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ORIGINAL PAPER

Representation and inequalities involving continuous linear functionals and fractional derivatives

Marc Jornet[1] · **Juan J. Nieto**[2]

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Abstract
We investigate how continuous linear functionals can be represented in terms of generic operators and certain kernels (Peano kernels), and we study lower bounds for the operators as a consequence, in the space of square-integrable functions. We apply and develop the theory for the Riemann–Liouville fractional derivative (an inverse of the Riemann–Liouville integral), where inequalities are derived with the Gaussian hypergeometric function. This work is inspired by the recent contributions by Fernandez and Buranay (J Comput Appl Math 441:115705, 2024) and Jornet (Arch Math, 2024).

Keywords Representation of functionals · Inequalities · Differintegral operators · Riemann-Liouville fractional calculus

Mathematics Subject Classification 26A33 · 26D10 · 47A67

1 Introduction

The classical Peano kernel, or Peano-Sard, theorem represents continuous linear functionals in terms of ordinary derivatives and a kernel function. This is not only a theorem within pure mathematics; it finds many applications when bounding errors from numerical approximation methods, such as interpolation and integration [2, 3, 16].

Fractional calculus generalizes ordinary derivatives to arbitrary indices, hence it is intuitive that versions of the Peano kernel theorem should exist in such a context,

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✉ Marc Jornet
marc.jornet@uv.es

Juan J. Nieto
juanjose.nieto.roig@usc.es

1 Departament de Matemàtiques, Universitat de València, 46100 Burjassot, Spain

2 CITMAga, Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

depending on the selected fractional derivative [1, 8, 17]. These works were initiated in the nineties by Diethelm, for the Riemann-Liouville derivative [4–6], and for the Caputo derivative in the specific case of quadrature-integration formulae [7]. Very recently, Fernandez and Buranay obtained and applied the Peano kernel theorem in the Caputo fractional calculus, in a general sense [10]. Inspired by this contribution and inequalities from the classical Peano kernel [11], a new tight lower bound has just been obtained for the L^2-norm of the Caputo derivative [13].

In this paper, we carry out an abstraction of the representation of continuous linear functionals, in terms of "generic" operators, that extends ideas from previous literature, such as [4, 10]. As a consequence, tight lower bounds in the L^2-sense for the operators are derived, to complement [11, 13]. This part corresponds to Sect. 2. We then adapt the theory to the particular situation of the Riemann–Liouville fractional derivative: First, by revisiting results from [4] in Sect. 3, and second, by obtaining novel operator's lower bounds in the Riemann-Liouville context in Sect. 4, like in [13]. The inequalities are expressed with the Gaussian hypergeometric function. Important comments about [13], concerning the tightness of inequalities, are also given at the end of the section. Finally, Sect. 5 lists some unanswered issues, for future investigation.

Our work assumes knowledge of the basic fractional operators, see [8, 18], as well as all the context and notations introduced in [13].

2 Peano kernel in the context of abstract operators and a lower bound

We state and prove an abstraction of [4, Theorem 2.2] and [10, Theorem 2.1].

Theorem 2.1 *Let $D : \Xi[a,b] \to \mathrm{L}^1[a,b]$ be an operator, not necessarily linear, where $\Xi[a,b] \subseteq \mathcal{C}[a,b]$. Let $I : \mathrm{L}^1[a,b] \to \mathrm{L}^1[a,b]$ be another operator, expressed in terms of a kernel $R \in \mathcal{C}([a,b]^2)$ as*

$$Iy(t) = \int_a^b R(t,s)y(s)\mathrm{d}s. \tag{2.1}$$

Assume that, for every $x \in \Xi[a,b]$, one has

$$p_x = x - IDx \in \mathcal{C}[a,b]. \tag{2.2}$$

Consider continuous linear functionals $L \in \mathcal{L}(\mathcal{C}[a,b])$ such that

$$Lp_x = 0, \tag{2.3}$$

for all $x \in \Xi[a,b]$. Then, on $\Xi[a,b]$, L can be represented as an integral in terms of a kernel K (the Peano kernel) and the operator D, as follows:

$$Lx = \int_a^b K(s)Dx(s)\mathrm{d}s, \quad K(s) = L[R(\cdot,s)], \quad K \in \mathcal{C}[a,b]. \tag{2.4}$$

Proof Consider the canonical representation of L, in terms of $g_L \in \mathrm{NBV}[a,b]$ [4, Theorem 2.1]:

$$Lz = \int_a^b z(t)\mathrm{d}g_L(t), \tag{2.5}$$

for $z \in \mathcal{C}[a,b]$. Then, for $x \in \Xi[a,b]$,

$$\begin{aligned} Lx &= L[IDx + p_x] = L[IDx] + Lp_x \\ &= L[IDx] = \int_a^b IDx(t)\mathrm{d}g_L(t), \end{aligned}$$

by (2.2) and (2.3). We combine this formula with (2.1):

$$Lx = \int_a^b \int_a^b R(t,s)Dx(s)\mathrm{d}s\mathrm{d}g_L(t).$$

We justify Fubini's theorem as follows. By Jordan's decomposition theorem, $g_L = g_L^+ - g_L^-$, where g_L^+ and g_L^- are right-continuous and non-decreasing functions. We let $g_L^\sharp = g_L^+ + g_L^-$, of use to bound integrals. Since $R \in \mathcal{C}([a,b]^2)$ and $Dx \in \mathrm{L}^1[a,b]$, we deduce

$$\int_a^b \int_a^b |R(t,s)||Dx(s)|\mathrm{d}s\mathrm{d}g_L^\sharp(t) \le \|R\|_\infty \left(g_L^\sharp(b) - g_L^\sharp(a)\right) \|Dx\|_1 < \infty.$$

