# CONFORMALLY EINSTEIN LORENTZIAN LIE GROUPS: EXTENSIONS OF THE EUCLIDEAN AND POINCARÉ GROUPS

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Abstract. We describe all Lorentzian semi-direct extensions of the Euclidean and Poincaré groups which are conformally Einstein.

## 1. Introduction

Einstein metrics, being critical for the Einstein-Hilbert functional, are central in Geometry and Physics. Four-dimensional Einstein Lorentzian metrics are not yet well-understood although much progress has been made under some additional conditions. Lie groups with left-invariant Lorentzian metrics play a distinguished role in the analysis, since the Einstein field equations reduce to a purely algebraic system which nevertheless may be a very involved one. Connected and simply connected four-dimensional Lie groups are isomorphic to the product Lie groups $SU(2)\times\mathbb{R}$, $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$, or to one of the solvable semi-direct extensions $G_3\rtimes\mathbb{R}$ of the three-dimensional unimodular Lie groups $\widetilde{E}(2)$, $E(1,1)$, $H_3$ and $\mathbb{R}^3$ corresponding to the simply connected Euclidean, Poincaré, Heisenberg and Abelian Lie groups, respectively (see, for example [3, 21], and [2] for a different approach).

### 1.1. **Einstein Lorentzian Lie groups.**

Four-dimensional Einstein Lorentzian Lie groups were described in [11]. They split into three categories: symmetric spaces, plane waves and left-invariant metrics which do not correspond to any of the above. Einstein Lorentzian symmetric spaces are of constant sectional curvature in the irreducible case [9]. In the non-irreducible case, either they are products $N_1(c)\times N_2(c)$ of two surfaces with the same constant sectional curvature or they correspond to some special cases of Cahen-Wallach symmetric spaces [9].

A *pp-wave* is a transversally flat spacetime $(M,g)$ admitting a null parallel vector field $\ell$ (i.e., the curvature endomorphism satisfies $R(\ell^\perp,\ell^\perp)=0$). In the special case that the covariant derivative of the curvature is also transversally flat (i.e., $\nabla_{\ell^\perp}R=0$) $(M,g)$ is referred to as a *plane wave*. The metric tensor of a *pp*-wave is locally described in adapted coordinates $(u,v,x^1,x^2)$ so that it takes the form

$$g = du\circ dv + H(v,x^1,x^2)dv\circ dv + dx^1\circ dx^1 + dx^2\circ dx^2\,,$$

and the parallel vector field is given by $\ell=\partial_u$. Furthermore plane waves correspond to the special case where the function $H(v,x^1,x^2)$ is a quadratic polynomial in the spacelike coordinates $\vec{x}=(x^1,x^2)$. Cahen-Wallach symmetric spaces are a special class of plane waves determined by a function $H(v,x^1,x^2)=\sum a_{ij}x^ix^j+b_kx^k$, where $a_{ij},b_k\in\mathbb{R}$. Four-dimensional *pp*-waves were discovered in a purely

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mathematical context by Brinkmann [7]. In Physics, plane waves and *pp*-waves appeared in General Relativity, where they play an important role (see [6, 17, 22]).

Homogeneous plane waves in dimension four are described in terms of a $2 \times 2$ skew-symmetric matrix $F$ and a $2 \times 2$ symmetric matrix $A_0$ so that the defining function $H(v, x^1, x^2)$ takes the form $H = \vec{x}^T A(v)\vec{x}$, where the matrix $A(v)$ is given by (see [5])

$$A(v) = e^{vF} A_0 e^{-vF}, \quad \text{or} \quad A(v) = \frac{1}{(v+b)^2} e^{\log(v+b)F} A_0 e^{-\log(v+b)F}.$$

Furthermore, homogeneous plane wave metrics are Einstein (and thus Ricci-flat) if and only if $A_0$ is trace-free.

The remaining situations are locally homothetic to the semi-direct extensions $\mathbb{R}^3 \rtimes \mathbb{R}$ determined by the following Lie algebra structures

$$\begin{array}{llll}
(i) & [e_1, e_4] = -2e_1, & [e_2, e_4] = e_2 + \sqrt{3}e_3, & [e_3, e_4] = -\sqrt{3}e_2 + e_3, \\
(ii) & [u_1, u_4] = -u_1 + \delta u_2, & [u_2, u_4] = 5u_2, & [u_3, u_4] = 2u_3, \ \delta \neq 0, \\
(iii) & [u_1, u_4] = 4u_1, & [u_2, u_4] = -2u_2 + \delta u_3, & [u_3, u_4] = \delta u_1 + u_3, \ \delta \neq 0,
\end{array}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis with $e_3$ timelike in case $(i)$, and $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$ in cases $(ii)$–$(iii)$.

Observe that the Lorentzian situation described above is in sharp contrast with the Riemannian one, where four-dimensional Einstein homogeneous Riemannian metrics are necessarily symmetric [19]. Lorentzian Einstein metrics on semi-direct extensions $\widetilde{E}(2) \rtimes \mathbb{R}$ or $E(1,1) \rtimes \mathbb{R}$ are also locally symmetric. Moreover they are flat or locally homothetic to the four-dimensional hyperbolic space $\mathbb{H}^4_1$ or to the product $\mathbb{H}^2_1 \times \mathbb{H}^2$, the last two cases occurring only on $E(1,1) \rtimes \mathbb{R}$ (see [28]).

### 1.2. Conformally Einstein metrics.

A pseudo-Riemannian manifold $(M^n, g)$ is *conformally Einstein* if there is an Einstein representative of the conformal class $[g]$. A conformally related metric $\overline{g} = \varphi^{-2} g$ is Einstein if and only if there exists a nowhere zero solution of the overdetermined PDE

$$(n-2)\operatorname{Hes}_\varphi + \varphi\rho = \tfrac{1}{n}\{(n-2)\Delta\varphi + \varphi\tau\}g, \tag{1}$$

where $\operatorname{Hes}_\varphi$ is the Hessian tensor of $\varphi$ and $\Delta\varphi = \operatorname{tr}_g \operatorname{Hes}_\varphi$ is the Laplacian of the conformal factor $\varphi$. It follows at once from Equation (1) that any two-dimensional pseudo-Riemannian manifold is conformally Einstein. Moreover, since any three-dimensional Einstein metric is of constant sectional curvature, one has that locally conformally Einstein metrics reduce to locally conformally flat ones in dimension three. Despite its apparent simplicity, the integrability of Equation (1) is surprisingly difficult and only few results are available in higher dimensions. In the physical context, Equation (1) is a generalization of Einstein's wave equation and it describes the propagation of gravitational waves in spacetime.

A basic observation to understand Equation (1) is that the quadratic curvature functional $g \mapsto \int_M \|W\|^2 \operatorname{dvol}_g$ given by the $L^2$-norm of the Weyl conformal curvature tensor is conformally invariant in dimension four. Hence so is its gradient, which is given by the *Bach tensor* [4]

$$\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \tfrac{1}{2} W[\rho], \tag{2}$$

where $W[\rho]$ denotes the contraction of the Weyl tensor and the Ricci tensor, $W[\rho]_{ij} = W_{iajb}\rho^{ab}$. Conformal gravity refers to gravity theories that are invariant under conformal transformations. The simplest theory in this category has the square of the norm of the Weyl tensor as the Lagrangian [25, 26]. Hence the equation of motion upon varying the metric is determined by $\mathfrak{B} = 0$. Locally conformally flat metrics are trivially Bach-flat, and so are Einstein metrics in dimension four since the Weyl tensor is trace-free. Due to the conformal invariance, being Bach-flat is a necessary condition in order to be conformally Einstein.

Another necessary condition to be conformally Einstein is derived from the fact that the Weyl tensor of any Einstein metric is divergence-free (equivalently, the Cotton tensor vanishes). Hence if a conformal metric $\overline{g} = e^{-2\sigma}g$ is Einstein, then $\overline{\operatorname{div}}\,\overline{W} = 0$, and a straightforward calculation now shows that $\operatorname{div}_4 W - W(\,\cdot\,,\,\cdot\,,\,\cdot\,,\nabla\sigma) = 0$. A pseudo-Riemannian manifold is *conformally Cotton-flat* if there is a conformal metric $\overline{g} = e^{-2\sigma}g$ which is Cotton-flat. Equivalently, there is a function $\sigma$ so that

$$\operatorname{div}_4 W - W(\,\cdot\,,\,\cdot\,,\,\cdot\,,\nabla\sigma) = 0. \tag{3}$$

More generally one says that $(M, g)$ is a *conformal $C$-space* if there is a (not necessarily gradient) vector field $X$ so that $\operatorname{div}_4 W - W(\,\cdot\,,\,\cdot\,,\,\cdot\,,X) = 0$ (see [18]).

The special significance of the conformally Cotton-flat property was given in [20], where it is shown that a weakly generic Bach-flat manifold is conformally Einstein if and only if it is conformally Cotton-flat, where *weakly genericity* means that the Weyl tensor, viewed as a map $W : TM \to \otimes^3 TM$, is injective. It must be remarked that there are many important spacetimes which are not weakly generic and thus the above criterion does not apply directly. This is, for instance, the case of *pp*-waves, where the Weyl curvature operator acting on the space of two-forms is two-step nilpotent. Four-dimensional *pp*-waves are conformally Einstein if and only if the Cotton tensor vanishes (equivalently, $\operatorname{div} W = 0$) as shown in [8]. Therefore one has that four-dimensional plane waves are conformally Einstein. We refer to [23] for a description of four-dimensional Bach-flat *pp*-waves.

### 1.3. **Semi-direct extensions of the Euclidean and Poincaré groups.**

A general program to classify conformally Einstein and Bach-flat Lorentzian Lie groups was proposed in [2] and initiated in [13], where semi-direct extensions of the Heisenberg group were considered. Our purpose in this paper is to determine all the conformally Einstein and Bach-flat left-invariant metrics on semi-direct extensions $\widetilde{E}(2) \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$ of the Euclidean and Poincaré groups. We therefore consider four-dimensional Lie groups $(G, \langle\cdot,\cdot\rangle)$ which have either a Euclidean or Poincaré symmetry, in the sense that the Euclidean group $\widetilde{E}(2)$ or the Poincaré group $E(1,1)$ are normal subgroups acting with cohomogeneity one on $(G, \langle\cdot,\cdot\rangle)$.

The three-dimensional Euclidean and Poincaré algebras, $\mathfrak{e}(2)$ and $\mathfrak{e}(1,1)$, are unimodular semi-direct extensions of the two-dimensional Abelian Lie algebra $\mathfrak{r}^2$ determined by an endomorphism $\varphi : \mathfrak{r}^2 \to \mathfrak{r}^2$. The endomorphism $\varphi$ has purely imaginary eigenvalues $\pm\sqrt{-1}$ in the Euclidean case and opposite eigenvalues $\pm 1$ in the Poincaré situation. Both cases may be treated simultaneously by describing the corresponding Lie algebras $\operatorname{span}\{e_1, e_2, e_3\}$ with Lie bracket

$$[e_3, e_1] = e_2, \quad [e_3, e_2] = -\varepsilon\, e_1, \qquad \varepsilon^2 = 1,$$

where $\varepsilon > 0$ in the Euclidean case and $\varepsilon < 0$ in the Poincaré Lie algebra.

The real affine Lie algebra $\mathfrak{aff}(\mathbb{R})$ is the non-Abelian two-dimensional Lie algebra determined by $[e_1, e_2] = e_2$. The complex affine Lie algebra $\mathfrak{aff}(\mathbb{C})$ is the four-dimensional Lie algebra determined by

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_2, e_3] = e_4, \quad [e_2, e_4] = -e_3.$$

Both algebras $\mathfrak{aff}(\mathbb{R})$ and $\mathfrak{aff}(\mathbb{C})$ are non-unimodular.

Semi-direct extensions of $\widetilde{E}(2)$ and $E(1,1)$ are isomorphic to the product Lie groups $\widetilde{E}(2) \times \mathbb{R}$ and $E(1,1) \times \mathbb{R}$ in the unimodular case, or they are isomorphic to the Lie groups $\mathit{Aff}(\mathbb{C})$ and $\mathit{Aff}(\mathbb{R}) \times \mathit{Aff}(\mathbb{R})$ in the non-unimodular case, respectively (see, for example, [3]).

