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Cohomological obstructions and weak crossed products over weak Hopf algebras

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ABSTRACT

Let H be a cocommutative weak Hopf algebra and let (B, φ_B) a weak left H-module algebra. In this paper, for a twisted convolution invertible morphism $\sigma : H^2 \to B$ we define its obstruction θ_σ as a Sweedler 3-cocycle with values in the center of B. We obtain that the class of this obstruction vanish in third Sweedler cohomology group $\mathcal{H}^3_{\varphi_{Z(B)}}(H, Z(B))$ if, and only if, there exists a twisted convolution invertible 2-cocycle $\alpha : H^2 \to B$ such that $H \otimes B$ can be endowed with a weak crossed product structure with α keeping a cohomological-like relation with σ. Then, as a consequence, the class of the obstruction of σ vanish if, and only if, there exists a cleft extension of B by H.

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0. Introduction

Crossed products of a Hopf algebra by an algebra have been widely studied in relation to extensions of algebras, generalizing classical results of semidirect products and extensions of groups, together with a generalization of group cohomology to the Hopf algebra setting. In [17] Sweedler defines the so-called Sweedler's cohomology for a cocommutative Hopf algebra H and a commutative H-module algebra B. In this paper he also shows that any cleft H-extension of algebras $B \hookrightarrow A$ (that is, roughly speaking, a split extension) can be realized as a crossed product given in terms of the action of H on B and a 2-cocycle $\sigma : H^2 \rightarrow B$. Moreover, cleft extensions of B are classified by the second cohomology group $\mathcal{H}^2(H, B)$. Several generalizations of these results were carried out by Doi and Takeuchi [7], Blattner, Cohen and Montgomery [4] and Blattner and Montgomery [5] by dropping out the conditions of cocommutativity and commutativity, and the associativity of the action $\varphi_B : H \otimes B \rightarrow B$ and thus, the use of Sweedler's cohomology. However some of its formalism is preserved: for an arbitrary Hopf algebra H and an arbitrary algebra B, a crossed product is given in terms of a measuring $\varphi_B : H \otimes B \rightarrow B$ and a formal 2-cocycle $\sigma : H^2 \rightarrow B$ that must also satisfy the twisted condition needed to substitute the associativity of φ_B. Moreover, two such crossed products are equivalent if the cocycles satisfy a cohomological-like equivalence. This last result was interpreted in an actual cohomological setting by Doi in [8], where he shows that cleft extensions of an algebra B by a cocommutative Hopf algebra H with the same action are classified by $\mathcal{H}^2(H, Z(B))$, where $Z(B)$ denotes the center of B. All these results can be interpreted in a symmetric monoidal category with base object K (see for example [1] and [11] for cleft extensions in a monoidal setting).

The next objective became to decide when an algebra B admits a cleft extension by H. Following some classical results of obstructions to extensions of groups (see, for example, [13]), Schauenburg finds in [16] a relation between the third Sweedler's cohomology group $\mathcal{H}^3(H, Z(B))$ and cleft extensions. For a measuring φ_B and a twisted morphism σ, he generalizes the notion of obstruction as Sweedler three cocycle θ_σ on H with values on the center of B and shows that the class $[\theta_\sigma] \in \mathcal{H}^3(H, Z(B))$ vanish if, and only if, φ_B and σ give rise to a crossed product on $H \otimes B$ and, at last, to a cleft extension.

With the apparition of weak Hopf algebras as generalizations of groupoid algebras (see [6]) all the theory of cleft extensions, Sweedler's cohomology and crossed products needed a deep review. Recall that the main point of a algebra-coalgebra H to be weak is that its unit does not need to be comultiplicative, nor its counit multiplicative. These apparently innocent generalizations conceptually imply the existence of two objects, different from the base object K in the ground monoidal category when H is actually weak, that somehow will play the role of K. From a practical point of view, this lack of (co)multiplicativity of the (co)unit forces to a change in the definition of regular morphisms, and thus to a change in the tackling of cleft extensions, cohomological interpretations of crossed products and a rethinking of cohomology and crossed products themselves. For the cocommutative case these problems were successfully solved in [2] and [3], where the authors explore the meaning of cleft extension and weak crossed product for a cocommutative weak Hopf algebra H weakly acting on an algebra B, and define Sweedler's cohomology in weak contexts. In order to achieve these objectives, they consider the unit in $Reg(H, B)$ as $\varphi_B \circ (H \otimes \eta_B)$ (and thus, regular morphisms depend on φ_B and we denote the set by $Reg_{\varphi_B}(H, B)$), where φ_B is the weak action of H on B, and η_B is the unit of B. Moreover, to study weak crossed products they consider a preunit instead of a unit, so they obtain an algebra as a subobject of $H \otimes B$, whose product is given in terms of φ_B and a twisted formal 2-cocycle $\sigma : H^2 \to B$. In such terms, they are able to define a cohomology theory for a cocommutative weak Hopf algebra H and a commutative H-module algebra B. Moreover, they identify cleft H-extensions of a weak H-module algebra B with products with convolution invertible 2-cocycle and classify them by $\mathcal{H}^2_{\varphi_{Z(B)}}(H, Z(B))$, this is, the second cohomology group. The relation of weak crossed products and cleft extensions for the non-cocommutative case was also studied in [12] by Guccione, Guccione and Valqui.

So once we have the proper concepts of cleft extensions, weak crossed products and Sweedler's cohomology for the weak setting, we just need to find out the role of obstructions in relation to cleft extensions and their cohomological meaning, and these are the main objectives of the present paper. In order to attain such objectives, we first make a wide review of weak crossed products, and we find that we just need a measuring $\varphi_B : H \otimes B \to B$ together with a twisted morphism $\sigma : H^2 \to B$ that does not need to be convolution invertible but a formal 2-cocycle to define a weak crossed product on $H \otimes B$. Moreover we obtain necessary and sufficient conditions for weak crossed products to be equivalent that, in particular, are given in terms of morphisms in $Reg_{\varphi_B}(H, B)$. We finally use these results in the particular case of a cocommutative weak Hopf algebra H and a weak H-module algebra (B, φ_B). We consider a twisted convolution invertible morphism $\sigma : H^2 \to B$ and define its cohomological obstruction θ_σ through the center of B. We obtain that this obstruction vanishes in $\mathcal{H}^3_{\varphi_{Z(B)}}(H, Z(B))$ if, and only if, there exists a twisted convolution invertible 2-cocycle $\alpha : H^2 \to B$ such that $H \otimes B$ can be endowed with a weak crossed product structure with α keeping a cohomological-like relation with σ. This result means, in terms of cleft extensions, that if (B, φ_B) is a weak H-module algebra with $\sigma : H^2 \to B$ twisted and convolution invertible then its obstruction vanishes if, and only if, there exists a cleft extension of B by H.

Throughout this paper C denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object K and symmetry isomorphism c. There is no loss of generality in assuming that C is strict because every monoidal category is monoidally equivalent to a strict one. Then, we may work as if the constrains were all identities. We also assume that in C every idempotent morphism splits, i.e., for any morphism $q : M \to M$ such that $q \circ q = q$ there exists an object N, called the image of q, and morphisms $i : N \to M$, $p : M \to N$ such that $q = i \circ p$ and $p \circ i = id_N$ (id_N denotes the identity morphism for N). The morphisms p and i will be called a factorization of q. Note that N, p and i are unique up to isomorphism. Given objects M, N, P and a morphism $f : N \to P$, we write $M \otimes f$ for $id_M \otimes f$ and $f \otimes M$ for $f \otimes id_M$. Finally, we write c instead $c_{M,N}$ if M and N are clear from the context.

We assume well known the notions of (unitary associative) algebra $A = (A, \eta, \mu)$ with unit η and product μ in C, (counitary coassociative) coalgebra $D = (D, \varepsilon, \delta)$ with counit ε and coproduct δ in C, commutative algebra, cocommutative coalgebra, morphism of algebras, morphism of coalgebras, tensor product of algebras and tensor product of coalgebras. If necessary we will write η_A (μ_A) instead of η (μ). Given an algebra $A = (A, \eta, \mu)$ we set $\mu^{op} := \mu \circ c$. Similarly, given a coalgebra $D = (D, \varepsilon, \delta)$, we set $\delta^{cop} := c \circ \delta$.

Let M be an object in C. For $n \geq 1$, we denote by M^n the n-fold tensor power $M \otimes \cdots \otimes M$. By M^0 we denote the unit object of C, i.e., $M^0 = K$. If A is an algebra and $n \geq 2$, m_A^n denotes the morphism $m_A^n : A^n \to A$ defined by $m_A^2 = \mu$ and by $m_A^k = m_A^{k-1} \circ (A^{k-2} \otimes \mu)$ for $k > 2$. On the other hand, if C is a coalgebra, with δ_{C^n} we denote the coproduct defined in the coalgebra C^n. Then by the coassociativity of δ and the naturality of c, for $k = 1, \cdots, n-1$, the identity $\delta_{C^n} = \delta_{C^{n-k} \otimes C^k}$ holds.

Finally, if A is an algebra, C is a coalgebra and $f : C \to A$, $g : C \to A$ are morphisms in C, we define the convolution product by $f * g = \mu \circ (f \otimes g) \circ \delta$.

1. Generalities on measurings and crossed products in a weak setting

In this section we resume some basic facts about the general theory of weak crossed products in C, introduced in [9], particularized for measurings over a weak Hopf algebra H. Firstly, we recall the notion of weak Hopf algebra, introduced in [6], and summarize some basic properties of these algebraic objects in a monoidal setting.

Definition 1.1. A weak bialgebra H is an object in C with an algebra structure (H, η, μ) and a coalgebra structure (H, ε, δ) such that the following axioms hold:

(a1) $\delta \circ \mu = (\mu \otimes \mu) \circ \delta_{H^2}$,
(a2) $\varepsilon \circ \mu \circ (\mu \otimes H) = (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (H \otimes \delta \otimes H) = (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (H \otimes (c \circ \delta) \otimes H)$,
(a3) $(\delta \otimes H) \circ \delta \circ \eta = (H \otimes \mu \otimes H) \circ (\delta \otimes \delta) \circ (\eta \otimes \eta) = (H \otimes (\mu \circ c) \otimes H) \circ (\delta \otimes \delta) \circ (\eta \otimes \eta)$.

Moreover, if there exists a morphism $\lambda : H \to H$ in C (called the antipode of H) satisfying

(a4) $id * \lambda = ((\varepsilon \circ \mu) \otimes H) \circ (H \otimes c) \circ ((\delta \circ \eta) \otimes H)$,
(a5) $\lambda * id = (H \otimes (\varepsilon \circ \mu)) \circ (c \otimes H) \circ (H \otimes (\delta \circ \eta))$,
(a6) $\lambda * id * \lambda = \lambda$,

we will say H is a weak Hopf algebra.

We say that H is commutative, if it is commutative as algebra and we say that H is cocommutative if it is cocommutative as coalgebra.