We can thus interchange the order of integration, to arrive at (2.4):

$$Lx = \int_a^b \int_a^b R(t,s)\mathrm{d}g_L(t)Dx(s)\mathrm{d}s = \int_a^b K(s)Dx(s)\mathrm{d}s,$$

where $R(\cdot, s)$ is continuous for the representation of K via L:

$$K(s) = \int_a^b R(t,s)\mathrm{d}g_L(t). \tag{2.6}$$

Finally, the continuity of K is true by the dominated convergence theorem. Indeed, if $s \to s_1 \in [a,b]$, then $R(t,s) \to R(t,s_1)$ and $R(t,s) \le \|R\|_\infty$, with

$$\int_a^b \|R\|_\infty \mathrm{d}g_L^\sharp(t) = \|R\|_\infty (g_L^\sharp(b) - g_L^\sharp(a)) < \infty,$$

because $R \in \mathcal{C}([a,b]^2)$; therefore

$$\lim_{s\to s_1} K(s) = \int_a^b \left(\lim_{s\to s_1} R(t,s)\right) \mathrm{d}g_L(t) = \int_a^b R(t,s_1)\mathrm{d}g_L(t) = K(s_1).$$

□

Remark 2.2 The set Ξ ensures (2.2). Nonetheless, D might be defined on a larger set than Ξ. For example, if D is the modified Caputo operator of non-integer order α [13], then $\Xi = \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]$, by [8, Theorem 3.8]. Here, $\lfloor\cdot\rfloor$ is the integer part.

This theorem can be used to derive sharp lower bounds for operators, see [13].

Lemma 2.3 *Let $L \in \mathcal{L}(\mathcal{C}[a,b])$. Consider a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{C}[a,b]$ such that*

$$\sup_{n\geq 1} \|x_n\|_\infty < \infty \tag{2.7}$$

and

$$x(t) = \lim_{n\to\infty} x_n(t) \tag{2.8}$$

pointwise on $[a,b]$, where $x \in \mathcal{C}[a,b]$. Then

$$Lx = \lim_{n\to\infty} Lx_n. \tag{2.9}$$

In particular, if $R \in \mathcal{C}([a,b]^2)$, then

$$L\left[\int_a^b R(\cdot,s)\mathrm{d}s\right] = \int_a^b L[R(\cdot,s)]\mathrm{d}s. \tag{2.10}$$

Proof The proof is simple, by the dominated convergence theorem. We have, by the canonical representation (2.5),

$$\begin{aligned} |Lx - Lx_n| &= \left|\int_a^b x(t)\mathrm{d}g_L(t) - \int_a^b x_n(t)\mathrm{d}g_L(t)\right| \\ &\leq \int_a^b |x(t)-x_n(t)|\mathrm{d}g_L^\sharp(t). \end{aligned}$$

The integrand $|x(t) - x_n(t)|$ tends to 0 pointwise by (2.8), as $n \to \infty$, and it is uniformly bounded on n by (2.7). Then (2.9) holds.

If $R \in \mathcal{C}([a,b]^2)$, we consider

$$x(t) = \int_a^b R(t,s)\mathrm{d}s$$

and the Riemann sums

$$x_n(t) = \sum_{i=1}^{r_n} R(t,s_i)(s_i - s_{i-1}),$$

for partitions $a = s_0 < s_1 < \ldots < s_{r_n} = b$, $r_n \geq 1$. Conditions (2.7) and (2.8) are satisfied, hence (2.9) and (2.10) hold. □

Corollary 2.4 *Consider the context of Theorem 2.1, with the additional property $DIK = K$ almost everywhere (we are assuming that D can be evaluated at IK, although it might be possible that $IK \notin \Xi$). Then, the following lower bound holds, for $x \in \Xi[a,b]$, and it is sharp on a possibly larger set than Ξ:*

$$\|Dx\|_2 \geq \frac{|Lx|}{\|L_t[R(t,\cdot)]\|_2}, \tag{2.11}$$

where L_t means that the functional L acts with respect to the variable t, while $\|\cdot\|_2$ is computed with respect to the second variable.

Proof By Cauchy-Schwarz inequality in (2.4),

$$|Lx| \leq \|K\|_2 \|Dx\|_2. \tag{2.12}$$

If $x^* = IK \in \mathcal{C}[a,b]$, then

$$Lx^* = \int_a^b K(s)DIK(s)\mathrm{d}s = \int_a^b K(s)^2 \mathrm{d}s = \|K\|_2^2, \tag{2.13}$$

by assumption. Such a function x^* is chosen to obtain equality in Cauchy-Schwarz inequality. Notice that it is "nearly" unique: If another $x^{**} \in \Xi$ meets equality in Cauchy-Schwarz inequality, then necessarily $K = Dx^{**}$, which implies $x^* = IK = IDx^{**} = x^{**} - p_{x^{**}}$, with $Lp_{x^{**}} = 0$. By combining (2.12) and (2.13), we derive

$$\|Dx\|_2 \geq \frac{|Lx|}{\sqrt{Lx^*}}. \tag{2.14}$$

Now, with (2.1) and (2.4), we compute

$$\begin{aligned} x^*(t) = IK(t) &= \int_a^b R(t,s)K(s)\mathrm{d}s = \int_a^b R(t,s)\int_a^b R(\tau,s)\mathrm{d}g_L(\tau)\mathrm{d}s \\ &= \int_a^b \int_a^b R(\tau,s)R(t,s)\mathrm{d}s\mathrm{d}g_L(\tau) = L_\tau\left[\int_a^b R(\tau,s)R(t,s)\mathrm{d}s\right]. \end{aligned}$$

