

Ao principio danse as definicións básicas dos contextos nos que imos estudar as trenzas, os módulos cruzados e os obxectos internos nas categorías de grupos, álxebras asociativas, álxebras de Lie e álxebras de Leibniz. Nesta parte definimos tamén o concepto de trenza para categorías semigrupais.

Unha vez feito isto, usamos as similitudes entre as distintas categorías para obter ditas nocións de trenzas, mostrando que para cada caso, as trenzas en módulos cruzados dan categorías equivalentes ás das trenzas en obxectos categóricos.

Para facer o caso de álxebras de Leibniz, usamos a categoría de Loday e Pirashvili sobre espazos vectoriais, introducindo así esta nova construción categórica que se trata de definir nun contexto máis amplo no último capítulo.

Neste último capitulo tamén traballamos coa internalización das álxebras de Lie e Leibniz. Baixo certas propiedades imos obter un functor Liezación entre estes obxectos, o cal podemos usar, xunto coa xeneralización da categoría de Loday e Pirashvili, para ver os obxectos de Leibniz como un caso particular de obxectos de Lie.

Nos preliminares (Capítulo 1), lembraremos algunhas definicións básicas necesarias para o resto da tese.

O termo "categorificación" aparece por primeira vez no ano 1995, mostrando dúas maneiras de pasar ideas da álxebra habitual ao ámbito das categorías.

Por unha banda temos a "internalización" ou "categorificación horizontal", a cal trata de pasar as ideas da álxebra usual a outras categorías usando diagramas conmutativos. Ao usar estes diagramas evítase o uso de elementos, xa que as frechas nunha categoría non teñen porque ser aplicacións. Un exemplo de internalización pódese ver na definición de categoría interna ou obxecto categórico, o cal trata de definir categoría pequena nunha categoría arbitraria con pullbacks.

Definición 1.1.1. *Consideremos* $\mathbf{C}$ *unha categoría con pullbacks.*

Unha categoría interna en $\mathbf{C}$ *consiste en dous obxectos* C_1 *(obxecto de morfismos) e* C_0 *(obxecto de obxectos) de* $\mathbf{C}$*, xunto cos seguintes morfismos* s, t, e, k*:*

$$\begin{tikzcd} C_0 \arrow[r, "e" description] & C_1 \arrow[l, shift right=1.5ex, "t"'] \arrow[l, shift left=1.5ex, "s"] & C_1 \times_{C_0} C_1, \arrow[l, "k"'] \end{tikzcd}$$

onde $C_1 \times_{C_0} C_1$ *é o pullback de* t *e* s*.*

s *é chamado morfismo orixe,* t *é chamado morfismo obxectivo,* e *é chamado morfismo identidade e* k *é chamado morfismo composición.*

Ademais, os morfismos teñen que cumprir os seguintes diagramas conmutativos, que representan as regras categóricas usuais (ver [6]):

(I1)
$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "\mathrm{Id}_{C_0}"'] & C_1 \arrow[d, "s"] \\ & C_0. \end{tikzcd}$$

(I2)
$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "\mathrm{Id}_{C_0}"'] & C_1 \arrow[d, "t"] \\ & C_0. \end{tikzcd}$$

(I3)
$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_1"] \arrow[d, "k"] & C_1 \arrow[d, "s"] \\ C_1 \arrow[r, "s"] & C_0. \end{tikzcd}$$

(I4)
$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "k"] & C_1 \arrow[d, "t"] \\ C_1 \arrow[r, "t"] & C_0. \end{tikzcd}$$

(I5) *Se* $(C_1 \times_{C_0} C_1) \times_{C_0} C_1$ *e* $C_1 \times_{C_0} (C_1 \times_{C_0} C_1)$ *son os pullbacks dados por:*

$$\begin{tikzcd}
(C_1 \times_{C_0} C_1) \times_{C_0} C_1 \arrow[r, "\pi_1'"] \arrow[d, "\pi_2'"'] & C_1 \times_{C_0} C_1 \arrow[d, "t\circ k"] \\
C_1 \arrow[r, "s"'] & C_0,
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "\pi_1''"] \arrow[d, "\pi_2''"'] & C_1 \arrow[d, "t"] \\
C_1 \times_{C_0} C_1 \arrow[r, "s\circ k"'] & C_0,
\end{tikzcd}$$

entón o seguinte diagrama é conmutativo:

$$\begin{tikzcd}
(C_1 \times_{C_0} C_1) \times_{C_0} C_1 \arrow[rr, "k\times_{C_0}\mathrm{Id}_{C_1}"] \arrow[d, "i"] & & C_1 \times_{C_0} C_1 \arrow[dd, "k"] \\
C_1 \times_{C_0} (C_1 \times_{C_0} C_1) \arrow[d, "\mathrm{Id}_{C_1} \times_{C_0} k"] & & \\
C_1 \times_{C_0} C_1 \arrow[rr, "k"] & & C_1.
\end{tikzcd}$$

I6) *Sexan* $C_0 \times_{C_0} C_1$ *e* $C_1 \times_{C_0} C_0$ *os pullbacks dados por:*

$$\begin{tikzcd}
C_0 \times_{C_0} C_1 \arrow[r, "p_1"] \arrow[d, "p_2"'] & C_0 \arrow[d, "\mathrm{Id}_{C_0}"] \\
C_1 \arrow[r, "s"] & C_0,
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} C_0 \arrow[r, "q_1"] \arrow[d, "q_2"'] & C_1 \arrow[d, "t"] \\
C_0 \arrow[r, "\mathrm{Id}_{C_0}"] & C_0.
\end{tikzcd}$$

Entón temos que verificar o seguinte diagrama:

$$\begin{tikzcd}
C_0 \times_{C_0} C_1 \arrow[r, "e\times_{C_0}\mathrm{Id}_{C_1}"] \arrow[dr, "p_2"'] & C_1 \times_{C_0} C_1 \arrow[d, "k"] & C_1 \times_{C_0} C_0 \arrow[l, "\mathrm{Id}_{C_1}\times_{C_0} e"'] \arrow[dl, "q_1"] \\
& C_1. &
\end{tikzcd}$$

Se se cumpren as condicións, referirémonos á categoría interna como a 6*-tupla* (C_1, C_0, s, t, e, k).