1.2. For any weak bialgebra, if we define the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$, $\overline{\Pi}_H^R \in End_{\mathsf{C}}(H)$, by $\Pi_H^L = ((\varepsilon\circ\mu)\otimes H)\circ(H\otimes c)\circ((\delta\circ\eta)\otimes H)$, $\Pi_H^R = (H\otimes(\varepsilon\circ\mu))\circ(c\otimes H)\circ(H\otimes(\delta\circ\eta))$, $\overline{\Pi}_H^L = (H\otimes(\varepsilon\circ\mu))\circ((\delta\circ\eta)\otimes H)$, $\overline{\Pi}_H^R = ((\varepsilon\circ\mu)\otimes H)\circ(H\otimes(\delta\circ\eta))$.

It is straightforward to show that they are idempotent and the following equalities hold:

$$\Pi_H^L\circ\overline{\Pi}_H^L = \Pi_H^L,\quad \Pi_H^L\circ\overline{\Pi}_H^R = \overline{\Pi}_H^R,\quad \Pi_H^R\circ\overline{\Pi}_H^L = \overline{\Pi}_H^L,\quad \Pi_H^R\circ\overline{\Pi}_H^R = \Pi_H^R, \tag{1}$$

$$\overline{\Pi}_H^L\circ\Pi_H^L = \overline{\Pi}_H^L,\quad \overline{\Pi}_H^L\circ\Pi_H^R = \Pi_H^R,\quad \overline{\Pi}_H^R\circ\Pi_H^L = \Pi_H^L,\quad \overline{\Pi}_H^R\circ\Pi_H^R = \overline{\Pi}_H^R. \tag{2}$$

On the other hand, let H_L be the image of the target morphism Π_H^L and let $p_H^L : H\to H_L$, $i_H^L : H_L\to H$ be the morphisms such that $i_H^L\circ p_H^L = \Pi_H^L$ and $p_H^L\circ i_H^L = id_{H_L}$. Then H_L is an algebra and a coalgebra via $\eta_{H_L} = p_H^L\circ\eta$, $\mu_{H_L} = p_H^L\circ\mu\circ(i_H^L\otimes i_H^L))$, $\varepsilon_{H_L} = \varepsilon\circ i_H^L$, $\delta_{H_L} = (p_H^L\otimes p_H^L)\circ\delta\circ i_H^L$.

For the morphisms target and source we have the following identities:

$$(H\otimes\overline{\Pi}_H^R)\circ\delta\circ\overline{\Pi}_H^R = \delta\circ\overline{\Pi}_H^R,\quad (\overline{\Pi}_H^L\otimes H)\circ\delta\circ\overline{\Pi}_H^L = \delta\circ\overline{\Pi}_H^L, \tag{3}$$

$$\mu\circ(H\otimes\Pi_H^L) = ((\varepsilon\circ\mu)\otimes H)\circ(H\otimes c)\circ(\delta\otimes H), \tag{4}$$

$$(\Pi_H^R\otimes H)\circ\delta = (H\otimes\mu)\circ(c\otimes H)\circ(H\otimes(\delta\circ\eta)) \tag{5}$$

$$\mu\circ(H\otimes\overline{\Pi}_H^L) = (H\otimes(\varepsilon\circ\mu))\circ(\delta\otimes H), \tag{6}$$

$$(\overline{\Pi}_H^L\otimes H)\circ\delta = (H\otimes\mu)\circ((\delta\circ\eta)\otimes H), \tag{7}$$

$$\delta\circ\eta = (\Pi_H^R\otimes H)\circ\delta\circ\eta = (H\otimes\Pi_H^L)\circ\delta\circ\eta = (H\otimes\overline{\Pi}_H^R)\circ\delta\circ\eta = (\overline{\Pi}_H^L\otimes H)\circ\delta\circ\eta, \tag{8}$$

$$\varepsilon\circ\mu = \varepsilon\circ\mu\circ(\Pi_H^R\otimes H) = \varepsilon\circ\mu\circ(H\otimes\Pi_H^L) = \varepsilon\circ\mu\circ(\overline{\Pi}_H^R\otimes H) = \varepsilon\circ\mu\circ(H\otimes\overline{\Pi}_H^L). \tag{9}$$

If H is a weak Hopf algebra in C, the antipode λ is unique, antimultiplicative, anticomultiplicative

$$\lambda\circ\mu = \mu\circ(\lambda\otimes\lambda)\circ c,\qquad \delta\circ\lambda = c\circ(\lambda\otimes\lambda)\circ\delta, \tag{10}$$

and leaves the unit and the counit invariant, i.e., $\lambda\circ\eta = \eta$ and $\varepsilon\circ\lambda = \varepsilon$ hold.

Also, it is straightforward to show that Π_H^L, Π_H^R satisfy the equalities

$$\begin{gathered}\Pi_H^L = id*\lambda,\;\; \Pi_H^R = \lambda*id,\;\; \Pi_H^L*id_H = id_H = id*\Pi_H^R,\\ \Pi_H^R*\lambda = \lambda = \lambda*\Pi_H^L,\;\; \Pi_H^L*\Pi_H^L = \Pi_H^L,\;\; \Pi_H^R*\Pi_H^R = \Pi_H^R.\end{gathered} \tag{11}$$

Finally we also have

$$\mu^{op}\circ(H\otimes\bar{\Pi}_H^L)\circ\delta = id_H = \mu^{op}\circ(\bar{\Pi}_H^R\otimes H)\circ\delta. \tag{12}$$

Now we recall the notions of measuring, left weak H-module algebra, and left H-module algebra.

Definition 1.3. Let H be a weak Hopf algebra and let B be an algebra. We say that the morphism $\varphi_B : H \otimes B \to B$ is a measuring if the following equality holds:

(b1) $\varphi_B \circ (H \otimes \mu) = \mu \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c \otimes B) \circ (\delta \otimes B \otimes B)$.

Set $u_1^{\varphi_B} = \varphi_B \circ (H \otimes \eta_B)$. If φ_B is a measuring satisfying

$$\text{(b2)} \quad \varphi_B \circ (\eta \otimes B) = id_B, \qquad \text{(b3)} \quad u_1^{\varphi_B} \circ \mu = \varphi_B \circ (H \otimes u_1^{\varphi_B}),$$

we will say that (B, φ_B) is a left weak H-module algebra. If we replace (b3) by $\varphi_B \circ (\mu \otimes B) = \varphi_B \circ (H \otimes \varphi_B)$ we will say that (B, φ_B) is a left H-module algebra.

If (B, φ_B) is a left weak H-module algebra the following conditions are satisfied:

$$\varphi_B \circ (\Pi_H^L \otimes B) = \mu \circ (u_1^{\varphi_B} \otimes B), \tag{13}$$

$$\varphi_B \circ (\overline{\Pi}_H^L \otimes B) = \mu^{op} \circ (u_1^{\varphi_B} \otimes B), \tag{14}$$

$$u_1^{\varphi_B} \circ \Pi_H^L = u_1^{\varphi_B}, \tag{15}$$

$$u_1^{\varphi_B} \circ \mu = u_1^{\varphi_B} \circ \mu \circ (H \otimes \Pi_H^L), \quad u_1^{\varphi_B} \circ \mu = u_1^{\varphi_B} \circ \mu \circ (H \otimes \overline{\Pi}_H^L). \tag{16}$$

For each $n \geq 1$ we define $\varphi_B^n : H^n \otimes B \to B$, recursively by $\varphi_B^1 = \varphi_B$ and $\varphi_B^n = \varphi_B \circ (H \otimes \varphi_B^{n-1})$. An inductive argument using Definition 1.3(b1) shows that

$$\varphi_B^n \circ (H^n \otimes \mu) = \mu \circ (\varphi_B^n \otimes \varphi_B^n) \circ (H^n \otimes c \otimes B) \circ (\delta_{H^n} \otimes B \otimes B). \tag{17}$$

For $n \geq 2$ we define $u_n^{\varphi_B} := \varphi_B \circ (m_H^n \otimes \eta)$ and, by Definition 1.3(b3), $u_n^{\varphi_B} = \varphi_B^{n-1} \circ (H^{n-1} \otimes u_1^{\varphi_B})$ holds. On the other hand, by a direct computation and [2, Proposition 2.11], we have that

$$u_n^{\varphi_B} = u_1^{\varphi_B} \circ m_H^n, \quad u_n^{\varphi_B} * u_n^{\varphi_B} = u_n^{\varphi_B}. \tag{18}$$

In the rest of this section H is a weak Hopf algebra and φ_B is a measuring.

Definition 1.4. For each morphism $\sigma : H^2 \to B$ we define the morphisms

$$P_{\varphi_B} : H \otimes B \to B \otimes H, \quad F_\sigma : H^2 \to B \otimes H, \quad G_\sigma : H^2 \to H \otimes B,$$

by $P_{\varphi_B} = (\varphi_B \otimes H) \circ (H \otimes c) \circ (\delta \otimes B)$, $F_\sigma = (\sigma \otimes \mu) \circ \delta_{H^2}$, and $G_\sigma = (\mu \otimes \sigma) \circ \delta_{H^2}$ respectively.

By [2, Proposition 3.3] and some easy computations we have the following result.

Proposition 1.5. *The morphism P_{φ_B} introduced in the previous definition satisfies*

$$(\mu \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (P_{\varphi_B} \otimes B) = P_{\varphi_B} \circ (H \otimes \mu). \tag{19}$$

The morphisms $\nabla_{BH}^{\varphi_B} : B \otimes H \to B \otimes H$ and $\nabla_{HB}^{\varphi_B} : H \otimes B \to H \otimes B$, defined by

$$\nabla_{BH}^{\varphi_B} = (\mu \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (B \otimes H \otimes \eta), \quad \nabla_{HB}^{\varphi_B} = (H \otimes \mu) \circ (((H \otimes \varphi_B) \circ (\delta \otimes \eta)) \otimes B)$$

are idempotent. We also have the following identities, in which $T^l = (u_1^{\varphi_B} \otimes H) \circ \delta$ and $T^r = (H \otimes u_1^{\varphi_B}) \circ \delta$:

$$P_{\varphi_B} \circ (H \otimes \eta) = T^l, \quad (B \otimes \varepsilon) \circ P_{\varphi_B} = \varphi_B, \quad \nabla_{BH}^{\varphi_B} \circ P_{\varphi_B} = P_{\varphi_B}, \tag{20}$$

$$\nabla_{BH}^{\varphi_B} = (\mu \otimes H) \circ (B \otimes T^l), \quad \nabla_{HB}^{\varphi_B} = (H \otimes \mu) \circ (T^r \otimes B), \tag{21}$$

$$\nabla_{BH}^{\varphi_B} \circ (\eta \otimes H) = T^l, \quad \nabla_{HB}^{\varphi_B} \circ (H \otimes \eta) = T^r, \tag{22}$$

$$(\mu \otimes H) \circ (B \otimes \nabla_{BH}^{\varphi_B}) = (B \otimes \nabla_{BH}^{\varphi_B}) \circ (\mu \otimes H), \quad (H \otimes \mu) \circ (\nabla_{HB}^{\varphi_B} \otimes B) = (\nabla_{HB}^{\varphi_B} \otimes B) \circ (H \otimes \mu), \tag{23}$$

$$(B \otimes \varepsilon) \circ \nabla_{BH}^{\varphi_B} = \mu \circ (B \otimes u_1^{\varphi_B}), \quad (\varepsilon \otimes B) \circ \nabla_{HB}^{\varphi_B} = \mu \circ (u_1^{\varphi_B} \otimes B), \tag{24}$$

$$(B \otimes \delta) \circ \nabla_{BH}^{\varphi_B} = (\nabla_{BH}^{\varphi_B} \otimes H) \circ (B \otimes \delta), \quad (\delta \otimes B) \circ \nabla_{HB}^{\varphi_B} = (H \otimes \nabla_{HB}^{\varphi_B}) \circ (\delta \otimes B), \tag{25}$$

$$\mu \circ (B \otimes \varphi_B) \circ (T^l \otimes B) = \varphi_B, \quad (\mu \otimes H) \circ (u_1^{\varphi_B} \otimes P_{\varphi_B}) \circ (\delta \otimes B) = P_{\varphi_B}. \tag{26}$$