Fubini's theorem is justified by the continuity of the integrand. In consequence,

$$Lx^* = L_t\left[L_\tau\left[\int_a^b R(\tau,s)R(t,s)\mathrm{d}s\right]\right].$$

By Lemma 2.3,

$$Lx^* = \|L_t[R(t,\cdot)]\|_2^2.$$

Then (2.14) gives (2.11). Inequality (2.11) is tight, because it is attained for $x^* \in \mathcal{C}[a,b]$ (it might be possible that $x^* \notin \Xi$). □

Remark 2.5 Note that (2.11) is essentially a consequence of (2.4), by Cauchy-Schwarz inequality. However, to demonstrate that (2.11) is sharp on a certain set (i.e., the existence of the minimizer $x^* \in \mathcal{C}[a,b]$ in the proof), Corollary 2.4 with its assumption $DIK = K$ is indispensable.

Remark 2.6 Cauchy-Schwarz inequality can be extended to arbitrary indices by Hölder's inequality. However, the fact that $\mathrm{L}^2[a,b]$ is endowed with an inner product permits computing $\|K\|_2$ and obtaining x^*. These calculations cannot be conducted in an arbitrary $\mathrm{L}^p[a,b]$, $p \neq 2$.

Example 2.7 In the context of Theorem 2.1 and Corollary 2.4, suppose that p_x is always a polynomial of degree less than 2. Consider

$$Lx = x(a) - 2x\left(\frac{a+b}{2}\right) + x(b), \tag{2.15}$$

which satisfies $Lp_x = 0$. Then, by Corollary 2.4, the following tight inequality is verified:

$$\|Dx\|_2 \geq \frac{\left|x(a) - 2x\left(\frac{a+b}{2}\right) + x(b)\right|}{\sqrt{\int_a^b \left(R(a,s) - 2R\left(\frac{a+b}{2}, s\right) + R(b,s)\right)^2 \mathrm{d}s}}. \tag{2.16}$$

For concrete operators, the integral $\int_a^b$ in the denominator might be computed explicitly, to avoid numerical integration of a non-differentiable R. Observe that (2.15) is related with numerical analysis: given the function x, consider the one-degree polynomial q that interpolates it at a and b; then

$$\left|\frac{Lx}{2}\right| = \left|q\left(\frac{a+b}{2}\right) - x\left(\frac{a+b}{2}\right)\right| = \text{interpolation error.}$$

When x is a polynomial of degree less than 2, then this error is zero. Notice that (2.16) gives rise to a bound for the interpolation remainder, in terms of the operator D.

3 Peano kernel for the Riemann–Liouville fractional derivative revisited

We show [4, Theorem 4.1] in the context of Theorem 2.1.

Theorem 3.1 *Let* ${}^{RL}I_a^\alpha : \mathrm{L}^1[a,b] \to \mathrm{L}^1[a,b]$ *be the Riemann–Liouville integral, of non-integer order* $\alpha > 1$, *where* $\lfloor \alpha \rfloor \leq \alpha$ *is the integer part,* $0 \leq a < b$, *and* Γ *is the gamma function:*

$${}^{RL}I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) \mathrm{d}s$$

(generalization of the Cauchy formula for repeated integration). Let ${}^{RL}D_a^\alpha : \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b] \to \mathrm{L}^1[a,b]$ *be the Riemann-Liouville fractional derivative, defined as*

$$ {}^{RL}D_a^\alpha x(t) = \frac{\mathrm{d}^{\lfloor\alpha+1\rfloor}}{\mathrm{d}t^{\lfloor\alpha+1\rfloor}} \, {}^{RL}I_a^{\lfloor\alpha+1\rfloor-\alpha} x(t) $$

(analytic continuation to define ${}^{RL}I_a^{-\alpha}$*). Consider the space*

$$ \Xi[a,b] = \{x : [a,b] \to \mathbb{R} \;/\; (\cdot - a)^{\alpha-\lfloor\alpha+1\rfloor} x \in \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]\} \subseteq \mathcal{C}[a,b]. \quad (3.1) $$

Define the modified operators ${}^{RL}\hat{D}_a^\alpha : \Xi[a,b] \to \mathrm{L}^1[a,b]$ *and* ${}^{RL}\hat{I}_a^\alpha : \mathrm{L}^1[a,b] \to \mathrm{L}^1[a,b]$ *as follows:*

$$ {}^{RL}\hat{D}_a^\alpha x = {}^{RL}D_a^\alpha \left[(\cdot - a)^{\alpha-\lfloor\alpha+1\rfloor} x \right] $$

and

$$ {}^{RL}\hat{I}_a^\alpha x = (\cdot - a)^{\lfloor\alpha+1\rfloor-\alpha} \cdot {}^{RL}I_a^\alpha x. $$

Let $L \in \mathcal{L}(\mathcal{C}[a,b])$ *that vanishes for polynomials of degree less than or equal to* $\lfloor\alpha\rfloor$*. Then*

$$ Lx = \int_a^b \hat{K}_a^\alpha(t) \cdot {}^{RL}\hat{D}_a^\alpha x(t) \mathrm{d}t $$

for $x \in \Xi[a,b]$*, where*

$$ \hat{K}_a^\alpha(t) = \frac{1}{\Gamma(\alpha)} L \left[(\cdot - a)^{\lfloor\alpha+1\rfloor-\alpha} (\cdot - t)_+^{\alpha-1} \right] \quad (3.2) $$

is in $\mathcal{C}[a,b]$ *and* $s_+ = \max\{s, 0\}$.