Pola outra banda temos a "categorificación vertical", a cal trata de traducir as ideas da álxebra habitual subíndoa ao nivel das categorías. Para facelo substitúe, por exemplo, conxuntos por categorías, elementos por obxectos, aplicacións por functores e formulas por isomorfismos naturais.

Un exemplo deste tipo de “categorificación” pode verse na definición de categoría semigrupal:

Definición 1.4.2. *Unha categoría semigrupal é un triplo $C = (\mathbf{C}, \otimes, a)$ onde $\boldsymbol{C}$ é unha categoría, $\otimes : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$ é un bifunctor e $a : \otimes \circ (\otimes \times \mathrm{Id}_{\boldsymbol{C}}) \to \otimes \circ (\mathrm{Id}_{\boldsymbol{C}} \times \otimes) \circ A$ é un isomorfismo natural chamado asociador, de modo que para todo $X, Y, Z \in \mathrm{Ob}(\boldsymbol{C})$ cúmprese o seguinte diagrama de coherencia asociativa (axioma pentagonal):*

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes W \arrow[dl, "a_{X,Y,Z} \otimes \mathrm{Id}_W"'] \arrow[dr, "a_{X \otimes Y, Z, W}"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "a_{X, Y \otimes Z, W}"'] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "a_{X,Y,Z \otimes W}"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\mathrm{Id}_X \otimes a_{Y,Z,W}"'] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

Diremos que unha categoría semigrupal é estrita se o isomorfismo a é o morfismo identidade. Neste caso temos $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.

As categorías monoidais foron presentadas por Jean Bénabou [7] e Saunders MacLane [45] para xeneralizar a idea do produto tensor en categorías arbitrarias.

É ben sabido que, no caso habitual do produto tensor para espazos vectoriais, hai un isomorfismo natural entre $V \otimes W$ e $W \otimes V$. Para estudar se esta propiedade tamén se cumpre nunha categoría monoidal arbitraria, é dicir, cando o produto tensor é (non estritamente) conmutativo, Joyal e Street definiron en [38] o concepto de trenza para categorías monoidais como un isomorfismo natural $\tau_{A,B} : A \otimes B \to B \otimes A$.

Definición 1.4.5. *Unha trenza nunha categoría monoidal C é un isomorfismo natural $\tau : \otimes \to \otimes \circ T$ tal que para todo $X, Y, Z \in \mathrm{Ob}(\boldsymbol{C})$ os seguintes diagramas de coherencia asociativa (axiomas hexagonais) conmutan:*

$$
\begin{tikzcd}
 & (X\otimes Y)\otimes Z \arrow[r, "\tau_{X\otimes Y,Z}"] \arrow[dl, "a_{X,Y,Z}"'] & Z\otimes(X\otimes Y) & \\
X\otimes(Y\otimes Z) \arrow[dr, "\mathrm{Id}_X\otimes\tau_{Y,Z}"'] & & & (Z\otimes X)\otimes Y \arrow[ul, "a_{Z,X,Y}"'] \\
 & X\otimes(Z\otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X\otimes Z)\otimes Y \arrow[ur, "\tau_{X,Z}\otimes\mathrm{Id}_Y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X\otimes(Y\otimes Z) \arrow[r, "\tau_{X,Y\otimes Z}"] \arrow[dl, "a^{-1}_{X,Y,Z}"'] & (Y\otimes Z)\otimes X & \\
(X\otimes Y)\otimes Z \arrow[dr, "\tau_{X,Y}\otimes\mathrm{Id}_Z"'] & & & Y\otimes(Z\otimes X) \arrow[ul, "a^{-1}_{Y,Z,X}"'] \\
 & (Y\otimes X)\otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y\otimes(X\otimes Z). \arrow[ur, "\mathrm{Id}_Y\otimes\tau_{X,Z}"'] &
\end{tikzcd}
$$

Usando esta mesma definición, definimos neste capítulo o concepto de trenza para as categorías semigrupais.

Definición 1.4.6. *Unha trenza sobre unha categoría semigrupal C é un isomorfismo natural $\tau :\ \otimes\!\to\ \otimes\circ T$ que satisfai os dous diagramas de coherencia asociativa dados na definición anterior.*

No capítulo 2, estudaremos as trenzas para módulos cruzados e categorías internas. Comezamos mostrando o caso dos grupos (Sección 2.1).

Cando tratamos de estudar o concepto de trenza para o caso mais simple de categorías monoidais (monoides categóricos, o cal é o mesmo que categorías pequenas monoidais estritas) atopamos que non todos os morfismos internos son isomorfismos internos, polo que a trenza non pode ser un morfismo interno arbitrario verificando propiedades simples. Para evitar este contratempo, podemos traballar con grupos categóricos (estritos), onde si temos que todos os morfismos internos son isomorfismos internos, en vez de con monoides categóricos, obtendo de modo inmediato a definición de grupos categóricos trenzados (véxase [9, 38]).