On the other hand, by a similar proof to the one used in [2, Proposition 3.4], it is possible to obtain the following identities:

$$(B \otimes \delta) \circ F_\sigma = (F_\sigma \otimes \mu) \circ \delta_{H^2}, \quad (\delta \otimes B) \circ G_\sigma = (\mu \otimes G_\sigma) \circ \delta_{H^2}. \tag{27}$$

Proposition 1.6. *The equality*

$$\mu \circ (B \otimes u_1^{\varphi_B}) \circ P_{\varphi_B} = \varphi_B, \tag{28}$$

holds. Let $\sigma : H^2 \to B$ be a morphism. If $\sigma * u_2^{\varphi_B} = \sigma$, the equality

$$\mu \circ (B \otimes u_1^{\varphi_B}) \circ F_\sigma = \sigma, \tag{29}$$

holds and, as a consequence, we have the following identities:

$$\nabla_{BH}^{\varphi_B} \circ F_\sigma = F_\sigma, \tag{30}$$

$$(B \otimes \varepsilon) \circ F_\sigma = \sigma. \tag{31}$$

Moreover, if $\sigma : H^2 \to B$ is a morphism satisfying (30) and (31), we have that $\sigma * u_2^{\varphi_B} = \sigma$.

Proof. Note that, (28) holds because $\mu\circ(B\otimes u_1^{\varphi_B})\circ P_{\varphi_B}\stackrel{(24)}{=}(B\otimes\varepsilon)\circ\nabla_{BH}^{\varphi_B}\circ P_{\varphi_B}\stackrel{(20)}{=}(B\otimes\varepsilon)\circ P_{\varphi_B}\stackrel{(20)}{=}\varphi_B$.

Trivially, $\mu\circ(B\otimes u_1^{\varphi_B})\circ F_\sigma=\sigma*u_2^{\varphi_B}$ holds and then we obtain (29). On the other hand, $\nabla_{BH}^{\varphi_B}\circ F_\sigma\stackrel{(27)}{=}((\mu\circ(B\otimes u_1^{\varphi_B})\circ F_\sigma)\otimes\mu)\circ\delta_{H^2}\stackrel{(29)}{=}F_\sigma$. Then, $(B\otimes\varepsilon)\circ F_\sigma\stackrel{(30)}{=}(B\otimes\varepsilon)\circ\nabla_{BH}^{\varphi_B}\circ F_\sigma\stackrel{(24)}{=}\mu\circ(B\otimes u_1^{\varphi_B})\circ F_\sigma\stackrel{(29)}{=}\sigma$. Finally, if σ satisfy (30) and (31), we have $\sigma\stackrel{(31)}{=}(B\otimes\varepsilon)\circ F_\sigma\stackrel{(30)}{=}(B\otimes\varepsilon)\circ\nabla_{BH}^{\varphi_B}\circ F_\sigma\stackrel{(24)}{=}\mu\circ(B\otimes u_1^{\varphi_B})\circ F_\sigma=\sigma*u_2^{\varphi_B}$. □

1.7. In a similar way to what was proven in the previous proposition, we have that $\mu\circ(u_1^{\varphi_B}\otimes B)\circ G_\sigma=u_2^{\varphi_B}*\sigma$. Then, we can obtain the following result:

Proposition 1.8. *Let $\sigma: H^2\to B$ be a morphism. If $u_2^{\varphi_B}*\sigma=\sigma$, the equality $\mu\circ(u_1^{\varphi_B}\otimes B)\circ G_\sigma=\sigma$, holds and, as a consequence, we have the following identities:*

$$\nabla_{HB}^{\varphi_B}\circ G_\sigma=G_\sigma,\quad(\varepsilon\otimes B)\circ G_\sigma=\sigma.\tag{32}$$

Moreover, if σ satisfies (32), $u_2^{\varphi_B}*\sigma=\sigma$ holds.

Remark 1.9. By the previous propositions, [12, Propositions 2.7 and 2.8], (1) and (2), if σ satisfies $\sigma*u_2^{\varphi_B}=\sigma$, we have

$$\sigma\circ(\mu\otimes H)\circ(H\otimes\Pi_H^R\otimes H)=\sigma\circ(H\otimes\mu)\circ(H\otimes\Pi_H^R\otimes H),\tag{33}$$

$$\sigma\circ(\mu\otimes H)\circ(H\otimes\bar{\Pi}_H^L\otimes H)=\sigma\circ(H\otimes\mu)\circ(H\otimes\bar{\Pi}_H^L\otimes H).\tag{34}$$

1.10. Let $\sigma: H^2\to B$ be a morphism such that $\sigma*u_2^{\varphi_B}=\sigma$. Under these conditions, we have a quadruple $\mathbb{B}_H=(B,H,\psi_H^B=P_{\varphi_B},\sigma_H^B=F_\sigma)$ as the ones introduced in [9] to define the notion of weak crossed product. For the quadruple $\mathbb{B}_H$ there exists a product in $B\otimes H$ defined by $\mu_{B\otimes_{\varphi_B}^\sigma H}=(\mu\otimes H)\circ(\mu\otimes F_\sigma)\circ(B\otimes P_{\varphi_B}\otimes H)$. Let $\mu_{A\times_{\varphi_B}^\sigma H}$ be the product $\mu_{B\times_{\varphi_B}^\sigma H}=p_{B\otimes H}^{\varphi_B}\circ\mu_{B\otimes_{\varphi_B}^\sigma H}\circ(i_{B\otimes H}^{\varphi_B}\otimes i_{B\otimes H}^{\varphi_B})$, where $B\times_{\varphi_B}^\sigma H$, $i_{B\otimes H}^{\varphi_B}: B\times_{\varphi_B}^\sigma H\to B\otimes H$ and $p_{B\otimes H}^{\varphi_B}: B\otimes H\to B\times_{\varphi_B}^\sigma H$ denote the image, the injection, and the projection associated to the factorization of $\nabla_{BH}^{\varphi_B}$. Following [9], we say that $\mathbb{B}_H$ satisfies the twisted condition if

$$(\mu\otimes H)\circ(B\otimes P_{\varphi_B})\circ(F_\sigma\otimes B)=(\mu\otimes H)\circ(B\otimes F_\sigma)\circ(P_{\varphi_B}\otimes H)\circ(H\otimes P_{\varphi_B})\tag{35}$$

and the cocycle condition holds if

$$(\mu\otimes H)\circ(B\otimes F_\sigma)\circ(F_\sigma\otimes H)=(\mu\otimes H)\circ(B\otimes F_\sigma)\circ(P_{\varphi_B}\otimes H)\circ(H\otimes F_\sigma).\tag{36}$$

Note that, if $\mathbb{B}_H$ satisfies the twisted condition, by [9, Proposition 3.4], and (30) we obtain the equality:

$$(\mu\otimes H)\circ(A\otimes F_\sigma)\circ(\nabla_{BH}^{\varphi_B}\otimes H)=(\mu\otimes H)\circ(B\otimes F_\sigma).\tag{37}$$

Taking into account that the cocommutativity condition for H is not necessary, we can repeat the proofs of [2, Theorems 3.12 and 3.13] getting the following theorem.

Theorem 1.11. *Let $\sigma : H^2 \to B$ be a morphism such that $\sigma * u_2^{\varphi_B} = \sigma$. Then, the following assertions hold.*

(i) *The quadruple $\mathbb{B}_H$ satisfies the twisted condition (35) iff σ satisfies the twisted condition*

$$\mu \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B}) = \mu \circ (B \otimes \varphi_B) \circ (F_\sigma \otimes B). \tag{38}$$

(ii) *The quadruple $\mathbb{B}_H$ satisfies the cocycle condition (36) iff σ satisfies the cocycle condition*

$$\mu \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes F_\sigma) = \mu \circ (B \otimes \sigma) \circ (F_\sigma \otimes B). \tag{39}$$

If the twisted and the cocycle conditions hold, the product $\mu_{B \otimes^\sigma_{\varphi_B} H}$ is associative and normalized with respect to $\nabla^{\varphi_B}_{BH}$, i.e., the following identities hold:

$$\begin{gathered} \nabla^{\varphi_B}_{BH} \circ \mu_{B \otimes^\sigma_{\varphi_B} H} = \mu_{B \otimes^\sigma_{\varphi_B} H} = \mu_{B \otimes^\sigma_{\varphi_B} H} \circ (\nabla^{\varphi_B}_{BH} \otimes \nabla^{\varphi_B}_{BH})) \\ \mu_{B \otimes^\sigma_{\varphi_B} H} \circ (\nabla^{\varphi_B}_{BH} \otimes B \otimes H) = \mu_{B \otimes^\sigma_{\varphi_B} H} = \mu_{B \otimes^\sigma_{\varphi_B} H} \circ (B \otimes H \otimes \nabla^{\varphi_B}_{BH}). \end{gathered} \tag{40}$$

Then, under these conditions $\mu_{B \times^\sigma_{\varphi_B} H}$ is associative as well (see [9, Proposition 3.7]). Hence we define:

Definition 1.12. If $\mathbb{B}_H$ satisfies (35) and (36) we say that $(B \otimes H, \mu_{B \otimes^\sigma_{\varphi_B} H})$ is a weak crossed product.