1.4. **Summary of results.** The main contributions of this paper provide classification results for conformally Einstein and Bach-flat metrics on semi-direct Lorentzian extensions of the Euclidean and Poincaré groups. We structure they as follows.

1.4.1. *Non-trivial conformally Einstein extensions.* A conformally Einstein metric is said to be *non-trivial* if it is neither Einstein, nor locally conformally flat, nor a plane wave. The main result of this paper describes all non-trivial conformally Einstein semi-direct Lorentzian extensions of the Euclidean and the Poincaré groups as follows.

**Theorem 1.1.** *Let $(G_3 \rtimes \mathbb{R}, \langle \cdot, \cdot \rangle)$ be a semi-direct extension of the Euclidean group $G_3 = \widetilde{E}(2)$ or the Poincaré group $G_3 = E(1,1)$ equipped with a left-invariant Lorentzian metric. Then, the metric is non-trivial conformally Einstein if and only if it is isomorphically homothetic to the Lie group corresponding to one of the following Lorentzian Lie algebras:*

(R) *The restriction of the metric to $\mathfrak{g}_3$ is Riemannian and the metric is determined by*

(R.i) $$[e_1, e_3] = -\lambda_2 e_2, \quad [e_1, e_4] = \alpha e_1 + \lambda_2 e_2, \quad [e_2, e_3] = e_1,$$ $$[e_2, e_4] = -e_1 + \alpha e_2, \quad \alpha > 0, \ \lambda_2 \in [-1, 1) \setminus \{0\}, \quad \textit{or}$$

(R.ii) $$[e_1, e_3] = e_2, \quad [e_1, e_4] = [e_2, e_4] = \tfrac{1}{2}\sqrt{\tfrac{3}{2}}(e_1 + e_2),$$ $$[e_2, e_3] = e_1, \quad [e_3, e_4] = \tfrac{1}{\sqrt{2}}(e_1 + e_2),$$

*where $\{e_1, \dots, e_4\}$ is an orthonormal basis with $e_4$ timelike.*

(L) *The restriction of the metric to $\mathfrak{g}_3$ is Lorentzian and the metric is determined by*

(L.i) $$[e_1, e_3] = -\lambda_2 e_2, \quad [e_1, e_4] = \alpha e_1 + \lambda_2 e_2, \quad [e_2, e_3] = e_1,$$ $$[e_2, e_4] = -e_1 + \alpha e_2, \quad \alpha > 0, \ \lambda_2 \in [-1, 1) \setminus \{0\}, \quad \textit{or}$$

(L.ii) $$[u_1, u_2] = u_1, \quad [u_1, u_4] = \tfrac{1}{2} u_1, \quad [u_2, u_3] = u_3,$$ $$[u_2, u_4] = -\tfrac{5}{4} u_1, \quad [u_3, u_4] = u_3, \quad \textit{or}$$

(L.iii) $$[u_1, u_2] = u_1, \quad [u_1, u_4] = 2\gamma u_1, \quad [u_2, u_3] = u_3,$$ $$[u_2, u_4] = \alpha u_1 + \beta u_3, \quad [u_3, u_4] = \gamma u_3,$$

*with $\alpha, \beta \geq 0$, $\gamma \neq 0$ and either $\beta \neq 0$ or $\alpha\gamma \neq -2$,*

*where $\{e_1, \dots, e_4\}$ is an orthonormal basis of the Lie algebra with $e_3$ timelike and $\{u_1, \dots, u_4\}$ is a pseudo-orthonormal basis of the Lie algebra with $\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = \langle u_4, u_4 \rangle = 1$.*

*Moreover, in all cases the left-invariant metrics are conformally equivalent to a Ricci-flat metric.*

**Remark 1.2.** The Ricci operator of the left-invariant metrics in Theorem 1.1–(R.i) has four different real eigenvalues $\{\alpha(2\alpha\pm(\lambda_2-1)), \alpha^2\pm\alpha\sqrt{\alpha^2+(\lambda_2-1)^2}\}$, so that the scalar curvature $\tau = 6\alpha^2$. These metrics are realized as left-invariant metrics on the Lie groups *Aff*$(\mathbb{R}) \times$ *Aff*$(\mathbb{R})$ or *Aff*$(\mathbb{C})$, depending on whether $\lambda_2 < 0$ or $\lambda_2 > 0$, respectively. The metrics are not weakly generic since the Weyl curvature operator acting on the space of two-forms is two-step nilpotent. Moreover, the covariant derivative of the Ricci and the curvature tensors is isotropic, i.e., $\|\nabla\rho\|^2 = \|\nabla R\|^2 = 0$.

Metric in Theorem 1.1–(R.ii) corresponds to a left-invariant metric on the product Lie group *Aff*$(\mathbb{R}) \times$ *Aff*$(\mathbb{R})$. The Ricci operator is diagonalizable with eigenvalues $\{-\frac{3}{2}, 0, \frac{3}{2}, \frac{3}{2}\}$, so that the scalar curvature $\tau = \frac{3}{2}$. The Weyl curvature operator $W : \Lambda^2 \to \Lambda^2$ is three-step nilpotent and thus not weakly generic.

**Remark 1.3.** The Lie algebras in Theorem 1.1–(L.i) have $\operatorname{tr}\operatorname{ad}(e_4) = -2\alpha$. Hence they correspond to different left-invariant metrics on the Lie groups *Aff*$(\mathbb{R})\times$*Aff*$(\mathbb{R})$ or *Aff*$(\mathbb{C})$, depending on whether $\lambda_2 < 0$ or $\lambda_2 > 0$, respectively. The Ricci operator has four different real eigenvalues $\{-\alpha(2\alpha \pm (\lambda_2 - 1)), -\alpha^2 \pm \alpha\sqrt{\alpha^2 + (\lambda_2 - 1)^2}\}$, so that the scalar curvature is given by $\tau = -6\alpha^2$. Metrics are not weakly generic since the Weyl curvature operator acting on the space of two forms is two-step nilpotent. Moreover the covariant derivative of the Ricci and the curvature tensors is isotropic, i.e., $\|\nabla\rho\|^2 = \|\nabla R\|^2 = 0$.

The Lie algebra in Theorem 1.1–(L.ii) has $\operatorname{tr}\operatorname{ad}(e_4) = -\frac{3}{2}$ and it corresponds to a left-invariant metric on *Aff*$(\mathbb{R}) \times$ *Aff*$(\mathbb{R})$. The Ricci operator is diagonalizable with eigenvalues $\{-\frac{3}{2}, -\frac{9}{8}, -\frac{3}{8}, -\frac{3}{8}\}$, so that the scalar curvature $\tau = -\frac{27}{8}$. Moreover the Weyl curvature operator $W : \Lambda^2 \to \Lambda^2$ is also diagonalizable with eigenvalues $\{-\frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\}$ and thus weakly generic.

The Lie algebras in Theorem 1.1–(L.iii) have $\operatorname{tr}\operatorname{ad}(e_4) = -3\gamma$ and they correspond to different left-invariant metrics on *Aff*$(\mathbb{R}) \times$ *Aff*$(\mathbb{R})$. The Ricci operator has only one real eigenvalue $-3\gamma^2$ which is a double root of the minimal polynomial if $\beta = 0$ and a triple root otherwise, and the scalar curvature is $\tau = -12\gamma^2$. (The case $\beta = 0$ and $\alpha\gamma = -2$ corresponds to a metric of constant sectional curvature). Moreover the Weyl curvature operator $W : \Lambda^2 \to \Lambda^2$ is two-step nilpotent. Furthermore the covariant derivative of the Ricci and the curvature tensors is isotropic, i.e., $\|\nabla\rho\|^2 = \|\nabla R\|^2 = 0$. Finally note that it is possible to express the Lie algebras in Theorem 1.1–(L.iii) using only two parameters. To do this, let $\alpha'$ be a new variable defined by $\alpha' = \alpha\gamma$. Since $\gamma \neq 0$, we can take a new basis $\tilde{u}_1 = \frac{1}{\gamma^2}u_1$, $\tilde{u}_2 = u_2$, $\tilde{u}_3 = \frac{1}{\gamma}u_3$, and $\tilde{u}_4 = \frac{1}{\gamma}u_4$, so that the initial Lie bracket transforms into

$$\begin{aligned} &[\tilde{u}_1, \tilde{u}_2] = \tilde{u}_1, && [\tilde{u}_1, \tilde{u}_4] = 2\tilde{u}_1, \quad [\tilde{u}_2, \tilde{u}_3] = \tilde{u}_3, \\ &[\tilde{u}_2, \tilde{u}_4] = \alpha'\tilde{u}_1 + \beta\tilde{u}_3, && [\tilde{u}_3, \tilde{u}_4] = \tilde{u}_3, \end{aligned}$$

while the inner product is rescaled by $\frac{1}{\gamma^2}\langle\cdot,\cdot\rangle$. With this Lie bracket one has the constraints $\beta \geq 0$ and $(\alpha', \beta) \neq (-2, 0)$, while $\alpha' \in \mathbb{R}$. Moreover, the initial inner product remains invariant since one works at the homothetic level.

**Remark 1.4.** Considering the Lie algebras (R.i) and (L.i) in Theorem 1.1, and using the work of [14] one has additional (not necessarily isomorphic) homotheties inside each one of those two classes. Note that a Lie algebra in (R.i) is never homothetic to a Lie algebra in (L.i) since the scalar curvature change sign. Since the Ricci operator has four different real eigenvalues in both cases, they determine the possible homothetic classes [14]. Hence, a straightforward calculation shows that two classes in either (R.i) or (L.1) determined by the parameters $(\alpha,\lambda_2)$ and $(\bar\alpha,\bar\lambda_2)$ are homothetic if and only if $\bar\alpha(\bar\lambda_2-1)^{-1}=\alpha(\lambda_2-1)^{-1}$. As a consequence, each homothetic class has a single representative with $\alpha>0$ and a fixed value $\lambda_2\in[-1,1)\setminus\{0\}$.

Thus, any metric given by (R.i) and (L.i) in Theorem 1.1 is homothetic (although not isomorphically homothetic) to a metric in $\mathit{Aff}(\mathbb{R})\times\mathit{Aff}(\mathbb{R})$ or in $\mathit{Aff}(\mathbb{C})$.

1.4.2. *Strictly Bach-flat extensions.* We say that a Bach-flat metric is *strict* if it is not conformally Einstein (and thus not a plane wave). The strictly Bach-flat semi-direct extensions of the Euclidean and Poincaré groups are given as follows.

**Theorem 1.5.** *Let $(G_3\rtimes\mathbb{R},\langle\cdot,\cdot\rangle)$ be a semi-direct extension of the Euclidean group $G_3=\widetilde{E}(2)$ or the Poincaré group $G_3=E(1,1)$ equipped with a left-invariant Lorentzian metric. Then, the metric is strictly Bach-flat if and only if it is isomorphically homothetic to the Lie group corresponding to one of the following Lorentzian Lie algebras:*

$$\text{(i)}\quad [e_1,e_2]=-e_2,\qquad [e_1,e_3]=e_3,\qquad [e_1,e_4]=\tfrac12\sqrt{\tfrac52}\,e_3,$$
$$[e_2,e_4]=\sqrt{\tfrac32}\,e_2,\quad [e_3,e_4]=\tfrac32\sqrt{\tfrac32}\,e_3,\quad \textit{or}$$

$$\text{(ii)}\quad [u_1,u_2]=\tfrac12u_3,\qquad [u_1,u_3]=-u_2,\quad [u_1,u_4]=\alpha u_2-\tfrac{\alpha^2+1}{4\alpha}u_3,$$
$$[u_2,u_4]=-\tfrac{\alpha}{2}u_3,\quad [u_3,u_4]=\alpha u_2,\quad \alpha>0,$$

*where $\{e_1,\dots,e_4\}$ is an orthonormal basis of the Lie algebra with $e_3$ timelike and $\{u_1,\dots,u_4\}$ is a pseudo-orthonormal basis of the Lie algebra with $\langle u_1,u_2\rangle=\langle u_3,u_3\rangle=\langle u_4,u_4\rangle=1$.*

**Remark 1.6.** The Lie algebra in Theorem 1.5–(i) is not unimodular. Moreover, the underlying Lie group is $\mathit{Aff}(\mathbb{R})\times\mathit{Aff}(\mathbb{R})$. Besides, the Ricci operator is diagonalizable with eigenvalues $\{-\frac{75}{16},-\frac{75}{16},-\frac{60}{16},-\frac{45}{16}\}$ and the scalar curvature $\tau=-\frac{255}{16}$. The Weyl curvature operator acting on the space of two-forms has eigenvalues $\frac{15}{16}$ and $-\frac{15}{8}$, the former with multiplicity four being a double root of the minimal polynomial. In particular this metric is weakly generic.