Definición 2.1.3. *Consideremos o grupo categórico $C = (C_1, C_0, s, t, e, k)$. Unha trenza para C é unha aplicación $\tau :\ C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, satisfacendo:*

$$\tau_{a,b} :\ ab \to ba, \tag{GrB1}$$

$$
\begin{tikzcd}
s(x)s(y) \arrow[r, "xy"] \arrow[d, "\tau_{s(x),s(y)}"'] & t(x)t(y) \arrow[d, "\tau_{t(x),t(y)}"] \\
s(y)s(x) \arrow[r, "yx"'] & t(y)t(x),
\end{tikzcd}
\tag{GrB2}
$$

$$\tau_{ab,c} = (\tau_{a,c}e(b))\circ(e(a)\tau_{b,c}), \tag{GrB3}$$

$$\tau_{a,bc} = (e(b)\tau_{a,c})\circ(\tau_{a,b}e(c)), \tag{GrB4}$$

para $a, b, c \in C_0$, $x, y \in C_1$.

Diremos que $(C_1, C_0, s, t, e, k, \tau)$ *é un grupo categórico trenzado.*

Por outra banda, en 1948 Whitehead [51] introduciu a noción de módulos cruzados de grupos como un modelo alxébrico para os espazos de homotopía de tipo 2 (i.e. espazos conexos con grupos de homotopía triviais en dimensión > 2). En 1984, Conduché [16] (véxase tamén [9]) introduciu a noción de módulo cruzado trenzado de grupos como un caso particular de 2-módulo cruzado de grupos.

Definición 2.1.6. *Consideremos o módulo cruzado de grupos* $(G \xrightarrow{\partial} H, \cdot)$. *Unha trenza (ou levantamento de Peiffer) para o módulo cruzado é unha aplicación* $\{-,-\} : H \times H \to G$ *verificando:*

$$\begin{aligned}
\partial\{h, h'\} &= [h, h'],\\
\{\partial g, \partial g'\} &= [g, g'],\\
\{\partial g, h\} &= g(h \cdot g^{-1}),\\
\{h, \partial g\} &= (h \cdot g)g^{-1},\\
\{h, h'h''\} &= \{h, h'\}(h' \cdot \{h, h''\}),\\
\{hh', h''\} &= (h \cdot \{h', h''\})\{h, h''\},
\end{aligned}$$

para $g, g' \in G$, $h, h', h'' \in H$, *onde* $[g, g'] = gg'g^{-1}g'^{-1}$.

Diremos que $(G \xrightarrow{\partial} H, \cdot, \{-,-\})$ *é un módulo cruzado trenzado de grupos.*

É ben coñecido que as categorías de módulos cruzados de grupos e os grupos categóricos son equivalentes, e Joyal e Street probaron en [38] que a noción de trenza para grupos categóricos proporcionaba unha categoría equivalente á categoría de módulos cruzados trenzados de grupos [9, 16].

As nocións de módulos cruzados de álxebras asociativas [18], álxebras de Lie [40] e álxebras de Leibniz [43] apareceron tratando de emular os módulos cruzados de grupos, e está probado que as categorías correspondentes son equivalentes ás súas categorías internas respectivas.

Tendo claro o que se fixo para grupos, no resto do Capítulo 2 estudamos as trenzas das categorías internas e módulos cruzados das categorías mencionadas: álxebras asociativas, álxebras de Lie e álxebras de Leibniz.

O caso de álxebras asociativas trátase na Sección 2.2. Neste caso pódese facer o mesmo que no caso de grupos, xa que a asociatividade permite traballar de modo natural coas trenzas de categorías semigrupais.

A definición de trenza introducida na tese para o caso de álxebras asociativas categóricas é a seguinte:

Definición 2.2.1. *Consideremos a K-álxebra asociativa $\mathcal{C} = (C_1, C_0, s, t, e, k)$.*

Unha trenza en $\mathcal{C}$ é unha aplicación K-bilineal $\tau : C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, satisfacendo:

$$\tau_{a,b} : ab \to ba, \tag{AsB1}$$

$$\begin{tikzcd} s(x)s(y) \arrow[r, "xy"] \arrow[d, "\tau_{s(x),s(y)}"'] & t(x)t(y) \arrow[d, "\tau_{t(x),t(y)}"] \\ s(y)s(x) \arrow[r, "yx"'] & t(y)t(x), \end{tikzcd} \tag{AsB2}$$

$$\tau_{ab,c} = (\tau_{a,c} e(b)) \circ (e(a) \tau_{b,c}), \tag{AsB3}$$

$$\tau_{a,bc} = (e(b) \tau_{a,c}) \circ (\tau_{a,b} e(c)), \tag{AsB4}$$

para $a, b, c \in C_0$, $x, y \in C_1$.

Diremos que $(C_1, C_0, s, t, e, k, \tau)$ é unha K-álxebra asociativa trenzada.

Dado que queremos que a categoría de módulos cruzados trenzados sexa equivalente á categoría de álxebras asociativas categóricas, introdúcese a seguinte definición para as trenzas dos módulos cruzados.

Definición 2.2.4. *Sexa $(M \xrightarrow{\partial} N, *= (*_1, *_2))$ un módulo cruzado de K-álxebras asociativas. Unha trenza (ou levantamento de Peiffer) é unha aplicación K-bilineal $\{-,-\} : N \times N \to M$ satisfacendo:*

$$\partial\{n, n'\} = [n, n'], \tag{BXAs1}$$

$$\{\partial m, \partial m'\} = [m, m'], \tag{BXAs2}$$

$$\{\partial m, n\} = -[n, m]_*, \tag{BXAs3}$$

$$\{n, \partial m\} = [n, m]_*, \tag{BXAs4}$$

$$\{n, n'n''\} = n' *_1 \{n, n''\} + \{n, n'\} *_2 n'', \tag{BXAs5}$$

$$\{nn', n''\} = n *_1 \{n', n''\} + \{n, n''\} *_2 n', \qquad \text{(BXAs6)}$$

con $m, m' \in M,\ n, n', n'' \in N$.