Proof We first notice that

$$ {}^{RL}I_a^\alpha : \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b] \to \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b] \quad (3.3) $$

is a well-defined operator. Indeed, one proceeds by induction for $0 \le m \le \lfloor\alpha+1\rfloor$ and the space $\mathrm{AC}^m[a,b]$. If $m = 0$ and $\mathrm{AC}^m[a,b] = \mathcal{C}[a,b]$, the result is clear. If we assume that the map is well-defined for $m-1$ and $u \in \mathrm{AC}^m[a,b]$, then

$$ \begin{aligned} {}^{RL}I_a^\alpha u(t) &= {}^{RL}I_a^\alpha \circ {}^{RL}I_a^1 u'(t) + u(0) {}^{RL}I_a^\alpha 1 \\ &= {}^{RL}I_a^1 \circ {}^{RL}I_a^\alpha u'(t) + u(0) \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}, \end{aligned} $$

see [18, Lemma 3.4]. Since $u' \in \mathrm{AC}^{m-1}[a,b]$ (if $m \geq 2$, otherwise $u' \in \mathrm{L}^1[a,b]$ if $m=1$), then by induction hypothesis, ${}^{RL}I_a^\alpha u' \in \mathrm{AC}^{m-1}[a,b]$ (if $m \geq 2$, otherwise in $\mathrm{L}^1[a,b]$ if $m=1$), which implies ${}^{RL}I_a^1 \circ {}^{RL}I_a^\alpha u' \in \mathrm{AC}^m[a,b]$. Also, $(t-a)^\alpha \in \mathrm{AC}^m[a,b]$. Hence ${}^{RL}I_a^\alpha u \in \mathrm{AC}^m[a,b]$, as wanted. An alternative proof of (3.3) is given in the classical book [15, Lemma 2.1].

Note that, by using (3.3), the Riemann–Liouville fractional derivative ${}^{RL}D_a^\alpha : \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b] \to \mathrm{L}^1[a,b]$ is well-defined. Also, when $x \in \Xi[a,b]$, we obtain that ${}^{RL}I_a^{\lfloor\alpha+1\rfloor-\alpha}[(\cdot-a)^{\alpha-\lfloor\alpha+1\rfloor}x] \in \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]$, by (3.3). This permits applying the known fundamental theorem of fractional calculus for ${}^{RL}I_a^\alpha \circ {}^{RL}D_a^\alpha$, check [8, Theorem 2.23]:

$$
\begin{aligned}
&{}^{RL}\hat{I}_a^\alpha \circ {}^{RL}\hat{D}_a^\alpha x = (\cdot-a)^{\lfloor\alpha+1\rfloor-\alpha} \cdot {}^{RL}I_a^\alpha \circ {}^{RL}D_a^\alpha \left[(\cdot-a)^{\alpha-\lfloor\alpha+1\rfloor}x\right] \\
&= (\cdot-a)^{\lfloor\alpha+1\rfloor-\alpha}\Bigg((\cdot-a)^{\alpha-\lfloor\alpha+1\rfloor}x \\
&\quad - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{(\cdot-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \\
&\quad \times \lim_{z\to a^+} {}^{RL}D_a^{\lfloor\alpha\rfloor-k} \circ {}^{RL}I_a^{\lfloor\alpha+1\rfloor-\alpha}\left[(\cdot-a)^{\alpha-\lfloor\alpha+1\rfloor}x\right]\Bigg) \\
&= x - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{(\cdot-a)^{\lfloor\alpha\rfloor-k}}{\Gamma(\alpha-k)} \\
&\quad \times \lim_{z\to a^+} {}^{RL}D_a^{\lfloor\alpha\rfloor-k} \circ {}^{RL}I_a^{\lfloor\alpha+1\rfloor-\alpha}\left[(\cdot-a)^{\alpha-\lfloor\alpha+1\rfloor}x\right] \\
&= x - p_x.
\end{aligned}
$$

Since p_x is a polynomial of degree less than or equal to $\lfloor\alpha\rfloor$ (this is precisely the reason of defining the modified operators), both (2.2) and (2.3) are met. On the other hand, for the modified Riemann-Liouville integral, we have

$$
R(t,s) = R_a^\alpha(t,s) = \frac{1}{\Gamma(\alpha)}(t-a)^{\lfloor\alpha+1\rfloor-\alpha}(t-s)_+^{\alpha-1} \in \mathcal{C}([a,b]^2).
$$

We can therefore use Theorem 2.1 and its conclusion (2.4), with $D = {}^{RL}\hat{D}_a^\alpha$ and $I = {}^{RL}\hat{I}_a^\alpha$. □

Remark 3.2 We notice that $\Xi[a,b]$ could be enlarged in Theorem 3.1, to

$$
\tilde{\Xi}[a,b] = \{x \in \mathcal{C}[a,b] \;/\; \exists\, {}^{RL}I_a^{\lfloor\alpha+1\rfloor-\alpha}[(\cdot-a)^{\alpha-\lfloor\alpha+1\rfloor}x] \in \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]\}.
$$

We used (3.1) for simplicity. In [4, Theorem 4.1], Diethelm employed the set

$$
\tilde{\tilde{\Xi}}[a,b] = \{x \in \mathcal{C}[a,b] \;/\; {}^{RL}\hat{D}_a^{\alpha-1}x \in \mathrm{AC}[a,b]\}.
$$

We have the following inclusions:

$$\Xi[a,b] \subseteq \tilde{\tilde{\Xi}}[a,b] \subseteq \tilde{\Xi}[a,b].$$

Indeed, if $x \in \Xi[a,b]$, then

$${}^{RL}I_a^{\lfloor\alpha+1\rfloor-\alpha}\left[(t-a)^{\alpha-\lfloor\alpha+1\rfloor}x\right] \in \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b],$$

by (3.3). This implies

$${}^{RL}\hat{D}_a^{\alpha-1}x = \frac{\mathrm{d}^{\lfloor\alpha\rfloor}}{\mathrm{d}t^{\lfloor\alpha\rfloor}}\,{}^{RL}I_a^{\lfloor\alpha+1\rfloor-\alpha}\left[(\cdot-a)^{\alpha-\lfloor\alpha+1\rfloor}x\right] \in \mathrm{AC}[a,b], \tag{3.4}$$

and the first inclusion follows. The second inclusion is clear by (3.4).