The Lie algebras in Theorem 1.5–(ii) are unimodular and the underlying Lie group is the product Lie group $\widetilde{E}(2)\times\mathbb{R}$. The Ricci operator corresponding to these left-invariant metrics has eigenvalues $\{0,\frac38\alpha^2,\pm\frac{\sqrt3}{8}\alpha^2\sqrt{-1}\}$ and the scalar curvature $\tau=\frac38\alpha^2$. The Weyl curvature operator $W:\Lambda^2\to\Lambda^2$ is three-step nilpotent.

**Remark 1.7.** It follows from the analysis in Section 2 and Section 3 that Bach-flat *pp*-waves which are neither Ricci-flat nor locally conformally flat appear as the limiting cases in Theorem 1.1 corresponding to (R.i) and (L.i) for $\alpha=0$,

$$[e_1,e_3]=-\lambda_2\,e_2,\quad [e_1,e_4]=\lambda_2\,e_2,\quad [e_2,e_3]=e_1,\quad [e_2,e_4]=-e_1,$$

where $\lambda_2\in(-1,1)\setminus\{0\}$ and $\{e_i\}$ is an orthonormal basis with either $e_3$ or $e_4$ timelike, or they correspond to the limiting case in (L.iii) for $\gamma=0$

$$[u_1,u_2]=u_1,\quad [u_2,u_3]=u_3,\quad [u_2,u_4]=\alpha u_1+\beta u_3,\quad \alpha\ge0,\ \beta\ge0, \tag{4}$$

where $\{u_i\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2\rangle = 1$. In the latter case, if $\alpha \neq 0$, the basis $\tilde{u}_1 = \alpha^2 u_1$, $\tilde{u}_2 = u_2$, $\tilde{u}_3 = \alpha u_3$, $\tilde{u}_4 = \alpha u_4$, transforms the initial Lie bracket into $[\tilde{u}_1, \tilde{u}_2] = \tilde{u}_1$, $[\tilde{u}_2, \tilde{u}_3] = \tilde{u}_3$, $[\tilde{u}_2, \tilde{u}_4] = \tilde{u}_1 + \beta\tilde{u}_3$, while the inner product is rescaled by $\alpha^2\langle\cdot,\cdot\rangle$. Hence, remaining in the same homothetic class, one may restrict the parameter $\alpha$ in Equation (4) to $\alpha \in \{0, 1\}$.

In all the situations above the underlying structure is a plane wave on the product Lie groups $E(1,1)\times\mathbb{R}$ or $\widetilde{E}(2)\times\mathbb{R}$. They correspond to the non-symmetric Bach-flat metrics where $\|\nabla\rho\|^2 = \|\nabla R\|^2 = 0$.

**Remark 1.8.** It follows from Theorem 1.1 and Theorem 1.5 that any left-invariant Bach-flat metric which is neither Einstein nor locally conformally flat on the product Lie group $E(1,1) \times \mathbb{R}$ is necessarily a plane wave as in Remark 1.7. The Bach-flat metrics on the product $\widetilde{E}(2) \times \mathbb{R}$ which are neither Einstein nor locally conformally flat are plane waves as in Remark 1.7 or the metrics in Theorem 1.5–(ii).

1.5. **Structure of the paper.** For a given three-dimensional Lie group $G_3 = \widetilde{E}(2)$ or $G_3 = E(1,1)$, Lorentzian left-invariant metrics on semi-direct extensions $G_3 \rtimes \mathbb{R}$ may induce left-invariant Riemannian, Lorentzian or degenerate metrics on $G_3$. Hence we consider all these possibilities separately. In Section §2 we analyze Lorentzian metrics inducing a Riemannian metric on $G_3$. We consider the Bach tensor and determine all Bach-flat left-invariant metrics in this case. Moreover, for these Bach-flat metrics we examine the existence of Einstein metrics in the conformal class, obtaining a positive answer in all the cases. The non-trivial ones are collected as cases (R.i) and (R.ii) in Theorem 1.1. Lorentzian metrics on $G_3 \rtimes \mathbb{R}$ whose restriction to $G_3$ is Lorentzian are a richer class which supports strictly Bach-flat examples (cf. Theorem 1.5). Those which are non-trivially conformally Einstein are collected in Theorem 1.1–(L). Plane wave and *pp*-wave left-invariant metrics also appear in both scenarios, as pointed out in Remark 1.7. The case of left-invariant metrics whose restriction to $G_3$ is degenerate reduces to the previously investigated cases, as pointed out in [10].

Finally we recall that the Bach-flatness condition $\mathfrak{B} = 0$ equals to a system of polynomial equations on the structure constants (given by the components $\mathfrak{B}_{ij}$ of the Bach tensor) which one has to solve in order to obtain a complete classification. Given a set $\mathcal{S}$ of polynomials $\mathfrak{B}_{ij} \in \mathbb{R}[x_1, \dots, x_n]$, an $n$-tuple of real numbers $\vec{a} = (a_1, \dots, a_n)$ is a solution of $\mathcal{S}$ if and only if $\mathfrak{B}_{ij}(\vec{a}) = 0$ for all $i$, $j$. It is a fundamental observation to recognize that $\vec{a}$ is a solution of $\mathcal{S}$ if and only if it is a solution of $\mathcal{I} = \langle\mathfrak{B}_{ij}\rangle$, the ideal generated by the $\mathfrak{B}_{ij}$'s: if two sets of polynomials generate the same ideal, the corresponding zero sets must be identical. The theory of Gröbner bases provides a well-known strategy to solve rather large polynomial systems obtaining "better" polynomials that belong to the ideal generated by the initial polynomial system (see [15] for more information on Gröbner bases). The calculations of Gröbner basis have been done using SINGULAR [16], and doubly checked with MATHEMATICA. In all cases we specify the ordering used to compute such basis so that the results can be reproduced.

## 2. SEMI-DIRECT EXTENSIONS WITH RIEMANNIAN LIE GROUPS $\widetilde{E}(2)$ OR $E(1,1)$

Let $\mathfrak{g}_3$ be a three-dimensional unimodular Lie algebra with a positive definite inner product $\langle\cdot,\cdot\rangle$. Let $L$ denote the structure operator defined by Milnor in [27] so that there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}_3$ diagonalizing $L$.

Let $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ be a semi-direct extension and consider the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ with $e_4$ timelike so that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_i, e_4] = \sum_{j=1}^{3} \alpha_i^j e_j,$$

with $i = 1, 2, 3$ and $\alpha_i^j \in \mathbb{R}$.

The structure operator of the Poincaré and Euclidean Lie algebras has one-dimensional kernel, so one may assume without loss of generality that $\lambda_3 = 0$ and $\lambda_1 \lambda_2 \neq 0$. Then, using the Jacobi identity, one has that the corresponding structure of the Lie algebra is given by

$$\begin{aligned} &[e_1, e_3] = -\lambda_2 e_2, && [e_1, e_4] = \gamma_1 e_1 + \gamma_2 \lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \\ &[e_2, e_4] = -\gamma_2 \lambda_1 e_1 + \gamma_1 e_2, && [e_3, e_4] = \gamma_3 e_1 + \gamma_4 e_2. \end{aligned}$$

Since $\operatorname{tr} \operatorname{ad}(e_4) = -2\gamma_1$ and $\operatorname{tr} \operatorname{ad}(e_i) = 0$ for $i \neq 4$, one has that the Lie group $G_3 \rtimes \mathbb{R}$ is unimodular if and only if $\gamma_1 = 0$, in which case it is isomorphic to the product Lie group $G_3 \times \mathbb{R}$. Observe that, even in this case, the left-invariant metric is not necessarily the product one.

In what follows we determine all the conformally Einstein and Bach-flat left-invariant metrics on $\widetilde{E}(2) \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$. Since $\lambda_1 \neq 0$, we can assume $\lambda_1 = 1$ and work in the homothetic class of the initial metric, just taking the orthogonal basis $\hat{e}_i = \frac{1}{\lambda_1} e_i$. Moreover, to streamline all calculations, we will separate our analysis into two cases depending on whether $\gamma_4 = 0$ or $\gamma_4 \neq 0$.

### 2.1. Case $\boldsymbol{\gamma_4 = 0}$.

We consider separately the two possibilities corresponding to $\gamma_3 = 0$ and $\gamma_3 \neq 0$.

#### 2.1.1. *Case* $\gamma_4 = \gamma_3 = 0$.

In this case, analyzing the components of the Bach tensor $\mathfrak{B}_{ij} = \mathfrak{B}(e_i, e_j)$, a straightforward calculation shows that the possible non-zero components are determined by

$$\begin{aligned} -6\mathfrak{B}_{11} &= (\gamma_2^2 - 1)(\lambda_2 - 1)\left(\gamma_1^2(5\lambda_2 + 7) - (\gamma_2^2 - 1)(3\lambda_2^3 + 2\lambda_2^2 + 2\lambda_2 + 5)\right), \\ 3\mathfrak{B}_{12} &= 4\gamma_1\gamma_2(\gamma_2^2 - 1)(\lambda_2^3 - 1), \\ 6\mathfrak{B}_{22} &= (\gamma_2^2 - 1)(\lambda_2 - 1)\left(\gamma_1^2(7\lambda_2 + 5) - (\gamma_2^2 - 1)(5\lambda_2^3 + 2\lambda_2^2 + 2\lambda_2 + 3)\right), \\ -6\mathfrak{B}_{33} &= (\gamma_2^2 - 1)(\lambda_2 - 1)^2\left(3\gamma_1^2 + (\gamma_2^2 + 3)(\lambda_2^2 + \lambda_2 + 1)\right), \\ 3\mathfrak{B}_{34} &= 2\gamma_2(\gamma_2^2 - 1)(\lambda_2 - 1)^2(\lambda_2^2 + \lambda_2 + 1), \\ -6\mathfrak{B}_{44} &= (\gamma_2^2 - 1)(\lambda_2 - 1)^2\left(\gamma_1^2 + (3\gamma_2^2 + 1)(\lambda_2^2 + \lambda_2 + 1)\right). \end{aligned}$$

The vanishing of the component $\mathfrak{B}_{44}$ implies that either $\lambda_2 = 1$ or $\gamma_2 = \pm 1$, and clearly the Bach tensor vanishes identically in any of those two cases. Moreover, if $\lambda_2 = 1$ then the corresponding left-invariant metric is locally conformally flat and locally symmetric. Also, it is locally isometric to a product manifold $N(c) \times \mathbb{R}$, where $N(c)$ is a Lorentzian three-manifold of constant sectional curvature $c = \gamma_1^2$.

From now on we take $\gamma_2 = \varepsilon$, with $\varepsilon = \pm 1$, and suppose that $\lambda_2 \neq 1$, so the corresponding left-invariant metrics are determined by

$$\begin{aligned} &[e_1, e_3] = -\lambda_2 e_2, && [e_1, e_4] = \gamma_1 e_1 + \varepsilon \lambda_2 e_2, \\ &[e_2, e_3] = e_1, && [e_2, e_4] = -\varepsilon e_1 + \gamma_1 e_2. \end{aligned} \tag{5}$$

Note that $(e_1, e_2, e_3, e_4) \mapsto (-e_1, e_2, -e_3, e_4)$ defines and isometry interchanging $\varepsilon$ with $-\varepsilon$, while $(e_1, e_2, e_3, e_4) \mapsto (-e_1, e_2, -e_3, -e_4)$ interchanges $\gamma_1$ with $-\gamma_1$. Besides, the homothety $(e_1, e_2, e_3, e_4) \mapsto \frac{1}{\lambda_2}(e_2, -e_1, e_3, e_4)$ interchanges $(\gamma_1, \lambda_2, \varepsilon)$ with $(\frac{\gamma_1}{\lambda_2}, \frac{1}{\lambda_2}, \varepsilon)$. Hence, without loss of generality, we can take $\varepsilon = 1$ and assume $\gamma_1 \geq 0$ and $\lambda_2 \in [-1, 1) \setminus \{0\}$. Moreover, a direct calculation shows that the above metrics are never Einstein, and they are locally conformally flat if and only if $\gamma_1 = 0$ and $\lambda_2 = -1$. Besides, the eigenvalues of the Ricci operator are given by $\{\gamma_1(2\gamma_1 \pm (\lambda_2 - 1)), \gamma_1^2 \pm \gamma_1\sqrt{\gamma_1^2 + (\lambda_2 - 1)^2}\}$.