Aquí, $[n, m]_* = n *_1 m - m *_2 n$ *e* $[x, y] = xy - yx$.

$(M \xrightarrow{\partial} N, *, \{-, -\})$ *é un módulo cruzado trenzado de* K*-álxebras.*

Tense, ao igual que no caso de grupos, unha equivalencia entre as categorías trenzadas.

A noción de trenza para o caso de K-álxebras de Lie xa foi introducida por Ulualan [50]. Por outro lado, Ellis [20] (véxase tamén [46]) definiu a noción de 2-módulo cruzado de K-álxebras de Lie.

Na Sección 2.3, motivaremos a definición dada por Ulualan [50] para módulos cruzados trenzados de álxebras de Lie utilizando a nosa definición de trenza para módulos cruzados de álxebras asociativas, e daremos unha definición máis sinxela cando $\mathrm{char}(K) \neq 2$. Tamén discutiremos unha definición diferente de módulo cruzado trenzado de álxebras de Lie mostrando a súa relación co caso asociativo. Usamos unha definición lixeiramente diferente da dada para módulos cruzados trenzados por Ulualan, xa que queremos un paralelismo entre os exemplos de módulos cruzados trenzados de grupos e os de módulos trenzados de K-álxebras de Lie. Con esta definición algo cambiada tamén buscamos ter un caso particular de 2-módulo cruzado (véxase [46]), como pasa no exemplo de grupos.

As definicións de trenza neste caso son as seguintes.

Definición 2.3.1 ([50]). *Sexa* $C = (C_1, C_0, s, t, e, k)$ *unha* K*-álxebra de Lie categórica.*

Unha trenza en C *é unha aplicación* K*-bilineal* $\tau : C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$, *satisfacendo:*

$$\tau_{a,b} : [a, b] \to [b, a], \qquad \text{(LieT1)}$$

$$\begin{array}{ccc} [s(x), s(y)] & \xrightarrow{[x,y]} & [t(x), t(y)] \\ {\scriptstyle \tau_{s(x),s(y)}}\downarrow & & \downarrow{\scriptstyle \tau_{t(x),t(y)}} \\ [s(y), s(x)] & \xrightarrow[{[y,x]}]{} & [t(y), t(x)], \end{array} \qquad \text{(LieT2)}$$

$$\tau_{[a,b],c} = [\tau_{a,c}, e(b)] + [e(a), \tau_{b,c}], \qquad \text{(LieB3)}$$

$$\tau_{a,[b,c]} = [e(b), \tau_{a,c}] + [\tau_{a,b}, e(c)], \qquad \text{(LieB4)}$$

para $a, b, c \in C_0$, $x, y \in C_1$.

Diremos que $(C_0, C_1, s, t, e, k, \tau)$ *é unha* K*-álxebra de Lie categórica trenzada.*

Definición 2.3.7. *Sexa* $\mathcal{X} = (M \xrightarrow{\partial} N, \cdot)$ *un módulo cruzado de* K*-álxebras de Lie.*

Unha trenza (ou levantamento de Peiffer) sobre o módulo cruzado $\mathcal{X}$ *vén dado por unha aplicación* K*-bilineal* $\{-,-\} : N \times N \to M$ *satisfacendo:*

$$\partial\{n, n'\} = [n, n'], \tag{BXLie1}$$

$$\{\partial m, \partial m'\} = [m, m'], \tag{BXLie2}$$

$$\{\partial m, n\} = -n \cdot m, \tag{BXLie3}$$

$$\{n, \partial m\} = n \cdot m, \tag{BXLie4}$$

$$\{n, [n', n'']\} = \{[n, n'], n''\} - \{[n, n''], n'\}, \tag{BXLie5}$$

$$\{[n, n'], n''\} = \{n, [n', n'']\} - \{n', [n, n'']\}, \tag{BXLie6}$$

para $m, m' \in M$, $n, n', n'' \in N$.

Diremos que $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *é un módulo cruzado trenzado de* K*-álxebras de Lie.*

Temos tamén unha equivalencia entre as categorías cando $\mathrm{char}(K) \neq 2$.

As K-álxebras de Leibniz aparecen nas matemáticas como o caso "non-antisimétrico" das álxebras de Lie. Con isto na mente, na Sección 2.4, mostraremos como estender a idea de trenza dende o caso de Lie o de Leibniz.

A pesar de que as K-álxebras de Lie forman unha subvariedade da variedade das K-álxebras de Leibniz, Loday e Pirashvili atoparon en [44] que as K-álxebras de Leibniz pódense ver como unha subcategoría correflectiva plena dun certo tipo de obxectos de Lie (internalización das K-álxebras de Lie). Para facelo, eles introduciron un novo produto tensor na categoría de aplicacións lineais, no cal aplicaron a internalización das K-álxebras de Lie. Esta realización demostrou ser moi útil para estudar diferentes problemas en K-álxebras de Leibniz.

Nesta sección, mostraremos a internalización da noción de módulo cruzado de obxectos de Lie cunha acción de Lie esquerda de nunha categoría arbitraria. Tamén definiremos trenzas para módulos cruzados de obxectos de Lie e obxectos de Lie categóricos. Logo aplicaremos esta definición á categoría $\mathcal{LM}_K$ de Loday-Pirashvili, e obteremos os conceptos de trenza para módulos cruzados de álxebras de Leibniz e álxebras categóricas de Leibniz.