The next natural question that arises is if it is possible to endow $\mu_{B \times^\sigma_{\varphi_B} H}$ with a unit, and hence with an algebra structure. As we recall in [9], we need to use the notion of preunit to obtain this unit. In our setting, $\nu : K \to B \otimes H$ is a preunit for the associative product $\mu_{B \otimes^\sigma_{\varphi_B} H}$ if $\mu_{B \otimes^\sigma_{\varphi_B} H} \circ (B \otimes H \otimes \nu) = \mu_{B \otimes^\sigma_{\varphi_B} H} \circ (\nu \otimes B \otimes H)$ and $\nu = \mu_{B \otimes^\sigma_{\varphi_B} H} \circ (\nu \otimes \nu)$ hold (see [9, Definition 2.3, Remark 2.4]). Following [9, Theorem 3.11] (see also [10, Definition 1.4]) we will say that $(B \otimes H, \mu_{B \otimes^\sigma_{\varphi_B} H})$ is a weak crossed product with preunit $\upsilon : K \to B \otimes H$ if

$$(\mu \otimes H) \circ (B \otimes F_\sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes \upsilon) = \nabla^{\varphi_B}_{BH} \circ (\eta \otimes H), \tag{41}$$

$$(\mu \otimes H) \circ (B \otimes F_\sigma) \circ (\upsilon \otimes H) = \nabla^{\varphi_B}_{BH} \circ (\eta \otimes H) \tag{42}$$

and

$$(\mu \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (\upsilon \otimes B) = (\mu \otimes H) \circ (B \otimes \upsilon) \tag{43}$$

hold.

Note that, by [9, Theorem 3.11], if $(B \otimes H, \mu_{B\otimes^{\sigma}_{\varphi_B} H})$ is a weak crossed product with preunit υ, the morphism υ is a preunit for the associative product $\mu_{B\otimes^{\sigma}_{\varphi_B} H}$ and $\nabla^{\varphi_B}_{BH} = \mu_{B\otimes^{\sigma}_{\varphi_B} H}\circ(\upsilon\otimes B\otimes H)$. Also, by [9, Corollary 3.12], we know that, if υ is a preunit for $(B\otimes H, \mu_{B\otimes^{\sigma}_{\varphi_B} H})$, the object $B\times^{\sigma}_{\varphi_B} H$ is an algebra with product $\mu_{B\times^{\sigma}_{\varphi_B} H}$ and unit $\eta_{B\times^{\sigma}_{\varphi_B} H} = p^{\varphi_B}_{B\otimes H}\circ\upsilon$. Finally, $\mu_{B\otimes^{\sigma}_{\varphi_B} H}$ is normalized with respect to $\mu_{B\otimes^{\sigma}_{\varphi_B} H}\circ(\upsilon\otimes B\otimes H)$ because $\nabla^{\varphi_B}_{BH} = \mu_{B\otimes^{\sigma}_{\varphi_B} H}\circ(\upsilon\otimes B\otimes H)$ (see [9, Theorem 3.11]).

Remark 1.13. Note that, if υ is a preunit for the weak crossed product $(B\otimes H, \mu_{B\otimes^{\sigma}_{\varphi_B} H})$, by (43) the following equality holds:

$$\nabla^{\varphi_B}_{BH}\circ\upsilon = \upsilon \tag{44}$$

Therefore the preunit of a weak crossed product, if it exists, is unique because if $(B\otimes H, \mu_{B\otimes^{\sigma}_{\varphi_B} H})$ admits two preunits υ_1, υ_2, we have $\eta_{B\times^{\sigma}_{\varphi_B} H} = p^{\varphi_B}_{B\otimes H}\circ\upsilon_1 = p^{\varphi_B}_{B\otimes H}\circ\upsilon_2$ and then $\upsilon_1 = \nabla^{\varphi_B}_{BH}\circ\upsilon_1 = \nabla^{\varphi_B}_{BH}\circ\upsilon_2 = \upsilon_2$.

The following proposition is a tool to establish the conditions under which the morphism $\upsilon = \nabla^{\varphi_B}_{BH}\circ(\eta\otimes\eta)$ is a preunit for a weak crossed product $(B\otimes H, \mu_{B\otimes^{\sigma}_{\varphi_B} H})$.

Proposition 1.14. *Let σ be a morphism as in Theorem 1.11. Then, the following equalities hold.*

$$\sigma\circ(\eta\otimes H) = \sigma\circ c\circ(H\otimes\overline{\Pi}^{L}_{H})\circ\delta, \tag{45}$$

$$\sigma\circ(H\otimes\eta) = \sigma\circ(H\otimes\Pi^{R}_{H})\circ\delta. \tag{46}$$

Proof. The equality (45) follows from (34) and (12), and (46) is a consequence of (33) and (11). □

Definition 1.15. Let $\sigma : H^2 \to B$ be a morphism. We say that σ satisfies the normal condition if

$$\sigma\circ(\eta\otimes H) = \sigma\circ(H\otimes\eta) = u^{\varphi_B}_{1}. \tag{47}$$

Therefore, if $\sigma * u^{\varphi_B}_{2} = \sigma$, by Proposition 1.14, σ is normal iff $\sigma\circ c\circ(H\otimes\overline{\Pi}^{L}_{H})\circ\delta = \sigma\circ(H\otimes\Pi^{R}_{H})\circ\delta = u^{\varphi_B}_{1}$.

Theorem 1.16. *Let σ be a morphism as in Theorem 1.11 and assume that*

$$\nabla^{\varphi_B}_{BH}\circ(B\otimes\eta) = P_{\varphi_B}\circ(\eta\otimes B), \tag{48}$$

holds. If $\mathbb{B}_H$ satisfies the twisted and the cocycle conditions (35) and (36), the morphism $\upsilon = \nabla^{\varphi_B}_{BH}\circ(\eta\otimes\eta)$ is a preunit for the weak crossed product associated to $\mathbb{B}_H$ iff

$$F_\sigma\circ(\eta\otimes H) = F_\sigma\circ(H\otimes\eta) = \nabla^{\varphi_B}_{BH}\circ(\eta\otimes H). \tag{49}$$

Proof. It suffices to check that the left side of (41) equals $F_\sigma\circ(H\otimes\eta)$, the left side of (42) equals $F_\sigma\circ(\eta\otimes H)$ and the left side of (43) equals $P_{\varphi_B}\circ(\eta\otimes B)$. But the first equality holds by (35) and (30); the second one, using that $(\mu\otimes H)\circ(B\otimes F_\sigma)\circ(P_{\varphi_B}\otimes H)\circ(H\otimes\eta\otimes H)=F_\sigma$; and the third one by (19). □

As a consequence of Theorem 1.16, and by Proposition 1.14 we have:

Corollary 1.17. *Let $\sigma : H^2 \to B$ be a morphism and let $\mathbb{B}_H$ be the associated quadruple such that the assumptions of Theorem 1.16 hold. Then, $\upsilon = \nabla_{BH}^{\varphi_B}\circ(\eta\otimes\eta)$ is a preunit for the weak crossed product associated to $\mathbb{B}_H$ iff σ satisfies the normal condition (47).*

Proof. Considering (22), the proof follows from the equalities $F_\sigma\circ(\eta\otimes H) = ((\sigma\circ c\circ(H\otimes\overline{\Pi}_H^L)\circ\delta)\otimes H)\circ\delta$ and $F_\sigma\circ(H\otimes\eta) = ((\sigma\circ(H\otimes\Pi_H^R)\circ\delta)\otimes H)\circ\delta$, which hold by (7), (5) and the naturality of c. □

Therefore, by the previous results, we obtain the complete characterization of weak crossed products associated to a measuring with preunit $\upsilon = \nabla_{BH}^{\varphi_B}\circ(\eta\otimes\eta)$.

Corollary 1.18. *Let $\sigma : H^2 \to B$ be a morphism and let $\mathbb{B}_H$ be the associated quadruple such that the assumptions of Theorem 1.16 hold. Then the following statements are equivalent:*

(i) *The pair $(B\otimes H, \mu_{B\otimes^\sigma_{\varphi_B}H})$ is a weak crossed product with preunit $\upsilon = \nabla_{BH}^{\varphi_B}\circ(\eta\otimes\eta)$.*
(ii) *The morphism σ satisfies the twisted condition (38), the cocycle condition (39) and the normal condition (47).*

Remark 1.19. If (B, φ_B) is a left weak H-module algebra the equality (48) follows from (21), (14), (8) and by the naturality of c.

2. Equivalent weak crossed products

The general theory of equivalent weak crossed products was presented in [10]. In this section we remember the criterion obtained in [10] that characterises the equivalence between two weak crossed products and we give the translation of this criterion to the particular setting of weak crossed products induced by measurings. We shall start by introducing the notion of equivalence of weak crossed products induced by measurings. As in the previous one, in this section H denotes a weak Hopf algebra.

Definition 2.1. Let $\varphi_B,\ \phi_B : H\otimes B\to B$ be measurings and let $\sigma,\ \tau : H^2\to B$ be morphisms such that $\sigma * u_2^{\varphi_B} = \sigma$, $\tau * u_2^{\phi_B} = \tau$. Assume that σ, τ satisfy (38), (39) and suppose that υ is a preunit for the weak crossed product $(B\otimes^\sigma_{\varphi_B} H, \mu_{B\otimes^\sigma_{\varphi_B}H})$, and u is a preunit for the weak crossed product $(B\otimes^\tau_{\phi_B} H, \mu_{B\otimes^\tau_{\phi_B}H})$. We say that

$(B\otimes H, \mu_{B\otimes_{\varphi_B}^{\sigma} H})$ and $(B\otimes H, \mu_{B\otimes_{\phi_B}^{\tau} H})$ are equivalent weak crossed products if there is an isomorphism $\Upsilon : B\times_{\varphi_B}^{\sigma} H \to B\times_{\phi_B}^{\tau} H$ of algebras, left B-modules and right H-comodules, where the left actions are defined by $\varphi_{B\times_{\varphi_B}^{\sigma} H} = p_{B\otimes H}^{\varphi_B}\circ(\mu\otimes H)\circ(B\otimes i_{B\otimes H}^{\varphi_B})$, $\varphi_{B\times_{\phi_B}^{\tau} H} = p_{B\otimes H}^{\phi_B}\circ(\mu\otimes H)\circ(B\otimes i_{B\otimes H}^{\phi_B})$, and the right coactions are $\rho_{B\times_{\varphi_B}^{\sigma} H} = (p_{B\otimes H}^{\varphi_B}\otimes H)\circ(B\otimes\delta)\circ i_{B\otimes H}^{\varphi_B}$, $\rho_{B\times_{\phi_B}^{\tau} H} = (p_{B\otimes H}^{\phi_B}\otimes H)\circ(B\otimes\delta)\circ i_{B\otimes H}^{\phi_B}$.

In our setting the general criterion [10, Theorem 1.7] that characterizes equivalent weak crossed products admits the following formulation.