Let $\{E_1, E_2, E_3, E_4\}$ be the left-invariant orthonormal frame on the Lie groups $\widetilde{E}(2) \rtimes \mathbb{R}$ or $E(1, 1) \rtimes \mathbb{R}$, depending on whether $\lambda_2 > 0$ or $\lambda_2 < 0$, obtained by left-translating the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of the corresponding Lie algebra. If $\gamma_1 = 0$ the Ricci operator is two-step nilpotent and $\ell = E_3 + E_4$ is a null parallel vector field. Besides, one has that $\ell^\perp = \operatorname{span}\{E_1, E_2, \ell\}$ satisfies $R(\ell^\perp, \ell^\perp) = 0$ and $\nabla_{\ell^\perp} R = 0$. Thus, the underlying structure is a plane wave (see Remark 1.7).

If $\gamma_1 \neq 0$ the four eigenvalues of the Ricci operator are distinct and we show below that the metrics in Equation (5) are conformally Einstein, thus corresponding to Theorem 1.1–(R.i). Recall that a metric is conformally Einstein if and only if there exists a locally defined nowhere zero function $\varphi$ such that $\overline{g} = \varphi^{-2} g$ satisfies Equation (1). Moreover, setting $\varphi = e^\sigma$ one has that the gradient of the function $\sigma$ satisfies the conformal Cotton-flat equation (3) since $\overline{g} = e^{-2\sigma} g$ is Cotton-flat. Considering the gradient $\xi = \nabla\sigma$, a direct calculation shows that $\nabla\varphi = \varphi\xi$ and

$$\operatorname{Hes}_\varphi(X, Y) = \varphi\{\langle X, \xi\rangle\langle Y, \xi\rangle + \langle\nabla_X \xi, Y\rangle\}.$$

To present a proper analysis of Equation (1) on a Lorentzian Lie group $(G, \langle\cdot,\cdot\rangle)$, we consider the symmetric $(0, 2)$-tensor field

$$\begin{aligned}\mathfrak{CE}(X, Y) &= 2\operatorname{Hes}_\varphi(X, Y) + \varphi\rho(X, Y) - \tfrac{1}{4}\{2\Delta\varphi + \varphi\tau\}\langle X, Y\rangle \\ &= 2\varphi\{\langle X, \xi\rangle\langle Y, \xi\rangle + \langle\nabla_X \xi, Y\rangle\} + \varphi\rho(X, Y) - \tfrac{1}{4}\{2\Delta\varphi + \varphi\tau\}\langle X, Y\rangle\end{aligned}$$

and evaluate it on the basis of left-invariant vector fields obtained from the corresponding basis of the Lie algebra.

Returning to the metrics given by Equation (5) and using the left-invariant orthonormal frame $\{E_1, E_2, E_3, E_4\}$ introduced above, a direct calculation shows that those left-invariant metrics are conformally Cotton-flat. Indeed, all vector fields satisfying the conformal $C$-space condition are given by $\xi = \phi E_3 + (\phi - \gamma_1)E_4$, where $\phi$ is a smooth function on the Lie group. Moreover, the vector field $\xi$ is a gradient, $\xi = \nabla\sigma$, if the smooth function $\phi$ satisfies $d\phi(E_1) = d\phi(E_2) = 0$ and $d\phi(E_3) + d\phi(E_4) = 0$.

Setting $\varphi = e^\sigma$ one has that the Hessian $\operatorname{Hes}_\varphi$, when expressed in the basis of left-invariant vector fields $\{E_i\}$, takes the form

$$\operatorname{Hes}_\varphi = \varphi \begin{pmatrix} \gamma_1(\phi - \gamma_1) & \frac{1}{2}\gamma_1(1 - \lambda_2) & 0 & 0 \\ \frac{1}{2}\gamma_1(1 - \lambda_2) & \gamma_1(\phi - \gamma_1) & 0 & 0 \\ 0 & 0 & \phi^2 + d\phi(E_3) & \phi(\gamma_1 - \phi) - d\phi(E_3) \\ 0 & 0 & \phi(\gamma_1 - \phi) - d\phi(E_3) & (\gamma_1 - \phi)^2 + d\phi(E_3) \end{pmatrix}$$

and thus the Laplacian $\Delta\varphi = -\gamma_1(3\gamma_1 - 4\phi)\varphi$. Moreover, the Ricci tensor in the same basis takes the form

$$\rho = \begin{pmatrix} 2\gamma_1^2 & \gamma_1(\lambda_2-1) & 0 & 0 \\ \gamma_1(\lambda_2-1) & 2\gamma_1^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\lambda_2-1)^2 & \frac{1}{2}(\lambda_2-1)^2 \\ 0 & 0 & \frac{1}{2}(\lambda_2-1)^2 & -2\gamma_1^2 - \frac{1}{2}(\lambda_2-1)^2 \end{pmatrix},$$

and the scalar curvature $\tau = 6\gamma_1^2$. Hence, the conformally Einstein equation when expressed on the basis of left-invariant vector fields reduces to

$$\mathfrak{CE} = 2\varphi\left(d\phi(E_3) + \phi(\phi-\gamma_1) - \tfrac{1}{4}(\lambda_2-1)^2\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

which shows that the conformal metric determined by the gradient vector field $\xi = \phi E_3 + (\phi - \gamma_1)E_4$, given by a solution of the equations

$$d\phi(E_1) = d\phi(E_2) = 0, \quad d\phi(E_4) = -d\phi(E_3), \quad d\phi(E_3) = -\phi(\phi-\gamma_1) + \tfrac{1}{4}(\lambda_2-1)^2,$$

is Einstein. Thus, metrics in Equation (5) are conformally Einstein and a straightforward calculation shows that they are conformally Ricci-flat.

2.1.2. *Case* $\gamma_4 = 0, \gamma_3 \neq 0$. In the polynomial ring $\mathbb{R}[\gamma_2, \gamma_3, \gamma_4, \lambda_2, \gamma_1]$ with the lexicographical order we consider the ideal $\langle \mathfrak{B}_{ij} \cup \{\gamma_4\}\rangle$, where $\mathfrak{B}_{ij} = \mathfrak{B}(e_i, e_j)$ are the polynomials given by the components of the Bach tensor in the orthonormal basis $\{e_i\}$. Computing a Gröbner basis for that ideal we get 19 polynomials that include

$$\begin{aligned} \mathbf{g}_1 &= \gamma_1^4\gamma_3(\gamma_1^2+1)^2(7\gamma_1^2+15)(9\gamma_1^2+25)(25\gamma_1^2+1), \\ \mathbf{g}_2 &= \gamma_3^2(19712\gamma_3^2\lambda_2^3 + 15975\gamma_1^8 + 83909\gamma_1^6 + 141589\gamma_1^4 + 73655\gamma_1^2). \end{aligned}$$

Since $\gamma_3\lambda_2 \neq 0$ we conclude that the Bach tensor cannot vanish in this case.

2.2. **Case $\boldsymbol{\gamma_4 \neq 0}$.** In this case we introduce a new variable $\tilde{\gamma}_4$ and take the polynomial $\gamma_4\tilde{\gamma}_4 - 1$ to express that the structure constant $\gamma_4$ is non-zero. In the polynomial ring $\mathbb{R}[\gamma_2, \lambda_2, \gamma_3, \tilde{\gamma}_4, \gamma_4, \gamma_1]$ we fix the lexicographical order and consider the ideal $\mathcal{I}_1$ generated by $\mathfrak{B}_{ij} \cup \{\gamma_4\tilde{\gamma}_4 - 1\}$. Computing a Gröbner basis for $\mathcal{I}_1$ we obtain 12 polynomials, being one of them

$$\mathbf{g}_{11} = (8\gamma_1^2-3)(\gamma_1^2+1)(5\gamma_1^2+21)(9\gamma_1^2+1)(32\gamma_1^2+75)\left(144\gamma_1^2 + (20\gamma_1^2-9)^2\right)(\gamma_3^2+\gamma_4^2).$$

Since $\gamma_4 \neq 0$, it follows that necessarily $8\gamma_1^2 - 3 = 0$. Now, computing a second Gröbner basis for the ideal $\mathcal{I}_2 = \langle \mathcal{I}_1 \cup \{8\gamma_1^2 - 3\}\rangle$ we get 10 polynomials, among which we find

$$\begin{aligned} \mathbf{g}_{21} &= (\lambda_2+1)(\gamma_3^2+\gamma_4^2), & \mathbf{g}_{23} &= (\gamma_3^2-\gamma_4^2)(\gamma_3^2+\gamma_4^2), \\ \mathbf{g}_{22} &= (2\gamma_4^2-1)(\gamma_3^2+\gamma_4^2), & \mathbf{g}_{24} &= (\gamma_2+2\gamma_1\gamma_3\gamma_4)(\gamma_3^2+\gamma_4^2). \end{aligned}$$

Thus, we conclude that $\lambda_2 = -1$, $\gamma_1^2 = \frac{3}{8}$, $\gamma_3^2 = \gamma_4^2 = \frac{1}{2}$ and $\gamma_2 = -2\gamma_1\gamma_3\gamma_4$. Hence, $\gamma_1 = \frac{\varepsilon_1}{2}\sqrt{\frac{3}{2}}$, $\gamma_3 = \frac{\varepsilon_2}{\sqrt{2}}$, $\gamma_4 = \frac{\varepsilon_3}{\sqrt{2}}$ and $\gamma_2 = -\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{2}\sqrt{\frac{3}{2}}$, where $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 1$.

Now, an explicit calculation of the Bach tensor shows that it vanishes, and the corresponding left-invariant metrics are determined by

$$\begin{aligned}
&[e_1,e_3]=e_2, \quad [e_1,e_4]=\tfrac{\varepsilon_1}{2}\sqrt{\tfrac{3}{2}}(e_1+\varepsilon_2\varepsilon_3e_2),\\
&[e_2,e_3]=e_1, \quad [e_2,e_4]=\tfrac{\varepsilon_1}{2}\sqrt{\tfrac{3}{2}}(\varepsilon_2\varepsilon_3e_1+e_2), \quad [e_3,e_4]=\tfrac{1}{\sqrt{2}}(\varepsilon_2e_1+\varepsilon_3e_2).
\end{aligned} \tag{6}$$

Note that $(e_1,e_2,e_3,e_4)\mapsto(-e_1,e_2,-e_3,e_4)$ defines an isometry which interchanges $\varepsilon_3$ with $-\varepsilon_3$, while $(e_1,e_2,e_3,e_4)\mapsto(-e_1,-e_2,e_3,e_4)$ interchanges $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ with $(\varepsilon_1,-\varepsilon_2,-\varepsilon_3)$. Analogously, the isometry $(e_1,e_2,e_3,e_4)\mapsto(e_1,e_2,e_3,-e_4)$ interchanges $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ with $(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3)$. Hence, without loss of generality, we can assume $\varepsilon_1=\varepsilon_2=\varepsilon_3=1$, and the underlying Lie group is $\mathit{Aff}(\mathbb{R})\times\mathit{Aff}(\mathbb{R})$. Moreover, a direct calculation shows that the above metrics are neither locally conformally flat nor Einstein, and the Ricci operator has eigenvalues $\{-\frac{3}{2},0,\frac{3}{2},\frac{3}{2}\}$. Besides, proceeding as in §2.1.1, we show below that the metric in Equation (6) with $\varepsilon_1=\varepsilon_2=\varepsilon_3=1$ is conformally Einstein, hence corresponding to Theorem 1.1–(R.ii).