Definición 2.4.47. *Sexa (C_1, C_0, s, t, e, k) unha K-álxebra de Leibniz categórica.*

Unha trenza é un par (τ, ψ) de aplicacións K-bilineais $\tau, \psi \colon C_0 \times C_0 \to C_1$, $(a, b) \mapsto \tau_{a,b}$ e $(a, b) \mapsto \psi_{a,b}$, satisfacendo:

$$\tau_{a,b} \colon [a, b] \to -[a, b] \quad e \quad \psi_{a,b} \colon [a, b] \to -[a, b], \tag{LeibT1}$$

$$\begin{tikzcd}
{[s(x), s(y)]} \arrow[r, "{[x,y]}"] \arrow[d, "\tau_{s(x),s(y)}"] & {[t(x), t(y)]} \arrow[d, "\tau_{t(x),t(y)}"] \\
{-[s(x), s(y)]} \arrow[r, "{-[x,y]}"] & {-[t(x), t(y)],}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[s(x), s(y)]} \arrow[r, "{[x,y]}"] \arrow[d, "\psi_{s(x),s(y)}"] & {[t(x), t(y)]} \arrow[d, "\psi_{t(x),t(y)}"] \\
{-[s(x), s(y)]} \arrow[r, "{-[x,y]}"] & {-[t(x), t(y)],}
\end{tikzcd} \tag{LeibT2}$$

$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \tau_{[a,c],b}, \tag{LeibT3}$$

$$\psi_{a,[b,c]} = \tau_{[a,b],c} - \psi_{[a,c],b}, \tag{LeibT4}$$

$$\tau_{a,[b,c]} = \tau_{[a,b],c} - \psi_{[a,c],b}, \tag{LeibT5}$$

$$\psi_{a,[b,c]} = \psi_{[a,b],c} - \psi_{[a,c],b}, \qquad a, b, c \in C_0,\ x, y \in C_1. \tag{LeibT6}$$

Diremos que $(C_1, C_0, s, t, e, k, (\tau, \psi))$ é unha K-álxebra de Leibniz trenzada.

Definición 2.4.29. *Sexa $\mathcal{X} = (M \xrightarrow{\partial} N, (\cdot_1, \cdot_2))$ un módulo cruzado de K-álxebras de Leibniz.*

Unha trenza (ou levantamento de Peiffer) sobre $\mathcal{X}$ é un par $(\{-,-\}, \langle -,- \rangle)$ de aplicacións K-bilineais $\{-,-\}, \langle -,- \rangle \colon N \times N \to M$, con $(n, n') \mapsto \{n, n'\}$ e $(n, n') \mapsto \langle n, n' \rangle$, satisfacendo:

$$\partial\{n, n'\} = [n, n'] = \partial\langle n, n' \rangle, \tag{BXLeib1}$$

$$\{\partial m, \partial m'\} = [m, m'] = \langle \partial m, \partial m' \rangle, \tag{BXLeib2}$$

$$\{\partial m, n\} = m \cdot_2 n = \langle \partial m, n \rangle, \tag{BXLeib3}$$

$$\{n, \partial m\} = n \cdot_1 m = \langle n, \partial m \rangle, \tag{BXLeib4}$$

$$\{n, [n', n'']\} = \{[n, n'], n''\} - \{[n, n''], n'\}, \tag{BXLeib5}$$

$$\langle n, [n', n''] \rangle = \{[n, n'], n''\} - \langle [n, n''], n' \rangle, \tag{BXLeib6}$$

$$\{n, [n', n'']\} = \{[n, n'], n''\} - \langle [n, n''], n' \rangle, \tag{BXLeib7}$$

$$\langle n, [n', n''] \rangle = \langle [n, n'], n'' \rangle - \langle [n, n''], n' \rangle, \tag{BXLeib8}$$

para $n, n', n'' \in N$, $m, m' \in M$.

Diremos que $(M \overset{\partial}{\longrightarrow} N, (\cdot_1, \cdot_2), (\{-,-\}, \langle -,- \rangle))$ *é un módulo cruzado trenzado de* K*-álxebras de Leibniz.*

As categorías resultantes son equivalentes, ao igual que no resto de casos. Tense, tamén, que as trenzas para o caso de Lie aparecen como un caso particular de estas últimas ($\psi_{a,b} = -\tau_{b,a}$, $\langle n, n' \rangle = -\{n', n\}$).

Na Sección 2.5, veremos o produto tensorial non abeliano de grupos como un exemplo de módulo cruzado trenzado de grupos. Ademais, coa nosa definición de trenza para módulos cruzados de álxebras de Lie, obtemos igualmente un exemplo de trenza utilizando o produto tensorial non abeliano de álxebras de Lie. O mesmo ocorre coa nosa definición de trenza para módulos cruzados de álxebras de Leibniz.

Na seguinte parte da tese (Capítulo 3) tratamos o tema das extensións centrais en módulos cruzados trenzados de K-álxebras de Lie.

Os conceptos de extensión central de grupos ou álxebras de Lie son moi relevantes en matemáticas, e teñen un rol fundamental en varias áreas da física. Estes conceptos foron estendidos ás categorías de módulos cruzados de grupos e módulos cruzados de K-álxebras de Lie. Este estudo no caso dos módulos cruzados iniciouse en [48] para grupos e en [13] para K-álxebras de Lie, e séguese a estudar hoxe en día.

Como os módulos cruzados de grupos ou K-álxebras de Lie son xeneralizacións de grupos e álxebras de Lie, é esencial buscar nestas categorías extensións dos resultados clásicos de grupos e K-álxebras de Lie.

En [26], Fukushi deu unha versión trenzada dos resultados en extensións centrais universais de módulos cruzados de grupos dados por Norrie en [48]. Fukushi encontrou unha trenza natural na extensión central universal dun módulo cruzado de grupos que é compatible coa do propio módulo cruzado. Esta maneira de traballar sería o que nesta tese se denomina extensión $\mathfrak{U}$-central universal, pero non define o concepto de centro ou conmutador na propia categoría trenzada.