4 A new lower bound for the L^2-norm of the Riemann–Liouville fractional derivative

We employ the previous theory to obtain a new, sharp lower bound for the L^2-norm of the Riemann–Liouville fractional derivative. The Caputo version has just been proved in [13]. See the notations in that paper.

Theorem 4.1 *Fix any non-integer order $\alpha > 1$. Let $L \in \mathcal{L}(\mathcal{C}[a,b])$ that vanishes for polynomials of degree less than or equal to $\lfloor\alpha\rfloor$. If $y \in \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]$, then*

$$\left\|{}^{RL}D_a^{\alpha}y\right\|_2 \geq \left|L\left[(\cdot-a)^{\lfloor\alpha+1\rfloor-\alpha}y\right]\right| \times\left(L_t\left[L_s\left[\frac{\sigma\left[(t-a)^{\lfloor\alpha\rfloor}(s-a)^{\lfloor\alpha+1\rfloor}\cdot{}_2F_1\left(1,1-\alpha,1+\alpha,\frac{s-a}{t-a}\right)\right]}{\Gamma(\alpha)^3\alpha}\right]\right]\right)^{-1/2}, \tag{4.1}$$

where

$$\sigma : \mathcal{C}([a,b]^2) \to \mathcal{C}([a,b]^2),$$
$$\sigma[h](t,s) = \begin{cases} h(t,s), & s < t \\ h(s,t), & t \leq s \end{cases}$$

is a symmetrization operator and ${}_2F_1$ is the Gauss's or ordinary hypergeometric function. Equality in (4.1) is attained in $\mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]$, for typical canonical integrators g_L: if $g_L \in \mathcal{C}^{\lfloor\alpha+1\rfloor}[a,b]$, or if $Ly = \sum_{j=1}^m w_j y(s_j)$ with $g_L(s) = \sum_{j=1}^m w_j H_{s_j}(s)$ in terms of Heaviside functions H_{s_j} and weights $w_j \in \mathbb{R}$.

Proof Notice that

$$ {}^{RL}\hat{D}_a^\alpha \circ {}^{RL}\hat{I}_a^\alpha z = {}^{RL}D_a^\alpha \circ {}^{RL}I_a^\alpha z = z $$

almost everywhere, for all $z \in \mathrm{L}^1[a,b]$. This is stronger than the added assumption in Corollary 2.4.

We proceed as in the proof of Corollary 2.4, with

$$ x = (\cdot - a)^{\lfloor \alpha+1 \rfloor - \alpha} y \in \Xi[a,b]. $$

We have:

$$ \begin{aligned} &{}^{RL}I_a^\alpha \left[L\left[(\cdot - a)^{\lfloor \alpha+1 \rfloor - \alpha} (\cdot - t)_+^{\alpha-1} \right] \right](t) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} L\left[(\cdot - a)^{\lfloor \alpha+1 \rfloor - \alpha} (\cdot - \tau)_+^{\alpha-1} \right] \mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \int_a^b (s-a)^{\lfloor \alpha+1 \rfloor - \alpha} (s-\tau)_+^{\alpha-1} \mathrm{d}g_L(s) \mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^t (s-\tau)_+^{\alpha-1} (t-\tau)^{\alpha-1} \mathrm{d}\tau (s-a)^{\lfloor \alpha+1 \rfloor - \alpha} \mathrm{d}g_L(s) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \int_a^s (s-\tau)^{\alpha-1} (t-\tau)^{\alpha-1} \mathrm{d}\tau (s-a)^{\lfloor \alpha+1 \rfloor - \alpha} \mathrm{d}g_L(s) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^b \int_a^t (s-\tau)^{\alpha-1} (t-\tau)^{\alpha-1} \mathrm{d}\tau (s-a)^{\lfloor \alpha+1 \rfloor - \alpha} \mathrm{d}g_L(s) \\ &= \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)^2 \alpha} \int_a^t (s-a)^{\lfloor \alpha+1 \rfloor} \cdot {}_2F_1\left(1, 1-\alpha, 1+\alpha, \frac{s-a}{t-a}\right) \mathrm{d}g_L(s) \\ &\quad + \frac{(t-a)^{\alpha}}{\Gamma(\alpha)^2 \alpha} \int_t^b (s-a)^{\lfloor \alpha \rfloor} \cdot {}_2F_1\left(1, 1-\alpha, 1+\alpha, \frac{t-a}{s-a}\right) \mathrm{d}g_L(s). \end{aligned} $$