Let $\{E_i\}$ be the global orthonormal frame on $\mathit{Aff}(\mathbb{R})\times\mathit{Aff}(\mathbb{R})$ obtained by left-translating the orthonormal basis $\{e_i\}$ of the Lie algebra above. A direct calculation shows that the left-invariant metric is conformally Cotton-flat, as the vector field $\xi=\frac{1}{2}E_3-\sqrt{\frac{3}{2}}E_4$ solves the equation $\operatorname{div}_4 W-W(\cdot,\cdot,\cdot,\xi)=0$. Moreover, the vector field $\xi$ is a gradient, $\xi=\nabla\sigma$. Setting $\varphi=e^\sigma$ one has that the Hessian $\operatorname{Hes}_\varphi$, when expressed in the basis of left-invariant vector fields $\{E_i\}$, takes the form

$$\operatorname{Hes}_\varphi=-\tfrac{1}{4}\varphi\begin{pmatrix} 3 & 1 & \sqrt{3} & \frac{\sqrt{2}}{2}\\ 1 & 3 & \sqrt{3} & \frac{\sqrt{2}}{2}\\ \sqrt{3} & \sqrt{3} & -1 & -\sqrt{6}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\sqrt{6} & -6 \end{pmatrix},$$

from where it follows that $\Delta\varphi=-\frac{11}{4}\varphi$. The Ricci tensor, expressed in the same orthonormal basis $\{E_i\}$, takes the form

$$\rho=\tfrac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{3} & \frac{\sqrt{2}}{2}\\ 1 & 1 & \sqrt{3} & \frac{\sqrt{2}}{2}\\ \sqrt{3} & \sqrt{3} & -3 & -\sqrt{6}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\sqrt{6} & -4 \end{pmatrix}$$

and the scalar curvature $\tau=\frac{3}{2}$. A straightforward calculation now shows that all components of the conformally Einstein tensor $\mathfrak{CE}(X,Y)=2\operatorname{Hes}_\varphi(X,Y)+\varphi\rho(X,Y)-\frac{1}{4}\{2\Delta\varphi+\varphi\tau\}\langle X,Y\rangle$ vanish identically. Therefore, metric in Equation (6) is conformally Einstein and corresponds to Theorem 1.1–(R.ii). Moreover, a direct calculation shows that it is conformally Ricci-flat.

## 3. Semi-direct extensions with Lorentzian Lie groups $\widetilde{E}(2)$ or $E(1,1)$

We proceed as in Section 2 and consider the structure operator $L(X\times Y)=[X,Y]$ where the vector-cross product $\langle X\times Y,Z\rangle=\det(X,Y,Z)$ is now defined with respect to a Lorentzian inner product on the three-dimensional unimodular Lie algebra $\mathfrak{e}(2)$ or $\mathfrak{e}(1,1)$. Even though $L$ is self-adjoint, it is not necessarily diagonalizable due to the indefiniteness of the inner product. Instead, $L$ may have non-trivial Jordan normal form and, in the general setting, one of the following holds (see [29]):

Ia. *$L$ is diagonalizable.* Hence there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where $e_3$ is assumed to be timelike, such that $L(e_i) = \lambda_i e_i$.

Ib. *$L$ has complex eigenvalues.* Thus there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where $e_3$ is assumed to be timelike, such that

$$L = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}, \quad \beta \neq 0.$$

II. *$L$ has a double root of its minimal polynomial.* Then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ such that

$$L = \begin{pmatrix} \lambda_1 & 0 & 0 \\ \varepsilon & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad \varepsilon = \pm 1, \quad \text{where} \quad \langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = 1.$$

III. *$L$ has a triple root of its minimal polynomial.* Then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ such that

$$L = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \quad \text{where} \quad \langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = 1.$$

In what follows we analyze the Bach-flat condition depending on the structure operator $L$ of the unimodular subalgebra. Since the structure operator of $\mathfrak{e}(2)$ or $\mathfrak{e}(1,1)$ has one-dimensional kernel, one must consider the possible causality of $\ker L$.

3.1. **Diagonalizable structure operator $L$ with spacelike kernel.** Without loss of generality, we assume $\lambda_1 = 0$ and $\lambda_2\lambda_3 \neq 0$. Hence, the left-invariant metrics are determined by

$$\begin{aligned} &[e_1, e_2] = -\lambda_3 e_3, && [e_1, e_3] = -\lambda_2 e_2, && [e_1, e_4] = \gamma_1 e_2 + \gamma_2 e_3, \\ &[e_2, e_4] = \gamma_3 e_2 + \gamma_4 \lambda_3 e_3, && [e_3, e_4] = \gamma_4 \lambda_2 e_2 + \gamma_3 e_3, && \end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis with $e_3$ timelike. Since $\lambda_2 \neq 0$, we can take the orthogonal basis $\hat{e}_i = \frac{1}{\lambda_2} e_i$ and work in the homothetic class of the initial metric, so we assume $\lambda_2 = 1$. Moreover, we introduce an additional variable $\tilde{\lambda}_3$ to express that the structure constant $\lambda_3$ is non-zero by means of the polynomial $\lambda_3\tilde{\lambda}_3 - 1$. Let $\mathcal{I}_1$ be the ideal generated by $\mathfrak{B}_{ij} \cup \{\lambda_3\tilde{\lambda}_3 - 1\}$ in the polynomial ring $\mathbb{R}[\tilde{\lambda}_3, \lambda_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4]$, where $\mathfrak{B}_{ij} = \mathfrak{B}(e_i, e_j)$ are the polynomials given by the components of the Bach tensor in the orthonormal basis $\{e_i\}$. Computing a Gröbner basis for $\mathcal{I}_1$ with respect to the lexicographical order we get 93 polynomials, being one of them

$$\begin{aligned} \mathbf{g}_{11} = {} & \gamma_2 \left(\gamma_4^2 + 1\right) \left((\gamma_3 + \gamma_4)^2 + 1\right) \left((\gamma_3 - \gamma_4)^2 + 1\right) \\ & \times \big(1572649984\gamma_4^8 + 4284719328\gamma_4^6 + 25396875\gamma_3^4 + 3670829099\gamma_4^4 \\ & \quad + 98562450\gamma_3^2\gamma_4^2 + 109488750\gamma_3^2 + 970611630\gamma_4^2 + 11851875\big). \end{aligned}$$

Hence $\gamma_2 = 0$. Repeating the process for the ideal $\mathcal{I}_2 = \langle \mathcal{I}_1 \cup \{\gamma_2\} \rangle$ we obtain 24 polynomials, among which we find

$$\mathbf{g}_{21} = \gamma_1^2 \left(1575\gamma_3^8 + 9388\gamma_3^6 + 17498\gamma_3^4 + 10060\gamma_3^2 + 375\gamma_4^2 + 375\right),$$

so necessarily $\gamma_1 = 0$. Using the conditions $\gamma_1 = \gamma_2 = 0$ a straightforward calculation shows that the possible non-zero components of the Bach tensor are determined by

$$\begin{aligned}
6\mathfrak{B}_{11} &= (\gamma_4^2+1)(\lambda_3-1)^2\left(3\gamma_3^2-(\gamma_4^2-3)(\lambda_3^2+\lambda_3+1)\right),\\
3\mathfrak{B}_{14} &= 2\gamma_4(\gamma_4^2+1)(\lambda_3-1)^2(\lambda_3^2+\lambda_3+1),\\
6\mathfrak{B}_{22} &= (\gamma_4^2+1)(\lambda_3-1)\left(\gamma_3^2(5\lambda_3+7)+(\gamma_4^2+1)(3\lambda_3^3+2\lambda_3^2+2\lambda_3+5)\right),\\
3\mathfrak{B}_{23} &= 4\gamma_3\gamma_4(\gamma_4^2+1)(\lambda_3^3-1),\\
6\mathfrak{B}_{33} &= (\gamma_4^2+1)(\lambda_3-1)\left(\gamma_3^2(7\lambda_3+5)+(\gamma_4^2+1)(5\lambda_3^3+2\lambda_3^2+2\lambda_3+3)\right),\\
-6\mathfrak{B}_{44} &= (\gamma_4^2+1)(\lambda_3-1)^2\left(\gamma_3^2-(3\gamma_4^2-1)(\lambda_3^2+\lambda_3+1)\right).
\end{aligned}$$

Thus, $8\mathfrak{B}_{11}+2\gamma_4\mathfrak{B}_{14} = (\gamma_4^2+1)(\lambda_3-1)^2\left((2\lambda_3+1)^2+4\gamma_3^2+3\right)=0$ implies that $\lambda_3 = 1$ and therefore the Bach tensor vanishes. A direct calculation shows that the corresponding left-invariant Bach-flat metric is locally symmetric. Moreover, it is locally isometric to a locally conformally flat product $N(c)\times\mathbb{R}$, where $N(c)$ is a Lorentzian three-manifold with constant sectional curvature.

### 3.2. **Diagonalizable structure operator $L$ with timelike kernel.**

We assume $\lambda_3 = 0$ and $\lambda_1\lambda_2 \neq 0$ so that the left-invariant metrics are described by

$$\begin{aligned}
&[e_1,e_3] = -\lambda_2 e_2, && [e_1,e_4] = \gamma_1 e_1 + \gamma_2\lambda_2 e_2, \quad [e_2,e_3] = \lambda_1 e_1,\\
&[e_2,e_4] = -\gamma_2\lambda_1 e_1 + \gamma_1 e_2, && [e_3,e_4] = \gamma_3 e_1 + \gamma_4 e_2,
\end{aligned}$$

where $\{e_1,e_2,e_3,e_4\}$ is an orthonormal basis with $e_3$ timelike. Since $\lambda_1 \neq 0$, we work in the homothetic class of the initial metric assuming $\lambda_1 = 1$.

We start computing a Gröbner basis for the ideal generated by the components of the Bach tensor, $\mathcal{I}_1 = \langle \mathfrak{B}_{ij}\rangle \subset \mathbb{R}[\gamma_1,\,\gamma_2,\,\lambda_2,\,\gamma_3,\,\gamma_4]$, with respect to the lexicographical order. This basis consists of 58 polynomials, being one of them

$$\begin{aligned}
\mathbf{g}_{11} = {} & \gamma_4^6(\gamma_3^2+\gamma_4^2)(2\gamma_4^2+1)(9\gamma_4^2+16)(16\gamma_4^2+5)\\
&\times(25\gamma_4^2+4)(25\gamma_4^2+24)\left(16\gamma_4^4+(8\gamma_4^2-1)^2\right).
\end{aligned}$$

Thus, necessarily $\gamma_4 = 0$. Repeating the process for the ideal $\mathcal{I}_2 = \langle \mathcal{I}_1 \cup \{\gamma_4\}\rangle$ we obtain 24 polynomials that include

$$\mathbf{g}_{21} = \gamma_3^4\lambda_2^3(9\gamma_3^2+4)(25\gamma_3^2+16)(49\gamma_3^2+24).$$

Since $\lambda_2 \neq 0$, it follows that $\gamma_3 = 0$. Now, assuming $\gamma_3 = \gamma_4 = 0$, the possible non-zero components of the Bach tensor are determined by

$$\begin{aligned}
-6\mathfrak{B}_{11} &= (\gamma_2^2-1)(\lambda_2-1)\left(\gamma_1^2(5\lambda_2+7)-(\gamma_2^2-1)(3\lambda_2^3+2\lambda_2^2+2\lambda_2+5)\right),\\
3\mathfrak{B}_{12} &= 4\gamma_1\gamma_2(\gamma_2^2-1)(\lambda_2^3-1),\\
6\mathfrak{B}_{22} &= (\gamma_2^2-1)(\lambda_2-1)\left(\gamma_1^2(7\lambda_2+5)-(\gamma_2^2-1)(5\lambda_2^3+2\lambda_2^2+2\lambda_2+3)\right),\\
6\mathfrak{B}_{33} &= (\gamma_2^2-1)(\lambda_2-1)^2\left(3\gamma_1^2+(\gamma_2^2+3)(\lambda_2^2+\lambda_2+1)\right),\\
-3\mathfrak{B}_{34} &= 2\gamma_2(\gamma_2^2-1)(\lambda_2-1)^2(\lambda_2^2+\lambda_2+1),\\
6\mathfrak{B}_{44} &= (\gamma_2^2-1)(\lambda_2-1)^2\left(\gamma_1^2+(3\gamma_2^2+1)(\lambda_2^2+\lambda_2+1)\right).
\end{aligned}$$

Note that the vanishing of the component $\mathfrak{B}_{33}$ implies that either $\lambda_2 = 1$ or $\gamma_2 = \pm 1$ and, moreover, in any of those cases the corresponding left-invariant metric is Bach-flat. If $\lambda_2 = 1$, then a direct calculation shows that the left-invariant metric is locally symmetric and locally isometric to a locally conformally flat product $N(c) \times \mathbb{R}$, where $N(c)$ is a Riemannian three-manifold of constant sectional curvature.