No Capítulo 3 mostramos unha versión trenzada dos resultados de Casas e Ladra en [13] para módulos cruzados trenzados de K-álxebras de Lie; especificamente, estudase a extensión central universal na categoría de módulos cruzados trenzados $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$.

Na sección 3.1, proporcionamos a definición de extensións centrais na categoría de módulos cruzados de K-álxebras de Lie $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$ e introducimos a definición de extensións $\boldsymbol{B}$-centrais en $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$. Para facer isto, atopamos cal é a definición de centro e conmutador na categoría, usando a internalización deses conceptos dada por Huq en [35].

Definición 3.1.5. *Sexa* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *un módulo cruzado trenzado de álxebras de Lie.*

O $\boldsymbol{B}$*-centro de* $\mathcal{M}$ *é o submódulo cruzado trenzado*

$$Z_{\boldsymbol{B}}(\mathcal{M}) = (M^N \xrightarrow{\partial|_{M^N}} Z_{\boldsymbol{B}}(N), \cdot_Z, \{-,-\}_Z).$$

O $\boldsymbol{B}$*-centro é o centro [35] na categoría* $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$.

Definición 3.1.7. *Sexa* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *un módulo cruzado trenzado de álxebras de Lie. O* $\boldsymbol{B}$*-submódulo cruzado trenzado conmutador está dado por*

$$[\mathcal{M}, \mathcal{M}]_{\boldsymbol{B}} = (B_N(M) \xrightarrow{\partial|_{B_N(M)}} [N, N], \cdot_C, \{-,-\}_C).$$

O $\boldsymbol{B}$*-conmutador é o conmutador [35] na categoría* $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$.

Coa definición do $\boldsymbol{B}$-centro obtemos a definición de extensión $\boldsymbol{B}$-central; e coa definición de $\boldsymbol{B}$-conmutador obtemos a definición de $\boldsymbol{B}$-perfecto.

Definición 3.1.9. *Un módulo cruzado trenzado de álxebras de Lie* $\mathcal{M}$ *é* $\boldsymbol{B}$*-perfecto se coincide co seu* $\boldsymbol{B}$*-submódulo trenzado conmutador.*

Definición 3.1.10. *Unha extensión de módulos cruzados trenzados de álxebras de Lie está dada por un morfismo* $\mathcal{X} \xrightarrow{(f_1,f_2)} \mathcal{Y}$ *en* $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$ *cumprindo que* f_1 *e* f_2 *son morfismos sobrexectivos.*

Ademais, diremos que é $\boldsymbol{B}$*-central (central na categoría* $\boldsymbol{BX}(\boldsymbol{LieAlg}_K)$*) se o núcleo* $\ker(f_1, f_2)$ *é un submódulo cruzado trenzado de* $Z_{\boldsymbol{B}}(\mathcal{X})$*, i.e. o núcleo está "dentro" do* $\boldsymbol{B}$*-centro.*

Introdúcese tamén nesta sección a noción de extensión $\mathfrak{U}$-central de módulos cruzados trenzados de K-álxebras de Lie, onde $\mathfrak{U} : \boldsymbol{BX}(\boldsymbol{LieAlg}_K) \to \boldsymbol{X}(\boldsymbol{LieAlg}_K)$ é o functor de esquecemento, que o que fai é esquecer a trenza.

Definición 3.1.11. *Diremos que unha extensión* $\mathcal{X} \xrightarrow{(f_1, f_2)} \mathcal{Y}$ *de módulos cruzados trenzados de* K*-álxebras de Lie é unha extensión* $\mathfrak{U}$*-central se* $\mathfrak{U}(\mathcal{X}) \xrightarrow{\mathfrak{U}(f_1, f_2)} \mathfrak{U}(\mathcal{Y})$ *é central en* $\boldsymbol{X}(\boldsymbol{LieAlg}_K)$*, i.e.* $\ker\big(\mathfrak{U}(f_1, f_2)\big)$ *é un submódulo cruzado do centro de* $\mathfrak{U}(\mathcal{X})$*,* $Z\big(\mathfrak{U}(\mathcal{X})\big)$.

Na Sección 3.2, construímos a extensión $\boldsymbol{B}$-central universal para un módulo cruzado trenzado de K-álxebras de Lie $\boldsymbol{B}$-perfecto e demostramos que un módulo cruzado trenzado de K-álxebras de Lie admite unha extensión universal $\boldsymbol{B}$-central se e só se é $\boldsymbol{B}$-perfecto.

Lema 3.2.1. *Se* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *é un módulo cruzado trenzado de* K*-álxebras de Lie, entón* $N \otimes N \xrightarrow{\Phi_1} M$ *definido por* $n \otimes n' \mapsto \{n, n'\}$*, e* $N \otimes N \xrightarrow{\Phi_2} N$ *definido por* $n \otimes n' \mapsto [n, n']$*, son* K*-homomorfismos de Lie.*

Ademais, Φ_1 *e* Φ_2 *son simultaneamente sobrexectivos se e só se o módulo cruzado trenzado de* K*-álxebras de Lie* $\mathcal{M}$ *é* $\boldsymbol{B}$*-perfecto.*

Corolario 3.2.7. *Se* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *é un módulo cruzado trenzado de* K*-álxebras de Lie* $\boldsymbol{B}$*-perfecto, entón*

$$\mathcal{U} = (N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-]) \xrightarrow{\Phi=(\Phi_1,\Phi_2)} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\}) \tag{UBCE}$$

é a extensión $\boldsymbol{B}$*-central universal de* $\mathcal{M}$*, onde* Φ_1 *e* Φ_2 *están definidas no Lema 3.2.1.*

Teorema 3.2.11. *Un módulo cruzado trenzado de álxebras de Lie admite unha extensión* $\boldsymbol{B}$*-central universal se e só se é* $\boldsymbol{B}$*-perfecto.*

Na Sección 3.3, construímos a extensión $\mathfrak{U}$-central universal para módulos cruzados trenzados de álxebras de Lie que sexan perfectos, tras aplicar o functor $\mathfrak{U}$, facendo o mesmo estudo que se fixo en [26] no caso de grupos.