Theorem 2.2. *Let $\varphi_B,\ \phi_B : H\otimes B\to B$ be measurings. Let $\sigma,\ \tau : H^2\to B$ and $\upsilon,\ u : K\to B\otimes H$ be morphisms satisfying the conditions of the previous definition. Let $(B, H, P_{\varphi_B}, F_\sigma)$ and $(B, H, P_{\phi_B}, F_\tau)$ be the corresponding quadruples. The following assertions are equivalent:*

(i) *The weak crossed products $(B\otimes H, \mu_{B\otimes_{\varphi_B}^{\sigma} H})$ and $(B\otimes H, \mu_{B\otimes_{\phi_B}^{\tau} H})$ are equivalent.*

(ii) *There exist two morphisms $T, S : B\otimes H\to B\otimes H$, of left B-modules for the trivial action $\varphi_{B\otimes H} = \mu\otimes H$, and right H-modules for the trivial coaction $\rho_{B\otimes H} = B\otimes\delta$, satisfying the conditions*

$$T\circ\upsilon = u, \tag{50}$$

$$T\circ\mu_{B\otimes_{\varphi_B}^{\sigma} H} = \mu_{B\otimes_{\phi_B}^{\tau} H}\circ(T\otimes T), \tag{51}$$

$$S\circ T = \nabla_{BH}^{\varphi_B},\ \ T\circ S = \nabla_{BH}^{\phi_B}. \tag{52}$$

(iii) *There exist two morphisms $\theta, \gamma : H\to B\otimes H$ of right H-comodules for the trivial coaction satisfying the conditions*

$$\theta = \nabla_{BH}^{\varphi_B}\circ\theta, \tag{53}$$

$$(\mu\otimes H)\circ(B\otimes\theta)\circ\gamma = \nabla_{BH}^{\varphi_B}\circ(\eta\otimes H), \tag{54}$$

$$P_{\phi_B} = (\mu\otimes H)\circ(\mu\otimes\gamma)\circ(B\otimes P_{\varphi_B})\circ(\theta\otimes B), \tag{55}$$

$$F_\tau = (\mu\otimes H)\circ(B\otimes\gamma)\circ\mu_{B\otimes_{\varphi_B}^{\sigma} H}\circ(\theta\otimes\theta), \tag{56}$$

$$u = (\mu\otimes H)\circ(B\otimes\gamma)\circ\upsilon. \tag{57}$$

Recall that, by the proof of the part (i) ⇒ (ii) of [10, Theorem 1.7], the morphisms T and S also satisfy the identity

$$T\circ S\circ T = T,\quad S\circ T\circ S = S. \tag{58}$$

On the other hand, if υ and u are the preunits for $(B\otimes H, \mu_{B\otimes_{\varphi_B}^{\sigma} H})$ and $(B\otimes H, \mu_{B\otimes_{\phi_B}^{\tau} H})$ respectively, by (50), we have that $S\circ T\circ\upsilon = S\circ u$. Then, by (52) we have that $\nabla_{BH}^{\varphi_B}\circ\upsilon = S\circ u$ and applying (44) we obtain that

$$S \circ u = v \tag{59}$$

holds.

Proposition 2.3. *Let φ_B, ϕ_B, σ and τ be morphisms satisfying the conditions of Definition 2.1 and suppose that $\nabla_{BH}^{\varphi_B} \circ (\eta \otimes \eta)$ is a preunit for $(B \otimes H, \mu_{B \otimes_{\varphi_B}^{\sigma} H})$ and $\nabla_{BH}^{\phi_B} \circ (\eta \otimes \eta)$ is a preunit for $(B \otimes H, \mu_{B \otimes_{\phi_B}^{\tau} H})$. If $(B \otimes H, \mu_{B \otimes_{\varphi_B}^{\sigma} H})$ and $(B \otimes H, \mu_{B \otimes_{\phi_B}^{\tau} H})$ are equivalent weak crossed products, there exist morphisms $T, S : B \otimes H \to B \otimes H$ of left B-modules for the trivial action and right H-comodules for the trivial coaction such that*

$$\nabla_{BH}^{\phi_B} \circ (\eta \otimes \eta) = T \circ (\eta \otimes \eta), \quad \nabla_{BH}^{\varphi_B} \circ (\eta \otimes \eta) = S \circ (\eta \otimes \eta). \tag{60}$$

Proof. We have that following:

$$\nabla_{BH}^{\phi_B} \circ (\eta \otimes \eta) \overset{(50)}{=} T \circ \nabla_{BH}^{\varphi_B} \circ (\eta \otimes \eta) \overset{(52)}{=} T \circ S \circ T \circ (\eta \otimes \eta) \overset{(58)}{=} T \circ (\eta \otimes \eta),$$

and $\nabla_{BH}^{\varphi_B} \circ (\eta \otimes \eta) \overset{(59)}{=} S \circ \nabla_{BH}^{\phi_B} \circ (\eta \otimes \eta) \overset{(52)}{=} S \circ T \circ S \circ (\eta \otimes \eta) \overset{(58)}{=} S \circ (\eta \otimes \eta)$. □

Theorem 2.4. *Let φ_B, ϕ_B, σ, τ, v and u be morphisms satisfying the conditions of Definition 2.1. The following assertions are equivalent:*

(i) *The weak crossed products $(B \otimes H, \mu_{B \otimes_{\varphi_B}^{\sigma} H})$ and $(B \otimes H, \mu_{B \otimes_{\phi_B}^{\tau} H})$ are equivalent.*
(ii) *There exists two morphisms $h, h^{-1} : H \to B$ such that*

$$h^{-1} * h = u_1^{\varphi_B}, \tag{61}$$

$$h * h^{-1} * h = h, \quad h^{-1} * h * h^{-1} = h^{-1}, \tag{62}$$

$$\phi_B = \mu \circ (\mu \otimes h^{-1}) \circ (h \otimes P_{\varphi_B}) \circ (\delta \otimes B), \tag{63}$$

$$\tau = \mu \circ (B \otimes h^{-1}) \circ \mu_{B \otimes_{\varphi_B}^{\sigma} H} \circ (((h \otimes H) \circ \delta) \otimes ((h \otimes H) \circ \delta)), \tag{64}$$

$$u = ((\mu \circ (B \otimes h^{-1})) \otimes H) \circ (B \otimes \delta) \circ v. \tag{65}$$

Proof. First we will prove that (i) ⇒ (ii). By the proof of the part (i) ⇒ (ii) of [10, Theorem 1.7], there exists two morphisms $T, S : B \otimes H \to B \otimes H$ of left B-modules for the trivial action and right H-comodules for trivial coaction and satisfying the conditions (50), (51), (52) and (58) ([10, (32)]). Also, S preserves the preunit, i.e., (59) holds, and it is multiplicative ([10, (37)]). In the proof of the part (ii) ⇒ (iii) of [10, Theorem 1.7], the morphisms $\theta, \gamma : H \to B \otimes H$ where defined by $\theta = S \circ (\eta \otimes H)$ and $\gamma = T \circ (\eta \otimes H)$. Since S and T are left B-bilinear for the trivial action it is clear that

$$S = (\mu \otimes H) \circ (B \otimes \theta), \quad T = (\mu \otimes H) \circ (B \otimes \gamma). \tag{66}$$

Furthermore, by (ii) ⇒ (iii) of [10, Theorem 1.7], for θ and γ the equalities (53), (54), (55), (56), and (57) hold. Moreover, $(\mu \otimes H) \circ (B \otimes \theta) \circ \gamma = \nabla_{BH}^{\varphi_B} \circ (\eta \otimes H)$, $\gamma = \nabla_{BH}^{\phi_B} \circ \gamma$,

$P_{\varphi_B} = (\mu \otimes H) \circ (\mu \otimes \theta) \circ (B \otimes P_{\phi_B}) \circ (\gamma \otimes B)$, $F_\sigma = (\mu \otimes H) \circ (B \otimes \theta) \circ \mu_{B \otimes^\tau_{\phi_B} H} \circ (\gamma \otimes \gamma)$ and $\nu = (\mu \otimes H) \circ (B \otimes \theta) \circ u$ also hold. Define

$$h = (B \otimes \varepsilon) \circ \theta, \ \ h^{-1} = (B \otimes \varepsilon) \circ \gamma. \tag{67}$$

Then, by the condition of right H-comodule morphism for θ and γ, we have

$$\theta = (h \otimes H) \circ \delta, \quad \gamma = (h^{-1} \otimes H) \circ \delta. \tag{68}$$

The equality (61) follows from the comodule morphism condition for γ, (54) and the counit properties. Mimiking the proof of (61) one can check that

$$h * h^{-1} = u_1^{\phi_B}. \tag{69}$$

By (67) and the definition of θ, we have

$$h = (B \otimes \varepsilon) \circ S \circ (\eta \otimes H) = (B \otimes \varepsilon) \circ S \circ \nabla^{\varphi_B}_{BH} \circ (\eta \otimes H) = (\mu \otimes \varepsilon) \circ (B \otimes \theta) \circ \nabla^{\varphi_B}_{BH} \circ (\eta \otimes H),$$

where the second equality holds by (52) and (58), and the last one by (66). Since $\nabla^{\varphi_B}_{BH} \circ (\eta \otimes H) = (u_1^{\phi_B} \otimes H) \circ \delta$ and $(\mu \otimes \varepsilon) \circ (B \otimes \theta) = \mu \circ (B \otimes h)$, by (69), we obtain

$$h = u_1^{\phi_B} * h = h * h^{-1} * h.$$

In a similar way, we can prove that $h^{-1} * h * h^{-1} = h^{-1}$.

The equality (65) follows directly from (57) and (68). Moreover, composing in (55) with $B \otimes \varepsilon$, we prove (63) using (67) and (68). Finally, (64) holds because

$$\begin{aligned} \tau &\overset{(31)}{=} (B \otimes \varepsilon) \circ F_\tau \overset{(56)}{=} (\mu \otimes \varepsilon) \circ (B \otimes \gamma) \circ \mu_{B \otimes^\sigma_{\varphi_B} H} \circ (\theta \otimes \theta) \\ &\overset{(68)}{=} (\mu \otimes h^{-1}) \circ \mu_{B \otimes^\sigma_{\varphi_B} H} \circ (((h \otimes H) \circ \delta) \otimes ((h \otimes H) \circ \delta)). \end{aligned}$$

Conversely, to prove (ii) ⇒ (i), define $\theta = (h \otimes H) \circ \delta$ and $\gamma = (h^{-1} \otimes H) \circ \delta$. Then θ and γ are morphisms of right H-comodules, $h = (B \otimes \varepsilon) \circ \theta$ and $h^{-1} = (B \otimes \varepsilon) \circ \gamma$. To prove the equivalence between $(B \otimes H, \mu_{B \otimes^\sigma_{\varphi_B} H})$ and $(B \otimes H, \mu_{B \otimes^\tau_{\phi_B} H})$, we must show that (53), (54), (55), (56) and (57) hold. First note that, (57) follows from (65). Also, (53) follows from the coassociativity of δ, (21), (61) and (62). On the other hand, (54) follows by the coassociativity of δ, (61) and (22). Similarly, (55) follows by (63), the coassociativity of δ and the naturality of c. Finally, (56) holds by (64), the coassociativity of δ, the naturality of c and (27). □

Remark 2.5. Note that, in the conditions of (ii) of Theorem 2.4 composing with $H \otimes \eta$ in (63) we obtain the identity (69).