Now, if $\gamma_2 = \varepsilon$ with $\varepsilon^2 = 1$ and $\lambda_2 \neq 1$, then the corresponding left-invariant Bach-flat metric is given by

$$\begin{aligned} &[e_1, e_3] = -\lambda_2 e_2, \quad &&[e_1, e_4] = \gamma_1 e_1 + \varepsilon\lambda_2 e_2, \\ &[e_2, e_3] = e_1, &&[e_2, e_4] = \gamma_1 e_2 - \varepsilon e_1. \end{aligned} \tag{7}$$

The isometries and the homothety used when analyzing the left-invariant metrics described by Equation (5) show that one may assume $\varepsilon = 1$, $\gamma_1 \geq 0$ and $\lambda_2 \in [-1, 1) \setminus \{0\}$ in Equation (7).

Now, a straightforward calculation shows that the above metrics are never Einstein and they are locally conformally flat if and only if $\gamma_1 = 0$ and $\lambda_2 = -1$. The Ricci operator has eigenvalues $\{-\gamma_1(2\gamma_1 \pm (\lambda_2 - 1)), -\gamma_1^2 \pm \gamma_1\sqrt{\gamma_1^2 + (\lambda_2 - 1)^2}\}$. Let $\{E_1, \dots, E_4\}$ be the left-invariant orthonormal frame on the Lie group obtained by left-translating the orthonormal basis $\{e_1, \dots, e_4\}$ of the corresponding Lie algebra. If $\gamma_1 = 0$ the Ricci operator is two-step nilpotent and, in such a case, $\ell = E_3 + E_4$ is a null parallel vector field. Besides, the curvature tensor satisfies $R(\ell^\perp, \ell^\perp) = 0$ and $\nabla_{\ell^\perp} R = 0$, where $\ell^\perp = \operatorname{span}\{E_1, E_2, \ell\}$. Thus, the underlying structure is a plane wave on the product Lie group $E(1,1) \times \mathbb{R}$ (if $\lambda_2 < 0$) or $\widetilde{E}(2) \times \mathbb{R}$ (if $\lambda_2 > 0$). It is shown in Remark 1.7 that these left-invariant metrics may be viewed as the limiting case in Theorem 1.1–(L.i).

If $\gamma_1 \neq 0$ the four eigenvalues of the Ricci operator are distinct and next we show that, in this case, the metrics determined by Equation (7) are conformally Einstein proceeding as in §2.1.1. Let $\{E_i\}$ be the global orthonormal frame used above. A direct calculation shows that the metric is conformally Cotton-flat. Indeed, the vector field $\xi = \phi E_3 + (\phi + \gamma_1)E_4$ solves the equation $\operatorname{div}_4 W - W(\cdot, \cdot, \cdot, \xi) = 0$ for any smooth function $\phi$ on the Lie group. Moreover, it follows from (7) that $\xi$ is a gradient, $\xi = \nabla\sigma$, if the function $\phi$ satisfies $d\phi(E_1) = d\phi(E_2) = 0$ and $d\phi(E_3) + d\phi(E_4) = 0$. Setting $\varphi = e^\sigma$ one has that $\Delta\varphi = \gamma_1(3\gamma_1 + 4\phi)\varphi$ and the non-zero components of the conformally Einstein tensor $\mathfrak{CE}(X, Y) = 2\operatorname{Hes}_\varphi(X, Y) + \varphi\rho(X, Y) - \frac{1}{4}\{2\Delta\varphi + \varphi\tau\}\langle X, Y\rangle$ are given by

$$\mathfrak{CE}_{34} = \mathfrak{CE}_{43} = -\mathfrak{CE}_{33} = -\mathfrak{CE}_{44} = 2\varphi\left(d\phi(E_3) - \phi(\phi + \gamma_1) + \tfrac{1}{4}(\lambda_2 - 1)^2\right).$$

Hence, any smooth function $\phi$ satisfying

$$d\phi(E_1) = d\phi(E_2) = 0, \quad d\phi(E_4) = -d\phi(E_3), \quad d\phi(E_3) = \phi(\phi + \gamma_1) - \tfrac{1}{4}(\lambda_2 - 1)^2$$

gives rise to a solution $\varphi = e^\sigma$ of the conformally Einstein equation (1). Moreover, a straightforward calculation shows that the Ricci tensor of the conformal metric $\overline{g} = \varphi^{-2}g$ vanishes identically. This corresponds to metrics in Theorem 1.1–(L.i).

3.3. **Structure operator $L$ with complex eigenvalues.** Assume the structure operator $L$ to be of type Ib with eigenvalues $\lambda$ and $\alpha \pm \beta\sqrt{-1}$. Since the structure operator has a one-dimensional kernel, the real eigenvalue $\lambda$ must vanish. Moreover, the underlying Lie group is a semi-direct extension $E(1,1) \rtimes \mathbb{R}$ and the left-invariant metric is determined by:

$$\begin{aligned} &[e_1, e_2] = -\beta e_2 - \alpha e_3, &&[e_1, e_3] = -\alpha e_2 + \beta e_3, \quad &&[e_1, e_4] = \gamma_1 e_2 + \gamma_2 e_3, \\ &[e_2, e_4] = 2\gamma_3\beta e_2 + (\gamma_3 - \gamma_4)\alpha e_3, \quad &&[e_3, e_4] = (\gamma_3 - \gamma_4)\alpha e_2 + 2\gamma_4\beta e_3, && \end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{e}(1,1) \rtimes \mathfrak{r}$ with $e_3$ timelike. Since $\beta \neq 0$, we assume $\beta = 1$ working in the homothetic class of the initial metric just taking the orthogonal basis $\hat{e}_i = \frac{1}{\beta} e_i$.

We consider the ideal $\mathcal{I}_1 = \langle \mathfrak{B}_{ij} \rangle \subset \mathbb{R}[\gamma_1,\, \gamma_2,\, \gamma_3,\, \gamma_4,\, \alpha]$, where $\mathfrak{B}_{ij} = \mathfrak{B}(e_i, e_j)$ are the polynomials given by the components of the Bach tensor in the orthonormal basis $\{e_i\}$. Computing a Gröbner basis with respect to the lexicographical order we get 46 polynomials, among which we find

$$\begin{aligned}\mathbf{g}_{11} &= \alpha(2\alpha-1)(2\alpha+1)(16\alpha^2-5)(\alpha^2+1)^3(\alpha^2+16)(3\alpha^2+1)\\ &\quad\times(16\alpha^2+1)(36\alpha^2+1)(4\gamma_4^2+1)^2\left((\gamma_3-\gamma_4)^2+1\right).\end{aligned}$$

Hence, either $\alpha = 0$ or $\mathbf{p} = (2\alpha-1)(2\alpha+1)\left(16\alpha^2-5\right) = 0$. Now, we compute a second Gröbner basis for the ideal $\mathcal{I}_2 = \langle \mathfrak{B}_{ij} \cup \{\mathbf{p}\} \rangle$ in the polynomial ring $\mathbb{R}[\alpha, \gamma_1, \gamma_2, \gamma_3, \gamma_4]$, where we consider the lexicographical order. As a consequence, we obtain a basis with 52 polynomials which includes

$$\mathbf{g}_{21} = (4\gamma_4^2+1)^3(4096\gamma_4^4+3712\gamma_4^2+121)\left((\gamma_3-\gamma_4)^2+1\right).$$

Since $\mathbf{g}_{21} \neq 0$, any left-invariant Bach-flat metric must have $\alpha = 0$. Repeating the process for the ideal $\mathcal{I}_3 = \langle \mathfrak{B}_{ij} \cup \{\alpha\} \rangle \subset \mathbb{R}[\alpha, \gamma_2, \gamma_3, \gamma_4, \gamma_1]$, where we consider once again the lexicographical order, we get a Gröbner basis with 15 polynomials, being one of them

$$\mathbf{g}_{31} = \gamma_1^9(\gamma_1^2+1)(8\gamma_1^2+5)(4\gamma_4^2+1).$$

Hence, necessarily $\gamma_1 = 0$.

At this point we have the necessary conditions $\alpha = \gamma_1 = 0$. If $\gamma_2 = 0$ a straightforward calculation shows that the possible non-zero components of the Bach tensor are determined by

$$\begin{aligned}-3\mathfrak{B}_{11} &= 2\left((\gamma_3-\gamma_4)^2+1\right)\left(4\gamma_3^2+4\gamma_4^2+4\gamma_3\gamma_4-3\right),\\ 3\mathfrak{B}_{14} &= 8\left((\gamma_3-\gamma_4)^2+1\right)(\gamma_3-\gamma_4),\\ 3\mathfrak{B}_{22} &= 2\left((\gamma_3-\gamma_4)^2+1\right)\left(4\gamma_3^2-4\gamma_4^2+4\gamma_3\gamma_4-1\right),\\ 3\mathfrak{B}_{33} &= 2\left((\gamma_3-\gamma_4)^2+1\right)\left(4\gamma_3^2-4\gamma_4^2-4\gamma_3\gamma_4+1\right),\\ 3\mathfrak{B}_{44} &= 2\left((\gamma_3-\gamma_4)^2+1\right)\left(4\gamma_3^2+4\gamma_4^2-4\gamma_3\gamma_4-1\right),\end{aligned}$$

so the Bach tensor vanishes if and only if $\gamma_3 = \gamma_4 = \pm\frac{1}{2}$. Moreover, in this case, the left-invariant metric is Einstein and locally symmetric. Besides, a direct calculation of the Weyl curvature operator acting on the space of two-forms shows that it has non-zero eigenvalues $-\frac{4}{3}$ and $\frac{2}{3}$ with multiplicities two and four, respectively. Hence the Lorentzian Lie group is locally isometric to a product of two surfaces of constant curvature.

Now, if $\gamma_2 \neq 0$, we use this condition introducing an auxiliary variable $\tilde{\gamma}_2$ and adding the polynomial $\gamma_2\tilde{\gamma}_2 - 1$. Computing a last Gröbner basis for the ideal $\mathcal{I}_4 = \langle \mathfrak{B}_{ij} \cup \{\alpha, \gamma_1, \gamma_2\tilde{\gamma}_2 - 1\} \rangle \subset \mathbb{R}[\tilde{\gamma}_2, \gamma_2, \gamma_4, \gamma_3, \gamma_1, \alpha]$ with respect to the lexicographical order, we get 7 polynomials which include $\mathbf{g}_{41} = -26\gamma_3^2 + 8\gamma_4^2 + 3$, from where $\gamma_3 \neq 0$, and also

$$\mathbf{g}_{42} = \gamma_3(8\gamma_3^2-3), \quad \mathbf{g}_{43} = -\gamma_3(3\gamma_3-2\gamma_4) \quad \text{and} \quad \mathbf{g}_{44} = \gamma_2^2+\gamma_3^2-1.$$

Hence, $\gamma_2 = \frac{\varepsilon_1}{2}\sqrt{\frac{5}{2}}$, $\gamma_3 = \frac{\varepsilon_2}{2}\sqrt{\frac{3}{2}}$ and $\gamma_4 = \frac{3\varepsilon_2}{4}\sqrt{\frac{3}{2}}$, with $\varepsilon_1^2 = \varepsilon_2^2 = 1$, and a straightforward calculation shows that the Bach tensor vanishes. Thus, the corresponding left-invariant Bach-flat metrics are determined by

$$\begin{aligned} &[e_1,e_2]=-e_2, && [e_1,e_3]=e_3, && [e_1,e_4]=\tfrac{\varepsilon_1}{2}\sqrt{\tfrac{5}{2}}\,e_3,\\ &[e_2,e_4]=\varepsilon_2\sqrt{\tfrac{3}{2}}\,e_2, && [e_3,e_4]=\tfrac{3\varepsilon_2}{2}\sqrt{\tfrac{3}{2}}\,e_3. \end{aligned}$$

The isometry determined by $(e_1,e_2,e_3,e_4)\mapsto(e_1,e_2,-e_3,-e_4)$ interchanges the sign of $\varepsilon_2$, while $(e_1,e_2,e_3,e_4)\mapsto(e_1,e_2,e_3,-e_4)$ defines an isometry which interchanges simultaneously the sign of $\varepsilon_1$ and $\varepsilon_2$. So one may fix $\varepsilon_1=\varepsilon_2=1$. Finally we show that this Bach-flat metric cannot be a $C$-space and therefore it corresponds to that in Theorem 1.5–(i). Indeed, let $\xi$ be an arbitrary vector field on the Lie group $G$. Evaluating $\operatorname{div}_4 W - W(\,\cdot\,,\,\cdot\,,\,\cdot\,,\xi)$ at the neutral $e\in G$ (where $\xi_e=\sum_k \xi_k e_k\in\mathfrak{g}$) one has that $(\operatorname{div}_4 W)(e_1,e_2,e_1)-W(e_1,e_2,e_1,\xi)=\frac{15}{16}\,\xi_2$, which shows that the component $\xi_2=0$. Moreover, in this case, we compute

$$\begin{aligned} (\operatorname{div}_4 W)(e_1,e_3,e_3)-W(e_1,e_3,e_3,\xi)&=-\tfrac{5}{64}(8\sqrt{6}\,\xi_4-21),\\ (\operatorname{div}_4 W)(e_1,e_3,e_1)-W(e_1,e_3,e_1,\xi)&=-\tfrac{3}{32}(2\sqrt{10}\,\xi_4-5\sqrt{15}), \end{aligned}$$

which lead to a contradiction.