Fubini's theorem is applicable by the continuity of the integrand. Euler's integral formula for the hypergeometric function has been used [13, 14]. Then, by (3.2),

$$ \begin{aligned} x^*(t) &= {}^{RL}\hat{I}_a^\alpha \hat{K}_a^\alpha(t) \\ &= \frac{(t-a)^{\lfloor \alpha \rfloor}}{\Gamma(\alpha)^3 \alpha} \int_a^t (s-a)^{\lfloor \alpha+1 \rfloor} \cdot {}_2F_1\left(1, 1-\alpha, 1+\alpha, \frac{s-a}{t-a}\right) \mathrm{d}g_L(s) \\ &\quad + \frac{(t-a)^{\lfloor \alpha+1 \rfloor}}{\Gamma(\alpha)^3 \alpha} \int_t^b (s-a)^{\lfloor \alpha \rfloor} \cdot {}_2F_1\left(1, 1-\alpha, 1+\alpha, \frac{t-a}{s-a}\right) \mathrm{d}g_L(s) \\ &= L_s\left[\frac{\sigma\left[(t-a)^{\lfloor \alpha \rfloor} (s-a)^{\lfloor \alpha+1 \rfloor} \cdot {}_2F_1\left(1, 1-\alpha, 1+\alpha, \frac{s-a}{t-a}\right) \right]}{\Gamma(\alpha)^3 \alpha} \right]. \end{aligned} $$

We conclude with inequality (2.14). Equality is met for

$$
\begin{aligned}
y^*(t) &= (t-a)^{\alpha-\lfloor\alpha+1\rfloor}x^*(t)\\
&= \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)^3\alpha}\int_a^t (s-a)^{\lfloor\alpha+1\rfloor}\cdot {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{s-a}{t-a}\right)\mathrm{d}g_L(s)\\
&\quad+\frac{(t-a)^{\alpha}}{\Gamma(\alpha)^3\alpha}\int_t^b (s-a)^{\lfloor\alpha\rfloor}\cdot {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{t-a}{s-a}\right)\mathrm{d}g_L(s),
\end{aligned}
$$

where $x^*\in\mathcal{C}[a,b]$. The structure of y^* is analogous to [13, expressions (13) and (14)]. To justify $y^*\in \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]$ under suitable g_L, see the last paragraph of Remark 4.4. The key fact is that ${}_2F_1(1,1-\alpha,1+\alpha,\cdot)\in \mathrm{AC}^{\lfloor\alpha+1\rfloor}[0,1]$, because ${}_2F_1(1,1-\alpha,1+\alpha,\cdot)\in\mathcal{C}^\infty[0,1)$ and $D^{\lfloor\alpha+1\rfloor}{}_2F_1(1,1-\alpha,1+\alpha,\cdot)\in \mathrm{L}^1[0,1]$. □

Example 4.2 In the context of Theorem 4.1, consider (2.15), which annihilates polynomials of degree less than 2. Then, if $\mathcal{X}$ is the indicator function,

$$
\begin{aligned}
x^*(t) &= L_s\left[\frac{\sigma\left[(t-a)^{\lfloor\alpha\rfloor}(s-a)^{\lfloor\alpha+1\rfloor}\cdot {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{s-a}{t-a}\right)\right]}{\Gamma(\alpha)^3\alpha}\right]\\
&= \frac{1}{\Gamma(\alpha)^3\alpha}\Bigg(-2\left(\frac{b-a}{2}\right)^{\lfloor\alpha+1\rfloor}(t-a)^{\lfloor\alpha\rfloor}\\
&\quad\times {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{b-a}{2(t-a)}\right)\mathcal{X}_{(\frac{a+b}{2},b]}(t)\\
&\quad-2\left(\frac{b-a}{2}\right)^{\lfloor\alpha\rfloor}(t-a)^{\lfloor\alpha+1\rfloor}\\
&\quad\times {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{2(t-a)}{b-a}\right)\mathcal{X}_{[a,\frac{a+b}{2}]}(t)\\
&\quad+(b-a)^{\lfloor\alpha\rfloor}(t-a)^{\lfloor\alpha+1\rfloor}\cdot {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{t-a}{b-a}\right)\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
Lx^* &= L_t\left[L_s\left[\frac{\sigma\left[(t-a)^{\lfloor\alpha\rfloor}(s-a)^{\lfloor\alpha+1\rfloor}\cdot {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{s-a}{t-a}\right)\right]}{\Gamma(\alpha)^3\alpha}\right]\right]\\
&= \frac{1}{\Gamma(\alpha)^3\alpha}\Bigg(\left(1+2^{1-2\lfloor\alpha\rfloor}\right)(b-a)^{2\lfloor\alpha\rfloor+1}\frac{\Gamma(1+\alpha)\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(\alpha)}\\
&\quad-4(b-a)^{\lfloor\alpha\rfloor}\left(\frac{b-a}{2}\right)^{\lfloor\alpha\rfloor+1}\cdot {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{1}{2}\right)\Bigg).
\end{aligned}
$$

Therefore, by substituting the denominator in (4.1),

$$
\begin{aligned}
\left\|{}^{RL}D_a^{\alpha}y\right\|_2 \geq & \left|(b-a)^{\lfloor\alpha+1\rfloor-\alpha}y(b)-2\left(\frac{b-a}{2}\right)^{\lfloor\alpha+1\rfloor-\alpha}y\left(\frac{a+b}{2}\right)\right| \\
& \times\left\{\frac{1}{\Gamma(\alpha)^3\alpha}\left(\left(1+2^{1-2\lfloor\alpha\rfloor}\right)(b-a)^{2\lfloor\alpha\rfloor+1}\frac{\Gamma(1+\alpha)\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(\alpha)}\right.\right. \\
& \left.\left. -4(b-a)^{\lfloor\alpha\rfloor}\left(\frac{b-a}{2}\right)^{\lfloor\alpha\rfloor+1}\cdot{}_2F_1\left(1,1-\alpha,1+\alpha,\frac{1}{2}\right)\right)\right\}^{-1/2},
\end{aligned}
\tag{4.2}
$$