Definition 2.6. We will say that the pair of morphisms $h, h^{-1} : H \to B$ is a gauge transformation for a measuring φ_B if they satisfy (61) and (62).

By the previous Theorem 2.4 we know that, under suitable conditions, equivalent weak crossed products are related by gauge transformations. After the next discussion, we should be able to secure that the converse is also true.

2.7. Let (h, h^{-1}) be a gauge transformation for a measuring φ_B and let σ be a morphism satisfying the conditions of Definition 2.1. Suppose that υ is a preunit for the associated weak crossed product $(B \otimes H, \mu_{B\otimes^{\sigma}_{\varphi_B} H})$. Define θ and γ as in (68). Then θ and γ are morphisms of right H-comodules and (53) and (54) hold. Define $\varphi_B^h : H \otimes B \to B$ and $\sigma^h : H^2 \to B$ by

$$\varphi_B^h = \mu \circ (\mu \otimes h^{-1}) \circ (B \otimes P_{\varphi_B})) \circ (\theta \otimes B), \tag{70}$$

$$\sigma^h = \mu \circ (B \otimes h^{-1}) \circ \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (\theta \otimes \theta). \tag{71}$$

The morphism φ_B^h is a measuring and $P_{\varphi_B^h} = (\mu \otimes H) \circ (\mu \otimes \gamma) \circ (B \otimes P_{\varphi_B}) \circ (\theta \otimes B)$. Therefore, (55) and

$$u_1^{\varphi_B^h} = h * h^{-1} \tag{72}$$

hold. On the other hand, for F_{σ^h} we have the identity (56), i.e., $F_{\sigma^h} = (\mu \otimes H) \circ (B \otimes \gamma) \circ \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (\theta \otimes \theta)$, and the equality $\sigma^h * u_2^{\varphi_B^h} = \sigma^h$. Finally, the quadruple $(B, H, P_{\varphi_B^h}, F_{\sigma^h})$ satisfies the twisted and the cocycle conditions. Moreover, if $\upsilon^h := (\mu \otimes H) \circ (B \otimes \gamma) \circ \upsilon$, (57) holds trivially and we have that υ^h is a preunit for the weak crossed product $(B \otimes H, \mu_{B\otimes^{\sigma^h}_{\varphi_B^h} H})$.

Therefore, as a consequence of the previous facts, we have a theorem that generalizes to the monoidal setting [15, Theorem 5.4].

Theorem 2.8. *Let φ_B, ϕ_B, σ, τ, υ and u be morphisms satisfying the conditions of Definition 2.1. The weak crossed products $(B \otimes H, \mu_{B\otimes^{\sigma}_{\varphi_B} H})$ and $(B \otimes H, \mu_{B\otimes^{\tau}_{\phi_B} H})$ are equivalent iff there exists a gauge transformation (h, h^{-1}) for φ_B such that $\phi_B = \varphi_B^h$, $\tau = \sigma^h$ and $u = \upsilon^h$.*

3. Regular morphisms

Definition 3.1. Let H be a weak Hopf algebra and φ_B be a measuring. We say that a morphism $h : H \to B$ is a regular morphism if there exists $h^{-1} : H \to B$, called the convolution inverse of h, such that the pair (h, h^{-1}) is a gauge transformation for φ_B and

$$h * h^{-1} = u_1^{\varphi_B}, \tag{73}$$

holds. We will denote by $Reg_{\varphi_B}(H,B)$ the set of regular morphisms, that it is a group with product the convolution product and unit $u_1^{\varphi_B}$.

3.2. Let $\mathcal{P}_{\varphi_B}$ be the set of all pairs (ϕ_B,τ), where:

(i) The morphism ϕ_B is a measuring satisfying $u_1^{\phi_B}=u_1^{\varphi_B}$.
(ii) The morphism $\tau : H^2\to B$ is such that $\tau=\tau * u_2^{\phi_B}$ and the associated quadruple (B,H,P_{ϕ_B},F_τ) satisfies the twisted condition and the cocycle condition.
(iii) The associated weak crossed product $(B\otimes H,\mu_{B\otimes_{\phi_B}^{\tau}H})$ admits a preunit υ.

By the results proved in the previous section we know that $Reg_{\varphi_B}(H,B)$ acts on $\mathcal{P}_{\varphi_B}$. The action $R : Reg_{\varphi_B}(H,B)\times\mathcal{P}_{\varphi_B}\to\mathcal{P}_{\varphi_B}$, is defined by $R(h,(\phi_B,\tau))=(\phi_B^h,\tau^h)$.

Proposition 3.3. *Let (B,φ_B), (B,ϕ_B) be left weak H-module algebras.*

(1) *Let $h : H\to B$ be a morphism such that $h*u_1^{\varphi_B}=h=u_1^{\phi_B}*h$. Then, the following assertions are equivalent:*

$$\text{(i)}\quad h\circ\eta=\eta,\qquad \text{(ii)}\quad h\circ\Pi_H^L=u_1^{\phi_B},\qquad \text{(iii)}\quad h\circ\overline{\Pi}_H^L=u_1^{\varphi_B}.$$

If one (hence any) of the previous condition is satisfied we have:

$$(\mu\otimes H)\circ(B\otimes((h\otimes H)\circ\delta\circ\eta))=\nabla_{BH}^{\varphi_B}\circ(B\otimes\eta). \tag{74}$$

(2) *If $g : H\to B$ is a morphism such that $g*u_1^{\phi_B}=g=u_1^{\varphi_B}*g$, the following assertions are equivalent:*

$$\text{(iv)}\quad g\circ\eta=\eta,\qquad \text{(v)}\quad g\circ\Pi_H^L=u_1^{\varphi_B},\qquad \text{(vi)}\quad g\circ\overline{\Pi}_H^L=u_1^{\phi_B}.$$

If one (hence any) of the previous conditions is satisfied the identity

$$(\mu\otimes H)\circ(B\otimes((g\otimes H)\circ\delta\circ\eta))=\nabla_{BH}^{\phi_B}\circ(B\otimes\eta) \tag{75}$$

holds.
(3) *If there exists $h^{-1} : H\to B$ such that (h,h^{-1}) is a gauge transformation for φ_B and $h*h^{-1}=u_1^{\phi_B}$ holds, we have $h\circ\eta=\eta$ iff $h^{-1}\circ\eta=\eta$.*

Proof. By the properties of η and Definition 1.3(b2), we obtain that (ii) $\Rightarrow$ (i), and (iii) $\Rightarrow$ (i). Also, (i) $\Rightarrow$ (ii) holds because,

$h\circ\Pi_H^L=(u_1^{\phi_B}*h)\circ\Pi_H^L$ (by $h=u_1^{\phi_B}*h$)
$=\mu\circ((u_1^{\phi_B}\circ\mu\circ(H\otimes\Pi_H^L))\otimes h)\circ(H\otimes c)\circ((\delta\circ\eta)\otimes H)$ (by naturality of c, coassociativity of δ and (4))

$= \mu \circ ((\phi_B \circ (H \otimes u_1^{\phi_B})) \otimes h) \circ (H \otimes c) \circ ((\delta \circ \eta) \otimes H)$ (by Definition 1.3(b3) and (16))
$= \mu \circ ((\phi_B \circ (\overline{\Pi}_H^L \otimes B)) \otimes h) \circ (H \otimes c) \circ ((\delta \circ \eta) \otimes u_1^{\phi_B})$ (by naturality of c and (8))
$= \mu \circ ((\mu \circ c_{B,B} \circ (u_1^{\phi_B} \otimes B)) \otimes h) \circ (H \otimes c) \circ ((\delta \circ \eta) \otimes u_1^{\phi_B})$ (by (14))
$= \mu \circ c_{B,B} \circ ((h \circ \eta) \otimes u_1^{\phi_B})$ (by naturality of c, associativity of μ and $h = u_1^{\phi_B} * h$)
$= u_1^{\phi_B}$ (by (i), naturality of c, and properties of η).

On the other hand, the proof of (i) ⇒ (iii) is similar to (i) ⇒ (ii) using (6) and (13) instead (4) and (14) respectively. Therefore, we have (i) ⇔ (ii) ⇔ (iii). As a consequence of these equivalences, by (8), (iii) and (21), we obtain (74). The proof for 2 is similar and we leave the details to the reader.

Finally, assume that there exists $h^{-1} : H \to B$ such that (h, h^{-1}) is a gauge transformation for φ_B and $h * h^{-1} = u_1^{\phi_B}$ holds. If $h \circ \eta = \eta$, we have

$h^{-1} \circ \eta = (u_1^{\varphi_B} * h^{-1}) \circ \eta$ (by $u_1^{\varphi_B} * h^{-1} = h^{-1}$)
$= ((h \circ \overline{\Pi}_H^L) * h^{-1}) \circ \eta$ (by (iii))
$= u_1^{\phi_B} \circ \eta$ (by (8) and $u_1^{\phi_B} = h * h^{-1}$)
$= \eta$ (by Definition 1.3(b2)).

Conversely, if $h^{-1} \circ \eta = \eta$, by similar arguments, $h \circ \eta = (h * u_1^{\varphi_B}) \circ \eta = (h * (h^{-1} \circ \Pi_H^L)) \circ \eta = u_1^{\phi_B} \circ \eta = \eta$. □

As a particular instance of the previous proposition we have the following corollary.

Corollary 3.4. *Let (B, φ_B) be a left weak H-module algebra and let $h : H \to B$ be a morphism such that $h * u_1^{\varphi_B} = h = u_1^{\varphi_B} * h$. Then, the following assertions are equivalent:*

$$\text{(i)} \quad h \circ \eta = \eta, \qquad \text{(ii)} \quad h \circ \Pi_H^L = u_1^{\varphi_B}, \qquad \text{(iii)} \quad h \circ \overline{\Pi}_H^L = u_1^{\varphi_B}.$$

Moreover, if $h \in Reg_{\varphi_B}(H, B)$ with convolution inverse $h^{-1} : H \to B$, we have $h \circ \eta = \eta$ iff $h^{-1} \circ \eta = \eta$. Then, under these conditions, if $h \circ \eta = \eta$, the following assertions hold:

$$\text{(iv)} \quad h^{-1} \circ \eta = \eta, \qquad \text{(v)} \quad h^{-1} \circ \Pi_H^L = u_1^{\varphi_B}, \qquad \text{(vi)} \quad h^{-1} \circ \overline{\Pi}_H^L = u_1^{\varphi_B}.$$

Definition 3.5. Let φ_B be a measuring. With $Reg_{\varphi_B}^t(H, B)$ we will denote the set of morphisms $h : H \to B$ in $Reg_{\varphi_B}(H, B)$ such that $h \circ \eta = \eta$.