Proposición 3.3.3. *Se* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *é un módulo cruzado trenzado de* K*-álxebras de Lie verificando que* $\mathfrak{U}(\mathcal{M})$ *é perfecto, entón*

$$\mathcal{V} = (N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *, \{\!\{-,-\}\!\}) \xrightarrow{c=(c_1,c_2)} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\}), \tag{UUCE}$$

é a extensión $\mathfrak{U}$-central universal de $\mathcal{M}$.

E máis, o morfismo universal inicial é o mesmo que o de universalidade no caso non trenzado.

Corolario 3.3.7. *Un módulo cruzado trenzado de álxebras de Lie admite extensión $\mathfrak{U}$-central universal se e só se é perfecto como módulo cruzado.*

Na sección 3.4, estudamos a relación entre a extensión universal $\boldsymbol{B}$-central e a extensión universal $\mathfrak{U}$-central dun módulo cruzado trenzado de K-álxebras de Lie, demostrando que ámbalas dúas extensións centrais universais existen e coinciden para un módulo cruzado trenzado de K-álxebras de Lie $\boldsymbol{B}$-perfecto.

Lema 3.4.1. *Sexa $\mathcal{M}$ un módulo cruzado trenzado de álxebras de Lie. Entón $\mathcal{M}$ é $\boldsymbol{B}$-perfecto se e só se $\mathfrak{U}(\mathcal{M})$ é perfecto.*
De feito, temos que se $N = [N, N]$ entón $B_N(M) = D_N(M)$.

Teorema 3.4.3. *Sexa $\mathcal{M}$ un módulo cruzado trenzado de álxebras de Lie $\boldsymbol{B}$-perfecto.*

Entón a súa extensión $\boldsymbol{B}$-central universal $\mathcal{U} \overset{\Phi}{\twoheadrightarrow} \mathcal{M}$ e a súa extensión $\mathfrak{U}$-central universal $\mathcal{V} \overset{c}{\twoheadrightarrow} \mathcal{M}$ son isomorfas.

No capítulo 4 introducimos a categoría de Loday-Pirashvili para categorías arbitrarias. Neste capitulo tamén estudamos a internalización do concepto de álxebras de Lie e álxebras de Leibniz e a relación entre as categorías resultantes.

Como xa vimos na Sección 2.4, o estudo da internalización das álxebras de Lie é unha ferramenta moi poderosa. Esta internalización tamén permite probar diferentes propiedades en moitas categorías diferentes ao mesmo tempo, como son as superálxebras de Lie, as álxebras de Lie coloreadas, as álxebras de Lie diferenciais graduadas ou as álxebras Hom-Lie regulares [33]. Por exemplo, dúas propiedades importantes que caracterizan á variedade de álxebras de Lie sobre todas as variedades de álxebras non asociativas, a existencia de expoñentes alxébricos [29, 30] ou a representatividade das accións [28], verifícanse tamén nas categorías de obxectos de Lie sobre certos tipos de categorías monoidais [27, 34].

Quérese xeneralizar a construción de Loday e Pirashvili fóra da categoría de aplicacións lineais, definindo un novo produto tensorial en certas clases de categorías con operacións, coa menor cantidade de propiedades necesarias para iso, para obter a categoría de Loday-Pirashvili.

Na Sección 4.1 estudaremos as diferentes categorías tensoriais: categorías con operacións, categorías semigrupais (trenzadas) e categorías monoidais (trenzadas), e construiremos a súa categoría de Loday-Pirashvili.

Un exemplo desta construción é a dada para a categoría de Loday-Pirashvili de categorías semigrupais.

Teorema 4.1.8. *Consideremos unha categoría semigrupal* $(\mathbf{C}, \otimes, a)$ *cumprindo que* $\mathcal{C} = (\mathbf{C}, \otimes)$ *é* $\otimes$*-distributiva. Para* $\begin{matrix} A \\ \downarrow{\scriptstyle f} \\ B \end{matrix}, \begin{matrix} C \\ \downarrow{\scriptstyle g} \\ D \end{matrix}, \begin{matrix} E \\ \downarrow{\scriptstyle h} \\ F \end{matrix} \in \mathrm{Ob}(\mathrm{Hom}(\mathbf{C}))$ *definimos usando a propiedade universal do coproduto o morfismo*

$$\begin{tikzcd}
((A \otimes D) \oplus (B \otimes C)) \otimes F \arrow[r, "\iota_1"] \arrow[d, "\varepsilon^{-1}"'] & (((A \otimes D) \oplus (B \otimes C)) \otimes F) \oplus ((B \otimes D) \otimes E) \arrow[ddd, dashed, "\alpha_{f,g,h}"] & (B \otimes D) \otimes E \arrow[l, "\iota_2"'] \arrow[d, "a"] \\
((A \otimes D) \otimes F) \oplus ((B \otimes C) \otimes F) \arrow[d, "a \oplus a"'] & & B \otimes (D \otimes E) \arrow[d, "\mathrm{Id} \otimes \iota_2"] \\
(A \otimes (D \otimes F)) \oplus (B \otimes (C \otimes F)) \arrow[dr, "\mathrm{Id} \oplus (\mathrm{Id} \otimes \iota_1)"'] & & B \otimes ((C \otimes F) \oplus (D \otimes E)) \arrow[dl, "\iota_2"] \\
& (A \otimes (D \otimes F)) \oplus (B \otimes ((C \otimes F) \oplus (D \otimes E))) &
\end{tikzcd}$$

Entón $\widehat{a}_{f,g,h} = (\alpha_{f,g,h}, a_{B,D,F})$ *dá un asociador para* $(\mathrm{Hom}(\mathbf{C}), \widehat{\otimes})$.