### 3.4. **Structure operator $L$ with a double root of its minimal polynomial.**

If the structure operator $L$ is of type II, then there exists a pseudo-orthonormal basis $\{u_1,u_2,u_3,u_4\}$ of $\mathfrak{g}=\mathfrak{g}_3\rtimes\mathfrak{r}$, with $\langle u_1,u_2\rangle=\langle u_3,u_3\rangle=\langle u_4,u_4\rangle=1$, where $\mathfrak{g}_3=\operatorname{span}\{u_1,u_2,u_3\}$ and $\mathfrak{r}=\operatorname{span}\{u_4\}$, such that

$$[u_1,u_2]=\lambda_2u_3,\ [u_1,u_3]=-\lambda_1u_1-\varepsilon u_2,\ [u_2,u_3]=\lambda_1u_2,\ [u_i,u_4]=\sum_{j=1}^{3}\alpha_i^j u_j,$$

with $\varepsilon^2=1$ and $\alpha_i^j\in\mathbb{R}$. Since the structure operator has one-dimensional kernel, one of the eigenvalues must vanish and hence either $\lambda_1=0$ and $\lambda_2\neq0$, or $\lambda_1\neq0$ and $\lambda_2=0$.

3.4.1. *Case* $\lambda_1=0,\lambda_2\neq0$. In this case the structure of the Lie algebra is given by

$$\begin{aligned} &[u_1,u_2]=\lambda_2u_3, && [u_1,u_3]=-\varepsilon u_2, && [u_1,u_4]=\gamma_1u_2+\gamma_2u_3,\\ &[u_2,u_4]=\gamma_3u_2+\gamma_4\lambda_2u_3, && [u_3,u_4]=-\varepsilon\gamma_4u_2+\gamma_3u_3. \end{aligned}$$

A direct calculation shows that the components $\mathfrak{B}_{12}$ and $\mathfrak{B}_{34}$ of the Bach tensor are given by

$$\begin{aligned} -24\mathfrak{B}_{12}&=\gamma_4\lambda_2\left(\gamma_4^3\lambda_2+2\gamma_4(13\varepsilon\gamma_3^2-8\gamma_2^2\lambda_2-6\gamma_1\gamma_3\lambda_2-2\varepsilon\lambda_2^2)+4\gamma_2(11\gamma_3^2+8\lambda_2^2)\right)\\ &\quad-3(\gamma_3^2+\lambda_2^2)(\gamma_3^2+4\lambda_2^2),\\ \mathfrak{B}_{34}&=\varepsilon\gamma_3\gamma_4^2\lambda_2^2, \end{aligned}$$

and $\lambda_2\neq0$ implies that necessarily $\gamma_3=0$ and $\gamma_4\neq0$. Under these assumptions, the possible non-zero components of the Bach tensor are determined by

$$\begin{aligned}
-24\mathfrak{B}_{11} &= \lambda_2\left(9\gamma_4^2(2\gamma_1^2\lambda_2-\varepsilon\gamma_4^2)+(8\gamma_2^2+\gamma_4^2-4\varepsilon\lambda_2)(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2)\right),\\
-24\mathfrak{B}_{12} &= \lambda_2^2(\varepsilon\gamma_4^2+4\gamma_2\gamma_4-6\lambda_2)(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2),\\
3\mathfrak{B}_{13} &= \gamma_1\gamma_4\lambda_2^2(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2),\\
6\mathfrak{B}_{14} &= \lambda_2^2(2\gamma_2-\varepsilon\gamma_4)(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2),\\
-3\mathfrak{B}_{22} &= \gamma_4^2\lambda_2^3(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2),\\
-3\mathfrak{B}_{24} &= \gamma_4\lambda_2^3(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2),\\
-24\mathfrak{B}_{33} &= \lambda_2^2(\varepsilon\gamma_4^2-20\gamma_2\gamma_4+10\lambda_2)(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2),\\
24\mathfrak{B}_{44} &= \lambda_2^2(3\varepsilon\gamma_4^2-12\gamma_2\gamma_4-2\lambda_2)(\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2).
\end{aligned}$$

Since $\gamma_4\lambda_2\neq 0$, the vanishing of the Bach tensor is then determined by

$$\varepsilon\gamma_4^2-4\gamma_2\gamma_4+2\lambda_2=0 \quad\text{and}\quad 2\gamma_1^2\lambda_2-\varepsilon\gamma_4^2=0,$$

or equivalently, $\gamma_2=\frac{\varepsilon\gamma_4(\gamma_1^2+1)}{4\gamma_1^2}$ and $\lambda_2=\frac{\varepsilon\gamma_4^2}{2\gamma_1^2}$, with $\gamma_1\gamma_4\neq 0$, and hence the corresponding left-invariant Bach-flat metrics are given by

$$\begin{aligned}
&[u_1,u_2]=\tfrac{\varepsilon\gamma_4^2}{2\gamma_1^2}u_3, \quad [u_1,u_3]=-\varepsilon u_2, \qquad [u_1,u_4]=\gamma_1u_2+\tfrac{\varepsilon\gamma_4(\gamma_1^2+1)}{4\gamma_1^2}u_3,\\
&[u_2,u_4]=\tfrac{\varepsilon\gamma_4^3}{2\gamma_1^2}u_3, \quad [u_3,u_4]=-\varepsilon\gamma_4u_2.
\end{aligned}$$

Note that $\varepsilon$ and the parameter $\gamma_4$ can be eliminated at the homothetic level as follows. Considering the new basis $\{\tilde u_i\}$ defined by $\tilde u_1=-\frac{\gamma_1}{\gamma_4}u_1$, $\tilde u_2=-\frac{\gamma_1^3}{\gamma_4^3}u_2$, $\tilde u_3=\frac{\varepsilon\gamma_1^2}{\gamma_4^2}u_3$, and $\tilde u_4=\frac{\gamma_1^2}{\gamma_4^2}u_4$, the Lie bracket transforms into

$$\begin{aligned}
&[\tilde u_1,\tilde u_2]=\tfrac12\tilde u_3, \qquad [\tilde u_1,\tilde u_3]=-\tilde u_2, \quad [\tilde u_1,\tilde u_4]=\gamma_1\tilde u_2-\tfrac{\gamma_1^2+1}{4\gamma_1}\tilde u_3,\\
&[\tilde u_2,\tilde u_4]=-\tfrac{\gamma_1}{2}\tilde u_3, \quad [\tilde u_3,\tilde u_4]=\gamma_1\tilde u_2,
\end{aligned}$$

while the inner product is rescaled by $\frac{\gamma_1^4}{\gamma_4^4}\langle\cdot,\cdot\rangle$. Since we are working at the homothetic level we maintain the initial inner product remaining in the same homothetic class. Moreover, the isometry $(\tilde u_1,\tilde u_2,\tilde u_3,\tilde u_4)\mapsto(\tilde u_1,\tilde u_2,\tilde u_3,-\tilde u_4)$ interchanges $\gamma_1$ with $-\gamma_1$, so one may assume $\gamma_1>0$.

These metrics, whose underlying Lie group is the product Lie group $\widetilde{E}(2)\times\mathbb{R}$, do not correspond to any $C$-space and therefore we get the left-invariant metrics in Theorem 1.5–(ii). Indeed, for any vector field $\xi$ on the Lie group $G$, evaluating $\operatorname{div}_4 W-W(\,\cdot\,,\,\cdot\,,\,\cdot\,,\xi)$ at the neutral $e\in G$ (where $\xi_e=\sum_k\xi_k\tilde u_k\in\mathfrak{g}$) one has that $(\operatorname{div}_4 W)(\tilde u_4,\tilde u_2,\tilde u_3)-W(\tilde u_4,\tilde u_2,\tilde u_3,\xi)=\frac{\gamma_1^3}{32}$, which never vanishes.

3.4.2. *Case* $\lambda_1\neq 0,\lambda_2=0$. In this case the underlying group is $E(1,1)\rtimes\mathbb{R}$ and the left-invariant metric is determined by

$$\begin{aligned}
&[u_1,u_3]=-\lambda_1u_1-\varepsilon u_2, \qquad [u_1,u_4]=\gamma_1u_1+\gamma_2u_2, \quad [u_2,u_3]=\lambda_1u_2,\\
&[u_2,u_4]=-(2\varepsilon\gamma_2\lambda_1-\gamma_1)u_2, \quad [u_3,u_4]=\gamma_3u_1+\gamma_4u_2.
\end{aligned}$$

A direct analysis of the components of the Bach tensor shows that

$$\begin{aligned}
12\mathfrak{B}_{11} &= -48\varepsilon\lambda_1(\gamma_1^2+\lambda_1^2)(\gamma_2^2+1)+9\gamma_3^2(\gamma_2^2+1)+21\gamma_4^2(\gamma_1^2+\lambda_1^2)\\
&\quad +2\gamma_3\gamma_4(12\varepsilon\gamma_2^2\,\lambda_1-8\gamma_4^2+11\gamma_1\gamma_2+23\varepsilon\lambda_1),\\
12\mathfrak{B}_{14} &= -3\varepsilon\gamma_3\left(2\gamma_2\lambda_1(5\gamma_2\lambda_1-2\varepsilon\gamma_1)+\gamma_1^2+7\lambda_1^2\right)-3\gamma_4\lambda_1(\gamma_1^2+\lambda_1^2)\\
&\quad +16\gamma_3\gamma_4(\gamma_4\lambda_1+\varepsilon\gamma_3),\\
12\mathfrak{B}_{22} &= \gamma_3^2\left(84\gamma_2\lambda_1(\gamma_2\lambda_1-\varepsilon\gamma_1)+21\gamma_1^2+21\lambda_1^2-16\gamma_3\gamma_4\right),\\
12\mathfrak{B}_{24} &= \gamma_3\lambda_1\left(12\gamma_2\lambda_1(\gamma_2\lambda_1-\varepsilon\gamma_1)+3\gamma_1^2+3\lambda_1^2-16\gamma_3\gamma_4\right),
\end{aligned}$$

from where one easily obtains that

$$\lambda_1\mathfrak{B}_{22}-7\gamma_3\mathfrak{B}_{24}=8\gamma_3^3\gamma_4\lambda_1.$$

Since $\lambda_1\neq 0$, either $\gamma_3=0$ or $\gamma_4=0$. If $\gamma_3=0$ then

$$\lambda_1\mathfrak{B}_{11}+7\gamma_4\mathfrak{B}_{14}=-4\varepsilon\lambda_1^2(\gamma_1^2+\lambda_1^2)(\gamma_2^2+1)\neq 0,$$

while if $\gamma_4=0$ and $\gamma_3\neq 0$ then

$$7\varepsilon\gamma_3\mathfrak{B}_{14}+\mathfrak{B}_{22}=-\tfrac{21}{2}\gamma_3^2\lambda_1^2(\gamma_2^2+1)\neq 0,$$

so we conclude that no Bach-flat left-invariant metrics exist in this case.