Remark 3.6. Assume that (B, φ_B) is a left weak H-module algebra, and let $h, l \in Reg_{\varphi_B}^t(H, B)$. Since, by (8) and Corollary 3.4, $(h * l^{-1}) \circ \eta = ((h \circ \overline{\Pi}_H^L) * (l^{-1} \circ \Pi_H^L)) = (u_1^{\varphi_B} * u_1^{\varphi_B}) \circ \eta = u_1^{\varphi_B} \circ \eta = \eta$, $Reg_{\varphi_B}^t(H, B)$ is a subgroup of $Reg_{\varphi_B}(H, B)$.

Remark 3.7. The set $Reg^t_{\varphi_B}(H,B)$ also acts on $\mathcal{P}_{\varphi_B}$, i.e., we have a map $R' : Reg^t_{\varphi_B}(H,B)\times\mathcal{P}_{\varphi_B}\to\mathcal{P}_{\varphi_B}$, defined by $R'(h,(\phi_B,\tau)) = R(h,(\phi_B,\tau))$, where R is the action defined in 3.2.

Note that, if (B,φ_B) is a left weak H-module algebra, the measuring φ^h_B defined in (70) satisfies Definition 1.3(b2) because we have

$\varphi^h_B\circ(\eta\otimes B) = \mu\circ((\mu\circ((h\circ\overline{\Pi}^L_H)\otimes B))\otimes h^{-1})\circ(H\otimes P_{\varphi_B})\circ((\delta\circ\eta)\otimes B)$ (by (8))
$=\mu\circ((\mu\circ(u_1^{\varphi_B}\otimes B))\otimes h^{-1})\circ(H\otimes P_{\varphi_B})\circ((\delta\circ\eta)\otimes B)$ (by (i) $\Rightarrow$ (iii) of Corollary 3.4)
$=\mu\circ(\mu\otimes h^{-1})\circ(B\otimes P_{\varphi_B})\circ((P_{\varphi_B}\circ(\eta\otimes\eta))\otimes B)$ (by (20))
$=\mu\circ(B\otimes h^{-1})\circ P_{\varphi_B}\circ(\eta\otimes B)$ (by (19) and properties of η)
$=\mu\circ(\varphi_B\otimes B)\circ(H\otimes c)\circ(((H\otimes(h^{-1}\circ\Pi^L_H))\circ\delta\circ\eta)\otimes B)$ (by (8) and naturality of c)
$=\mu\circ(B\otimes u_1^{\varphi_B})\circ P_{\varphi_B}\circ(\eta\otimes B)$ (by Corollary 3.4(v) and naturality of c)
$=id_B$ (by (28) for ϕ_B and Definition 1.3(b2)).

Theorem 3.8. *Let (B,φ_B), (B,ϕ_B) be left weak H-module algebras and let σ,τ be morphisms satisfying the conditions of Definition 2.1 and the normal condition (47). The following assertions are equivalent:*

(i) *The weak crossed products $(B\otimes H,\mu_{B\otimes^\sigma_{\varphi_B}H})$ and $(B\otimes H,\mu_{B\otimes^\tau_{\phi_B}H})$ are equivalent.*
(ii) *There exists a gauge transformation (h,h^{-1}) for φ_B such that (69),*

$$h\circ\eta=\eta, \tag{76}$$

$$\mu\circ(B\otimes h)\circ P_{\phi_B} = \mu\circ(h\otimes\varphi_B)\circ(\delta\otimes B), \tag{77}$$

$$\mu\circ(B\otimes h)\circ F_\tau = \mu\circ(\mu\otimes\sigma)\circ(B\otimes P_{\varphi_B}\otimes H)\circ(((h\otimes H)\circ\delta)\otimes((h\otimes H)\circ\delta)), \tag{78}$$

hold.

Proof. We first prove (i) $\Rightarrow$ (ii). By Corollary 1.18, we know that $(B\otimes H,\mu_{B\otimes^\sigma_{\varphi_B}H})$ and $(B\otimes H,\mu_{B\otimes^\tau_{\phi_B}H})$ are weak crossed products with preunits $\nu=\nabla^{\varphi_B}_{BH}\circ(\eta\otimes\eta)$, $u=\nabla^{\phi_B}_{BH}\circ(\eta\otimes\eta)$, respectively. Define θ,γ as in the proof of Theorem 2.4 and h, h^{-1} by (67). Then, using that T,S are morphisms of left B-modules and (68) we have the following identities:

$$(B\otimes\varepsilon)\circ T=\mu\circ(B\otimes h^{-1}),\;\;(B\otimes\varepsilon)\circ S=\mu\circ(B\otimes h). \tag{79}$$

By (i) $\Rightarrow$ (ii) of Theorem 2.4, the pair (h,h^{-1}) is a gauge transformation for φ_B and the identities (63), (64) and (65) hold. Therefore, by Remark (2.5) we obtain that (69) holds. Moreover, (76) follows by (79), (60), the naturality of c, the counit properties and Definition 1.3(b2).

Now, by the proof of (i) ⇒ (ii) of Theorem 2.4 we know that S is multiplicative, i.e.,

$$S \circ \mu_{B\otimes^{\tau}_{\phi_B} H} = \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (S \otimes S). \tag{80}$$

In fact this is the version of identity (51) for S (see ([10, (37)])).

Then composing with $\eta \otimes H \otimes B \otimes \eta$ in the previous identity we have

$S \circ \mu_{B\otimes^{\tau}_{\phi_B} H} \circ (\eta \otimes H \otimes B \otimes \eta)$
$= (\mu \otimes H) \circ (\mu \otimes \theta) \circ (B \otimes (F_\tau \circ (H \otimes \eta))) \circ P_{\phi_B}$ (by (66) and properties of η)
$= (\mu \otimes H) \circ (\mu \otimes \theta) \circ (B \otimes (\nabla^{\phi_B}_{BH} \circ (\eta \otimes H))) \circ P_{\phi_B}$ (by (49))
$= (\mu \otimes H) \circ (B \otimes \theta) \circ P_{\phi_B}$ (by left B-linearity of $\nabla^{\phi_B}_{BH}$, properties of η and (20))

and

$\mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (S \otimes S) \circ (\eta \otimes H \otimes B \otimes \eta)$
$= \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (((h \otimes H) \circ \delta) \otimes (((\mu \otimes H) \circ (B \otimes ((h \otimes H) \circ \delta \circ \eta)))$ (by (66), (68), and properties of η)
$= \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (((h \otimes H) \circ \delta) \otimes (\nabla^{\varphi_B}_{BH} \circ (B \otimes \eta)))$ (by (74))
$= \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (((h \otimes H) \circ \delta) \otimes B \otimes \eta)))$ (by (40))
$= (\mu \otimes H) \circ (\mu \otimes (\nabla^{\varphi_B}_{BH} \circ (\eta \otimes H))) \circ (B \otimes P_{\varphi_B}) \circ (((h \otimes H) \circ \delta) \otimes B)$ (by (49))
$= (\mu \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (((h \otimes H) \circ \delta) \otimes B)$ (by left B-linearity of $\nabla^{\varphi_B}_{BH}$, properties of η and (20)).

Therefore,

$$(\mu \otimes H) \circ (B \otimes \theta) \circ P_{\phi_B} = (\mu \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (((h \otimes H) \circ \delta) \otimes B) \tag{81}$$

holds and, composing with $B \otimes \varepsilon$ in (81), we obtain (77) by the naturality of c, (67) and (20).

Finally, composing with $\eta \otimes H \otimes \eta \otimes H$ in (80), by (20), (21), (37), (66), (68) and the properties of η, we have $S \circ \mu_{B\otimes^{\tau}_{\phi_B} H} \circ (\eta \otimes H \otimes \eta \otimes H) = (\mu \otimes H) \circ (B \otimes ((h \otimes H) \circ \delta)) \circ F_\tau$ and, by (66), properties of η, and (68), we obtain $\mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (S \otimes S) \circ (\eta \otimes H \otimes \eta \otimes H) = \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (((h \otimes H) \circ \delta) \otimes ((h \otimes H) \circ \delta))$. As a consequence, $(\mu \otimes H) \circ (B \otimes ((h \otimes H) \circ \delta)) \circ F_\tau = \mu_{B\otimes^{\sigma}_{\varphi_B} H} \circ (((h \otimes H) \circ \delta) \otimes ((h \otimes H) \circ \delta))$ holds and, composing with $B \otimes \varepsilon$, we obtain (78) by the counit properties and (31).

Conversely, suppose that (ii) holds. In light of (ii) ⇒ (i) of Theorem 2.4 we only need to prove equalities (63), (64) and (65). Indeed, note that by Proposition 3.3, $h^{-1} \circ \eta = \eta$ because (76) holds. Then, (63) is satisfied because

$\mu \circ (\mu \otimes h^{-1}) \circ (h \otimes P_{\varphi_B}) \circ (\delta \otimes B)$
$= \mu \circ (B \otimes (\mu \circ (h \otimes h^{-1}))) \circ (P_{\phi_B} \otimes H) \circ (H \otimes c) \circ (\delta \otimes B)$ (by (77), coassociativity of δ and associativity of μ)

$= \mu\circ(B\otimes u_1^{\phi_B})\circ P_{\phi_B}$ (by naturality of c, coassociativity of δ and (69))
$= \phi_B$ (by (28) for ϕ_B).