for every $y\in \mathrm{AC}^2[a,b]$, and the inequality is sharp. For instance, if $a=0$, $b=2$ and $\alpha=1.8$, then

$$
\left\|{}^{RL}D_0^{1.8}y\right\|_2 \geq \frac{|2^{0.2}y(2)-2y(1)|}{1.0106067168445019\ldots},
$$

for all $y\in \mathrm{AC}^2[0,2]$. Informally, if a and b are arbitrary and $\alpha\to 2^-$, with $\alpha\neq 2$, then $(\cdot-a)^{\lfloor\alpha+1\rfloor-\alpha}\to(\cdot-a)^0=1$ and it is not written in the numerator of (4.1); this yields a lower bound for the second-order derivative when $y\in \mathrm{AC}^2[a,b]$,

$$
\|y''\|_2 \geq \sqrt{12}\frac{\left|y(a)-y\left(\frac{a+b}{2}\right)+y(b)\right|}{(b-a)^{3/2}},
$$

which is consistent with [11, Example 5]. In general, we notice that the structure of the Riemann-Liouville bound is quite different to that of the Caputo bound [13], by the factor $(b-a)^{\lfloor\alpha+1\rfloor-\alpha}$ and its discontinuity at integers as a function of α.

Example 4.3 Related to the previous example, some inequalities in real analysis can be derived. Considering (4.2), if $y(t)=t^{\gamma}$, for $\gamma\in(1,2)$, then

$$
\begin{aligned}
\left\|{}^{RL}D_a^{\alpha}y\right\|_2 &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\sqrt{\int_a^b t^{2(\gamma-\alpha)}\mathrm{d}t} \\
&= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\frac{b^{2(\gamma-\alpha)+1}-a^{2(\gamma-\alpha)+1}}{2(\gamma-\alpha)+1}
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\frac{b^{2(\gamma-\alpha)+1}-a^{2(\gamma-\alpha)+1}}{2(\gamma-\alpha)+1} \\
& > \left|(b-a)^{\lfloor\alpha+1\rfloor-\alpha}b^{\gamma}-2\left(\frac{b-a}{2}\right)^{\lfloor\alpha+1\rfloor-\alpha}\left(\frac{a+b}{2}\right)^{\gamma}\right| \\
& \quad \times\left\{\frac{1}{\Gamma(\alpha)^3\alpha}\left(\left(1+2^{1-2\lfloor\alpha\rfloor}\right)(b-a)^{2\lfloor\alpha\rfloor+1}\frac{\Gamma(1+\alpha)\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(\alpha)}\right.\right.
\end{aligned}
$$

$$-4(b-a)^{\lfloor\alpha\rfloor}\left(\frac{b-a}{2}\right)^{\lfloor\alpha\rfloor+1}\cdot {}_2F_1\left(1,1-\alpha,1+\alpha,\frac{1}{2}\right)\Bigg)\Bigg\}^{-1/2},$$

for all $0 \leq a < b$ and $\alpha \in (1,2)$. If $a=0$, $b=2$, $\alpha=1.8$, and $\gamma=1.9$, for example, then the numeric inequality $2.65772 > 2.26309$ is consistent. It is slightly worse than the Caputo bound 2.32765 from [13] (we notice that ${}^{RL}D_0^{\alpha}t^{\gamma} = {}^{C}D_0^{\alpha}t^{\gamma}$, so it is possible to compare). If $\gamma=1.5$, then the Riemann–Liouville inequality renders $1.86003 > 1.23590$, which is now better than the Caputo bound 1.11325.

Remark 4.4 **Important comments about** [13].

In the Caputo version from [13], we have $\Xi = \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]$. In that paper, it is seen that $x^* = IK \in \Xi$, hence $DIK = K$ almost everywhere and Corollary 2.4 holds. Computations are then carried out to elaborate on $\|L_t[R(t,\cdot)]\|_2$, like in Theorem 4.1.

On the other hand, if the modified Caputo operator $D = D^{\alpha}_{*,a}$ is used in [13], we still need $\Xi = \mathrm{AC}^{\lfloor\alpha+1\rfloor}[a,b]$, by (2.2) and [8, Theorem 3.8]. However, in such a case, $DIK = K$ is straightforward [8, Theorem 3.7], because $K \in \mathcal{C}[a,b]$, although the property $x^* = IK \in \Xi$ is then unknown. Proving that the minimizer x^* is in the domain Ξ is relevant. Observe that, in Ξ, both versions of the Caputo derivative coincide almost everywhere and have the same L^2-norm.

Based on [9, Theorem 2.5], another less-natural possibility for Ξ could be $\hat{\Xi} = \{x \in \mathcal{C}^{\lfloor\alpha\rfloor}[a,b] : \exists\, D^{\alpha}_{*,a}x \in \mathcal{C}[a,b]\}$. Notice that $x^* \in \hat{\Xi}$ is readily justified, for any L, by [18, Proposition 3.2 (8)]. We said "less-natural" because $t^{1.5} \notin \hat{\Xi}$ when $\alpha = 1.8$, but $t^{1.5} \in \Xi$, for example.

The space $\hat{\Xi}$ is certainly natural in differintegral equations $D^{\alpha}_{*,a}x(t) = f(t,x(t))$ with continuous function f. Indeed, $x \in \mathcal{C}^{\lfloor\alpha\rfloor}[a,b]$ comes from the associated Volterra integral problem, and $D^{\alpha}_{*,a}x = f(\cdot,x(\cdot)) \in \mathcal{C}[a,b]$ is clear.