Na Sección 4.2, falaremos das categorías aditivas e mostraremos que, con algúns supostos, podemos recuperar moitas propiedades nas aplicacións entre o produto tensorial e a operación "+".

Teorema 4.2.2. *Consideremos unha categoría con operación* $(\mathbf{C}, \otimes)$ *que sexa distributiva e aniquilada, onde* $\mathbf{C}$ *ten biprodutos. Entón,*

(i) $(f + g) \otimes h = (f \otimes h) + (g \otimes h)$ *para* $A \xrightarrow{f,g} B$, $C \xrightarrow{h} D$,

(ii) $f \otimes (g + h) = (f \otimes g) + (f \otimes h)$ *para* $A \xrightarrow{f} B$, $C \xrightarrow{g,h} D$.

A última sección (Sección 4.3) está dedicada a estudar a internalización dos obxectos de Leibniz e os obxectos de Lie nunha categoría $\mathcal{C}$, mostrando que o functor Liezación existe entre estas categorías.

Teorema 4.3.7. *Consideremos unha categoría semigrupal simétrica $C = (\mathbf{C}, \otimes, a, \tau)$ que verifica que $\mathbf{C}$ é unha categoría aditiva, $\otimes$-pechada e finitamente cocompleta. Sexa (L, μ) un obxecto de Leibniz considérese o coigualador Lieización*

$$L \otimes L \underset{\mu \circ \tau_{L,L}}{\overset{-\mu}{\rightrightarrows}} L \xrightarrow{\pi^L} \overline{L}.$$

Entón existe un único $\overline{\mu}$ que fai o seguinte diagrama conmutativo:

$$\begin{tikzcd} L \otimes L \arrow[r, "\pi^L \otimes \pi^L"] \arrow[d, "\mu"'] & \overline{L} \otimes \overline{L} \arrow[d, "\overline{\mu}"] \\ L \arrow[r, "\pi^L"] & \overline{L} \end{tikzcd}$$

Ademais, verifica as seguintes propiedades:

(i) $(\overline{L}, \overline{\mu})$ *é un obxecto de Lie.*

(ii) $(L, \mu) \xrightarrow{\pi^L} (\overline{L}, \overline{\mu})$ *é un morfismo entre obxectos de Lie.*

(iii) Temos un functor $\overline{(-)}\colon \mathrm{Leib}(C) \to \mathrm{Lie}(C)$ *definido para as frechas como*

$$\overline{(-)}\Big((L, \mu) \xrightarrow{f} (M, \xi) \Big) = (\overline{L}, \overline{\mu}) \xrightarrow{\overline{f}} (\overline{M}, \overline{\xi})$$

onde $\overline{f}$ *é o único morfismo inducido polo coigualador*

$$\begin{tikzcd} L \otimes L \arrow[r, shift left, "-\mu"] \arrow[r, shift right, "\mu \circ \tau_{L,L}"'] & L \arrow[r, "\pi^L"] \arrow[d, "f"'] & \overline{L} \arrow[d, dashed, "\overline{f}"] \\ & M \arrow[r, "\pi^M"] & \overline{M} \end{tikzcd}$$

(iv) O functor $\overline{(-)}$ *é adxunto pola esquerda do functor esquecemento U. É máis, temos a identidade* $\overline{(-)} \circ \mathrm{U} \cong \mathrm{Id}_{\mathrm{Lie}(C)}$.

A continuación, proporcionaremos unha mellor comprensión dos obxectos de Lie na categoría de Loday-Pirashvili de C. No caso particular dos espazos vectoriais, esta construción xeneraliza á dada en [44].

Proposición 4.3.11. *Sexa* $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ *unha categoría semigrupal simétrica onde* $\mathbf{C}$ *é aditiva,* $\otimes$*-distributiva e* $\otimes$*-aniquilada. Entón, un obxecto de Lie en* $\mathrm{LP}(\mathcal{C})$ *é equivalente a un triplo* $\left(\begin{smallmatrix} M \\ \downarrow f \\ N \end{smallmatrix}, *_N^M, \mu_N\right)$ *onde:*

(i) (N, μ_N) *é un obxecto de Lie en* $\mathbf{C}$*,*

(ii) $*_N^M : M \otimes N \to M$ *é un morfismo en* $\mathbf{C}$*.*

(iii) $(M, *_N^M)$ *é un* (N, μ_N)*-módulo pola dereita.*

(iv) f *é* $\big((N, \mu_N), *_N^M, \mu_N\big)$*-equivariante.*

Para concluír, demostraremos que a categoría de obxectos de Leibniz en $\mathcal{C}$ é unha subcategoría plena correflectiva dos obxectos de Lie na categoría de Loday-Pirashvili de $\mathcal{C}$.

Teorema 4.3.15. *Sexa* $\mathcal{C} = (\mathbf{C}, \otimes, a, \tau)$ *unha categoría semigrupal simétrica tal que* $\mathbf{C}$ *é aditiva* $\otimes$*-pechada e finitamente cocompleta. Entón* $\mathrm{Leib}(\mathcal{C})$ *é unha subcategoría correflectiva plena de* $\mathrm{Lie}\big(\mathrm{LP}(\mathcal{C})\big)$.

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Annexes

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