3.5. **Structure operator $L$ with a triple root of its minimal polynomial.** If the structure operator $L$ is of type III, then there exists a pseudo-orthonormal basis $\{u_1,u_2,u_3,u_4\}$ of $\mathfrak{g}=\mathfrak{g}_3\rtimes\mathfrak{r}$, with $\langle u_1,u_2\rangle=\langle u_3,u_3\rangle=\langle u_4,u_4\rangle=1$, where $\mathfrak{g}_3=\operatorname{span}\{u_1,u_2,u_3\}$ and $\mathfrak{r}=\operatorname{span}\{u_4\}$ so that

$$[u_1,u_2]=u_1+\lambda u_3,\quad [u_1,u_3]=-\lambda u_1,\quad [u_2,u_3]=\lambda u_2+u_3,\quad [u_i,u_4]=\sum_{j=1}^{3}\alpha_i^j u_j,$$

for certain $\alpha_i^j\in\mathbb{R}$. Moreover, $\lambda=0$ since the structure operator has a one-dimensional kernel and the corresponding Lie algebra structure is given by

$$\begin{aligned}
&[u_1,u_2]=u_1, && [u_1,u_4]=\gamma_1u_1, \quad [u_2,u_3]=u_3,\\
&[u_2,u_4]=\gamma_2u_1+\gamma_3u_3, && [u_3,u_4]=\gamma_4u_3.
\end{aligned}$$

In this case the underlying Lie group is the product $E(1,1)\times\mathbb{R}$. A direct calculation of the Bach tensor shows that the possible non-zero components are determined by

$$\begin{aligned}
-24\mathfrak{B}_{12} &= \gamma_4(\gamma_1-2\gamma_4)(\gamma_1+2\gamma_4)(2\gamma_1-\gamma_4),\\
12\mathfrak{B}_{22} &= (\gamma_1-2\gamma_4)\left(3\gamma_1\gamma_3^2-2\gamma_2\gamma_4^2+2\gamma_3^2\gamma_4+3\gamma_1-4\gamma_4\right),\\
6\mathfrak{B}_{23} &= \gamma_3\gamma_4(\gamma_1-2\gamma_4)(3\gamma_1-\gamma_4),\\
-6\mathfrak{B}_{24} &= \gamma_4(\gamma_1-2\gamma_4)(2\gamma_1-\gamma_4),\\
24\mathfrak{B}_{33} &= \gamma_4(\gamma_1-2\gamma_4)(2\gamma_1-\gamma_4)(3\gamma_1+2\gamma_4),\\
-24\mathfrak{B}_{44} &= \gamma_4(\gamma_1-2\gamma_4)^2(2\gamma_1-\gamma_4).
\end{aligned}\tag{8}$$

Hence, the vanishing of the component $\mathfrak{B}_{24}$ leads to three possibilities: $\gamma_4=0$, or $\gamma_1=2\gamma_4$, or $\gamma_1=\frac{\gamma_4}{2}$. However, since $\mathfrak{B}_{22}=\frac{1}{4}\gamma_1^2(\gamma_3^2+1)$ whenever $\gamma_4=0$, it is enough to analyze the cases $\gamma_1=2\gamma_4$ and $\gamma_1=\frac{\gamma_4}{2}\neq 0$.

3.5.1. *Case* $\gamma_1 = 2\gamma_4$. In this case, Equation (8) implies that the Bach tensor vanishes identically and the corresponding Lie algebra structure is given by

$$\begin{aligned} &[u_1,u_2]=u_1, && [u_1,u_4]=2\gamma_4 u_1, \quad [u_2,u_3]=u_3,\\ &[u_2,u_4]=\gamma_2 u_1+\gamma_3 u_3, && [u_3,u_4]=\gamma_4 u_3. \end{aligned} \tag{9}$$

The isometry determined by $(u_1,u_2,u_3,u_4)\mapsto(u_1,u_2,-u_3,u_4)$ interchanges $\gamma_3$ with $-\gamma_3$, while $(u_1,u_2,u_3,u_4)\mapsto(u_1,u_2,u_3,-u_4)$ interchanges $(\gamma_2,\gamma_3,\gamma_4)$ and $(-\gamma_2,-\gamma_3,-\gamma_4)$. Hence, we assume $\gamma_2\geq 0$, $\gamma_3\geq 0$, while $\gamma_4\in\mathbb{R}$. One checks that the above metrics are neither locally conformally flat nor Einstein, unless $\gamma_3=0$ and $\gamma_2\gamma_4=-2$, in which case the space has constant sectional curvature. The Ricci operator of the metrics above has an only real eigenvalue, given by $-3\gamma_4^2$.

Let $\{U_1,\dots,U_4\}$ be the pseudo-orthonormal basis of the Lie group obtained by left-translation of the basis of the Lie algebra above. If $\gamma_4=0$, the Ricci operator is two-step nilpotent and a straightforward calculation shows that the left-invariant vector field $U_1$ is recurrent, since $\nabla_{U_2}U_1=-U_1$ and $\nabla_{U_k}U_1=0$ for all $k\neq 2$. Considering the one-dimensional null distribution $\ell=\operatorname{span}\{U_1\}$ one has $\ell^\perp=\operatorname{span}\{U_1,U_3,U_4\}$ and $R(\ell^\perp,\ell^\perp)=0$, so it follows from the work in [22] that the underlying structure is a *pp*-wave. Moreover, it is also a plane wave since $\nabla_{\ell^\perp}R=0$. It is shown in Remark 1.7 that the above metrics may be viewed as a limiting case of Theorem 1.1–(L.iii).

If $\gamma_4\neq 0$ we proceed as in Section 2.1.1 to show that the metrics in Equation (9), which correspond to Theorem 1.1–(L.iii), are conformally Einstein. Using the global frame $\{U_1,\dots,U_4\}$ introduced above, a direct calculation shows that the metric is conformally Cotton-flat. Indeed, any vector field $\xi=\phi U_1+\gamma_4 U_4$ solves the equation $\operatorname{div}_4 W-W(\cdot,\cdot,\cdot,\xi)=0$ and it is a gradient, $\xi=\nabla\sigma$, for any function $\phi$ defined on the Lie group such that $d\phi(U_1)=d\phi(U_3)=d\phi(U_4)=0$. We set $\varphi=e^\sigma$ so that $\Delta\varphi=4\gamma_4^2\varphi$. A straightforward calculation shows that the only non-zero component of $\mathfrak{CE}(X,Y)=2\operatorname{Hes}_\varphi(X,Y)+\varphi\rho(X,Y)-\frac14\{2\Delta\varphi+\varphi\tau\}\langle X,Y\rangle$ is given by

$$\mathfrak{CE}(U_2,U_2)=2\varphi\left(d\phi(U_2)+(\phi-1)\phi-\tfrac14\gamma_3^2+\tfrac12\gamma_2\gamma_4-1\right).$$

This shows that any function $\phi$ satisfying $d\phi(U_1)=d\phi(U_3)=d\phi(U_4)=0$ and $d\phi(U_2)=-(\phi-1)\phi+\frac14\gamma_3^2-\frac12\gamma_2\gamma_4+1$ gives rise to a conformal Einstein metric. Moreover, a straightforward calculation shows that the Ricci tensor of the conformal metric $\overline{g}=\varphi^{-2}g$ vanishes identically. These left-invariant metrics correspond to the case in Theorem 1.1–(L.iii).

3.5.2. *Case* $\gamma_1=\frac{\gamma_4}{2}$, $\gamma_4\neq 0$. From Equation (8) we get that the possible non-zero components of the Bach tensor are determined by

$$\mathfrak{B}_{22}=-\tfrac{1}{16}\gamma_4^2(7\gamma_3^2-4\gamma_2\gamma_4-5),\quad \mathfrak{B}_{23}=-\tfrac18\gamma_3\gamma_4^3.$$

Thus, necessarily $\gamma_3=0$, $\gamma_2=-\frac{5}{4\gamma_4}$ and the corresponding Lie algebra structure is given by

$$\begin{aligned} &[u_1,u_2]=u_1, && [u_1,u_4]=\tfrac{\gamma_4}{2}u_1, \quad [u_2,u_3]=u_3,\\ &[u_2,u_4]=-\tfrac{5}{4\gamma_4}u_1, && [u_3,u_4]=\gamma_4 u_3. \end{aligned}$$

Since $\gamma_4 \neq 0$, one may consider the new basis $\tilde{u}_1 = \frac{1}{\gamma_4^2}u_1$, $\tilde{u}_2 = u_2$, $\tilde{u}_3 = \frac{1}{\gamma_4}u_3$, and $\tilde{u}_4 = \frac{1}{\gamma_4}u_4$, so that the Lie bracket transforms into

$$\begin{aligned} &[\tilde{u}_1, \tilde{u}_2] = \tilde{u}_1, \qquad &&[\tilde{u}_1, \tilde{u}_4] = \tfrac{1}{2}\tilde{u}_1, \quad &&[\tilde{u}_2, \tilde{u}_3] = \tilde{u}_3, \\ &[\tilde{u}_2, \tilde{u}_4] = -\tfrac{5}{4}\tilde{u}_1, \quad &&[\tilde{u}_3, \tilde{u}_4] = \tilde{u}_3, \end{aligned} \tag{10}$$

while the inner product is rescaled by $\frac{1}{\gamma_4^2}\langle\cdot,\cdot\rangle$. Note that we can maintain the initial inner product remaining in the same homothetic class.

A straightforward calculation shows that the above metric is neither locally conformally flat nor Einstein, and the Ricci operator has eigenvalues $\{-\frac{3}{2}, -\frac{9}{8}, -\frac{3}{8}, -\frac{3}{8}\}$. Next we show that the left-invariant metric determined by Equation (10) is conformally Einstein. Let $\{\tilde{U}_1, \dots, \tilde{U}_4\}$ be the pseudo-orthonormal left-invariant global frame obtained by left-translating the pseudo-orthonormal basis $\{\tilde{u}_1, \dots, \tilde{u}_4\}$ of the Lie algebra. A direct calculation shows that the metric is conformally Cotton-flat. Indeed, the vector field $\xi = -\frac{3}{4}(\tilde{U}_1 - \tilde{U}_4)$ solves the equation $\operatorname{div}_4 W - W(\cdot,\cdot,\cdot,\xi) = 0$ and it is a gradient $\xi = \nabla\sigma$. Moreover, the Weyl curvature operator $W \colon \Lambda^2 \to \Lambda^2$ has eigenvalues $\{-\frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\}$ and thus it is weakly generic. Hence, it follows from the results in [20] that the metric is conformally Einstein. A straightforward calculation shows that the Ricci tensor of the conformal metric $\overline{g} = \varphi^{-2}g$ vanishes identically. It corresponds to the metric in Theorem 1.1–(L.ii).

## 4. Conclusions

Bach-flatness is a very restrictive condition for left-invariant Riemannian metrics on four-dimensional Lie groups [1, 12]. The Lorentzian situation is more subtle due to the fact that the restriction of the metric to the three-dimensional normal subgroup $G$ may be a positive definite, Lorentzian or degenerate metric.

We classify all left-invariant Bach-flat Lorentzian metrics on semi-direct extensions $\widetilde{E}(2) \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$. As a consequence it is shown that the class of conformally Einstein metrics which are not locally conformally flat reduces to plane waves and five generically non-homothetic classes (see Theorem 1.1), and refer to [2] for more examples of conformally Einstein metrics on spacetime groups. On the opposite, the class of strictly Bach-flat metrics, i.e., those which are not conformally Einstein, reduces to two non-homothetic families (see Theorem 1.5). We refer to [24] for other examples of strict Bach-flat homogeneous metrics and to [13] for conformally Einstein and strictly Bach-flat metrics on semi-direct extensions of the Heisenberg group.

Finally, note that all conformally Einstein semi-direct extensions of the Euclidean and Poincaré groups are conformal to Ricci flat metrics which are generically not *pp*-waves since the Weyl curvature operator acting on the space of two-forms is not nilpotent in most cases. Indeed, only cases (R.i), (L.i) and (L.iii) in Theorem 1.1 have two-step nilpotent Weyl curvature operator and they give rise to conformally Einstein *pp*-waves in the limiting cases as pointed out in Remark 1.7.

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