Finally, about [13, Lemma 3.2] and the last lines of the proof to justify that the minimizer x^* belongs to Ξ, we notice that ${}_2F_1(A,B,C,1)$ in (3) is finite only when $C > A+B$, and ${}_2F_1(A,B,C,t) = \mathcal{O}(1/(1-t)^{A+B-C})$ on $[0,1]$ if $C < A+B$ [14, Section 15.4 (ii)]. In particular, if $n = \lfloor\alpha+1\rfloor$, then ${}_2F_1(1+n, 1-\alpha+n, 1+\alpha+n, 1)$ is finite whenever $\alpha > 1.5$, and ${}_2F_1(1+n, 1-\alpha+n, 1+\alpha+n, t) = \mathcal{O}(1/(1-t)^N)$ always holds for $\alpha > 1$, for an $0 < N = N_\alpha < 1$. Note that $1/(1-t)^N \in \mathrm{L}^1[0,1]$. This implies that ${}_2F_1(1, 1-\alpha, 1+\alpha, \cdot) \in \mathrm{AC}^n[0,1]$, as correctly stated in [13, sixth page], and when $\alpha > 1.5$, it is in $\mathcal{C}^n[0,1]$; see (4) in [13] and [18, Proposition 2.2]. On the other hand, if $g_L \in \mathcal{C}^n[0,T]$, or if $Lx = \sum_{j=1}^m w_j x(s_j)$ with $g_L(s) = \sum_{j=1}^m w_j H_{s_j}(s)$ in terms of Heaviside functions H_{s_j} and weights $w_j \in \mathbb{R}$, which are the typical cases, then it is clear that (13) and (14) in [13] are indeed in $\mathrm{AC}^n[0,T]$, for $\alpha > 1$. We include these points to better describe the proof of [13, Lemma 3.2], with an enhanced discussion on the hypergeometric function, and clarify types of common integrators g_L that ensure $x^* \in \Xi$.

5 Open problems

Some interesting questions are the following:

- Considering Remark 2.6, can we obtain closed-form lower bounds in $\mathrm{L}^p[a, b]$, for $p \neq 2$? This advance would be relevant for the Caputo and the Riemann–Liouville fractional derivatives.
- Can we adapt the results to other fractional operators? On the one hand, the condition $DIK = K$ limits the types of operators; for example, it is known that fractional operators with non-singular weight do not exhibit such a fundamental theorem of calculus. The normalized versions might be of use in such a case [12]. Recall that the property $DIK = K$ permits proving that inequality (2.11) is tight on a certain space, but it is not needed for the validity of (2.11). On the other hand, the explicit computation of $\|L_t[R(t, \cdot)]\|_2$ in (2.11) may not be possible, depending on the form of I.
- The selection of appropriate functionals L may permit bounding numerical errors, like in [10]. Do our results have applications in this sense, to improve classical formulae? Specifically, for interpolation and quadrature integration.
- Can our bounds be linked to previously known inequalities for the hypergeometric function?

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References

1. Baleanu, D., Fernandez, A.: On fractional operators and their classifications. Mathematics **7**(9), 830 (2019)
2. Brass, H., Förster, K.J.: On the application of the Peano representation of linear functionals in numerical analysis. In: Milovanovic, G.V. (ed.) Recent Progress in Inequalities, pp. 175–202. Kluwer, Amsterdam (1998)
3. Davis, P.J.: Interpolation and Approximation. Dover Publications, New York (1975)
4. Diethelm, K.: A fractional version of the Peano-Sard theorem. Numer. Funct. Anal. Optim. **18**, 745–757 (1997)
5. Diethelm, K.: Peano kernels of non-integer order. Z. Anal. Anwend. **16**(3), 727–738 (1997)
6. Diethelm, K.: Fractional error constants for quadrature formulas. In: Chui, C.K., Schumaker, L.L. (eds.) Approximation Theory IX, vol. 1, pp. 113–118. VandErbilt University Press, Nashville (1998)
7. Diethelm, K.: Estimation of quadrature errors in terms of Caputo-type fractional derivatives. Fract. Calc. Appl. Anal. **2**, 313–327 (1999)
8. Diethelm, K.: The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type. Lecture Notes in Mathematics, Springer, Berlin Heidelberg (2010)

9. Diethelm, K.: The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus. Fract. Calc. Appl. Anal. **15**(2), 304–313 (2012)
10. Fernandez, A., Buranay, S.C.: The Peano-Sard theorem for Caputo fractional derivatives and applications. J. Comput. Appl. Math. **441**, 115705 (2024)
11. Gavrea, I., Ivan, M.: An inequality for continuous linear functionals. Appl. Math. Lett. **23**(4), 381–384 (2010)
12. Jornet, M.: Theory on new fractional operators using normalization and probability tools. arXiv:2403.06198 (2024)
13. Jornet, M.: A new lower bound for the L^2-norm of the Caputo fractional derivative. Arch. Math. (2024). https://doi.org/10.1007/s00013-024-02033-6
14. Olver, F.W.J., Olde Daalhuis, A.B., Lozier D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V., Cohl, H.S., McClain, M.A.: NIST Digital Library of Mathematical Functions. https://dlmf.nist.gov/ (2024). Release 1.2.1 of 2024-06-15
15. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Yverdon (1993)
16. Sard, A.: Linear Approximation. No. 9 in Mathematical Surveys and Monographs, American Mathematical Society, Providence (1982)
17. Teodoro, G.S., Machado, J.T., De Oliveira, E.C.: A review of definitions of fractional derivatives and other operators. J. Comput. Phys. **388**, 195–208 (2019)
18. Webb, J.R.L.: Initial value problems for Caputo fractional equations with singular nonlinearities. Electron. J. Differ. Equ. **2019**(117), 1–32 (2019)