On the other hand,

$\mu\circ(B\otimes h^{-1})\circ\mu_{B\otimes^{\sigma}_{\varphi_B}H}\circ(((h\otimes H)\circ\delta)\otimes((h\otimes H)\circ\delta))$
$= \mu\circ(B\otimes(\mu\circ(h\otimes h^{-1})))\circ(F_\tau\otimes\mu)\circ\delta_{H^{\otimes 2}}$ (by naturality of c, coassociativity of δ, (78) and associativity of μ)
$= \mu\circ(B\otimes(h*h^{-1}))\circ F_\tau$ (by (27))
$= \mu\circ(B\otimes u_1^{\varphi_B})\circ F_\tau$ (by (69))
$= \tau$ (by (29))

and then (64) holds. Finally, we obtain (65) because

$((\mu\circ(B\otimes h^{-1}))\otimes H)\circ(B\otimes\delta)\circ\nabla_{BH}^{\varphi_B}\circ(\eta\otimes\eta)$
$= ((u_1^{\varphi_B}*h^{-1})\otimes H)\circ\delta\circ\eta$ (by (22) and by the coassociativity of δ)
$= ((h^{-1}\circ\overline{\Pi}_H^L)\otimes H)\circ\delta\circ\eta$ (by the gauge transformation condition and (8))
$= (u_1^{\phi_B}\otimes H)\circ\delta\circ\eta$ (by Proposition 3.3(vi))
$= \nabla_{BH}^{\phi_B}\circ(\eta\otimes\eta)$ (by (22)). □

3.9. As a consequence of the previous theorem, it is possible to define a groupoid, denoted by $\mathcal{G}_H^B$ whose objects are pairs (φ_B,σ), where (B,φ_B) is a left weak H-module algebra, $\sigma: H^2\to B$ is a morphism such that $u_2^{\varphi_B}*\sigma=\sigma=\sigma*u_2^{\varphi_B}$ and the associated quadruple $\mathbb{B}_H$ satisfies the twisted, cocycle and normal conditions. A morphism between two objects (φ_B,σ), (ϕ_B,τ) of $\mathcal{G}_H^B$ is defined by a morphism $h: H\to B$ for which there exists a morphism $h^{-1}: H\to B$ such that (h,h^{-1}) is a gauge transformation for φ_B satisfying the conditions (ii) of Theorem 3.8. The identity of (φ_B,σ) is $id_{(\varphi_B,\sigma)}=u_1^{\varphi_B}$ and, if $h:(\varphi_B,\sigma)\to(\phi_B,\tau)$, $g:(\phi_B,\tau)\to(\chi_B,\omega)$ are morphisms in $\mathcal{G}_H^B$, the composition, denoted by $g\circledcirc h$, is defined by $g\circledcirc h=g*h$ with $(g\circledcirc h)^{-1}=h^{-1}*g^{-1}$. We left the details to the reader (use (66), (68), that $(\mu\otimes H)\circ(B\otimes l\otimes H)\circ(B\otimes\delta)\circ(B\otimes\mu\otimes H)\circ(B\otimes h\otimes H)\circ(B\otimes\delta)=(\mu\otimes H)\circ(B\otimes(h*l)\otimes H)\circ(B\otimes\delta)$).

3.10. Let (B,φ_B) be a left weak H-module algebra and let σ be a morphism satisfying the conditions of Definition 2.1 and the normal condition (47). Let h be a morphism in $Reg^t_{\varphi_B}(H,B)$. Then (h,h^{-1}) is a gauge transformation for φ_B such that (73) and (76) hold. Define φ_B^h and σ^h as in (70) and (71) respectively. Then, by 2.7, φ_B^h is a measuring such that (72) holds. Therefore $u_1^{\varphi_B}=u_1^{\varphi_B^h}$ and then

$$\nabla_{BH}^{\varphi_B^h}=\nabla_{BH}^{\varphi_B}. \tag{82}$$

Moreover, by Remark 3.7, we know that φ_B^h satisfies (b2) of Definition 1.3. On the other hand, σ^h is such that $\sigma^h*u_2^{\varphi_B^h}=\sigma^h$ and satisfies the twisted condition (38), the cocycle condition (39) and $\nu^h = (\mu \otimes H) \circ (B \otimes ((h^{-1} \otimes H) \circ \delta)) \circ \nabla_{BH}^{\varphi_B} \circ (\eta \otimes \eta)$ is a preunit for the associated weak crossed product $(B \otimes H, \mu_{B \otimes_{\varphi_B^h}^{\sigma^h} H})$. Note that

$\nu^h = (h^{-1} \otimes H) \circ \delta \circ \eta$ (by coassociativity of δ and the condition of gauge transformation)
$= ((h^{-1} \circ \overline{\Pi}_H^L) \otimes H) \circ \delta \circ \eta$ (by (8))
$= (u_1^{\varphi_B} \otimes H) \circ \delta \circ \eta$ (by Corollary 3.4(vi))
$= \nabla_{BH}^{\varphi_B^h} \circ (\eta \otimes \eta)$ (by (22) and (82)).

Therefore, $\nu^h = \nabla_{BH}^{\varphi_B^h} \circ (\eta \otimes \eta) = \nabla_{BH}^{\varphi_B} \circ (\eta \otimes \eta) = \nu$. Also, φ_B^h satisfies (14) because:

$\varphi_B^h \circ (\overline{\Pi}_H^L \otimes B)$
$= \mu \otimes (\mu \otimes h^{-1}) \circ (B \otimes P_{\varphi_B})) \circ (((h \circ \overline{\Pi}_H^L) \otimes H) \circ \delta \circ \overline{\Pi}_H^L) \otimes B)$ (by (3) and (77) for h^{-1})
$= \mu \otimes (\mu \otimes h^{-1}) \circ (u_1^{\varphi_B} \otimes P_{\varphi_B})) \circ ((\delta \circ \overline{\Pi}_H^L) \otimes B)$ (by (ii) of Corollary 3.4)
$= \mu \otimes (B \otimes h^{-1}) \circ P_{\varphi_B} \circ (\overline{\Pi}_H^L \otimes B)$ (by (26))
$= \mu \circ ((\varphi_B \circ (\overline{\Pi}_H^L \otimes B)) \otimes h^{-1}) \circ (H \otimes c) \circ ((\delta \circ \overline{\Pi}_H^L) \otimes B)$ (by (3))
$= \mu \circ ((\mu \circ c \circ (u_1^{\varphi_B} \otimes B)) \otimes h^{-1}) \circ (H \otimes c) \circ ((\delta \circ \overline{\Pi}_H^L) \otimes B)$ (by (14) for φ_B)
$= \mu^{op} \circ ((h^{-1} \circ \overline{\Pi}_H^L) \otimes B)$ (by associativity of μ, naturality of c and the condition of gauge transformation)
$= \mu^{op} \circ (u_1^{\varphi_B^h} \otimes B)$ (by Corollary 3.4(vi) and $u_1^{\varphi_B} = u_1^{\varphi_B^h}$)

As a consequence, we obtain that (48) holds for φ_B^h. Therefore, by Corollary 1.17, we have that σ^h satisfies the normal condition (47). Finally, if B is commutative and H is cocommutative, the equality

$$\mu \circ (B \otimes h) \circ P_{\varphi_B} = \mu \circ (h \otimes \varphi_B) \circ (\delta \otimes B) \tag{83}$$

holds and then by the usual arguments, (83), (73) and (28) we have that $\varphi_B^h = \varphi_B$.

4. Hom-products, invertible morphisms and centers

In this section, for a weak Hopf algebra H and an algebra B, we will explore a new product in $Hom_{\mathsf{C}}(H^{\otimes n} \otimes B, B)$ that will permit us to extend some results about the factorization through the center of B, given in [14] for Hopf algebras, to the weak Hopf algebra setting.

Definition 4.1. Let φ and $\psi \in Hom_{\mathsf{C}}(H^n \otimes B, B)$. We define the product

$$\wedge : Hom_{\mathsf{C}}(H^n \otimes B, B) \times Hom_{\mathsf{C}}(H^n \otimes B, B) \to Hom_{\mathsf{C}}(H^n \otimes B, B)$$

between φ and ψ as $\varphi \wedge \psi := \varphi \circ (H^n \otimes \psi) \circ (\delta_{H^n} \otimes B)$.

Obviously, $\wedge$ is an associative product because δ_{H^n} is coassociative. We say that a morphism $\varphi \in Hom_{\mathsf{C}}(H^n \otimes B, B)$ is φ_B-invertible if there exists a morphism $\varphi^{\dagger} \in Hom_{\mathsf{C}}(H^n \otimes B, B)$ such that $\varphi \wedge \varphi^{\dagger} = \mu \circ (u_n^{\varphi_B} \otimes B)$.

Proposition 4.2. *Let φ_B be a measuring. For a morphism $\omega : H^n \to B$ define $\overline{\omega} := \mu \circ (\omega \otimes B)$ and $\overline{\omega}^{op} := \mu^{op} \circ (\omega \otimes B)$. Then, if $\omega, \theta \in Hom_{\mathsf{C}}(H^n, B)$ and $\gamma \in Hom_{\mathsf{C}}(H^n \otimes B, B)$ the following equalities hold:*

(i) $\overline{\omega} \wedge \overline{\theta} = \overline{\omega * \theta}$.
(ii) *If H is cocommutative,* $\overline{\omega}^{op} \wedge \gamma = \mu \circ (\gamma \otimes \omega) \circ (H^n \otimes c) \circ (\delta_{H^n} \otimes B)$.
(iii) *If H is cocommutative,* $\overline{\omega}^{op} \wedge \overline{\theta}^{op} = \overline{\theta * \omega}^{op}$.
(iv) *If H is cocommutative,* $\overline{\omega}^{op} \wedge \overline{\theta} = \overline{\theta} \wedge \overline{\omega}^{op}$.
(v) *If (B, φ_B) is a left weak H-module algebra,* $\overline{u_n^{\varphi_B}} \wedge \varphi_B^n = \varphi_B^n$.
(vi) *If H is cocommutative and (B, φ_B) is a left weak H-module algebra,* $\overline{u_n^{\varphi_B}}^{op} \wedge \varphi_B^n = \varphi_B^n$.
(vii) $\overline{\varphi_B \circ (H \otimes \omega)} \wedge (\varphi_B \circ (H \otimes \gamma)) = \varphi_B \circ (H \otimes (\overline{\omega} \wedge \gamma))$.
(viii) *If H is cocommutative,* $\overline{\varphi_B \circ (H \otimes \omega)}^{op} \wedge (\varphi_B \circ (H \otimes \gamma)) = \varphi_B \circ (H \otimes (\overline{\omega}^{op} \wedge \gamma))$.

Proof. The proof of (i) follows directly from the associativity of μ. If H is cocommutative, so is H^n and, by the naturality of c, we obtain (ii). By similar reasoning and using the associativity of μ we obtain (iii) and (iv). On the other hand,

$$\overline{u_n^{\varphi_B}} \wedge \varphi_B^n = \mu \circ (\varphi_B^n \otimes \varphi_B^n) \circ (H^n \otimes c \otimes B) \circ (\delta_{H^n} \otimes \eta \otimes B) \overset{(17)}{=} \varphi_B^n \circ (H^n \otimes (\mu \circ (\eta \otimes B))) = \varphi_B^n,$$

and then (v) holds. Similarly, using that H^n is cocommutative, the naturality of c and (17) we prove (vi). The identity, (vii) follows from the naturality of c and (17). Similarly, using that H is cocommutative, we obtain (viii). □

Remark 4.3. The equivalence of measurings (or, in particular, of weak actions) in terms of gauge transformations acquires a new meaning in terms of this product. Actually, if H is cocommutative, the action described in 3.2 on a measuring ϕ_B can be seen as a conjugation by gauge transformations in the following way: $\phi_B^h = \overline{h} \wedge \overline{h^{-1}}^{op} \wedge \phi_B$. Moreover observe that for a cocommutative weak Hopf algebra H and measurings φ_B and ϕ_B satisfying conditions of Theorem 2.4, we can re-write equality (63) using the Hom-product as $\phi_B = \overline{h} \wedge \overline{h^{-1}}^{op} \wedge \varphi_B$. Also in this way, equality (77) of Theorem 3.8 can be interpreted as $\overline{h}^{op} \wedge \phi_B = \overline{h} \wedge \varphi_B$, in coherence with the action of gauge transformations as a conjugation given above.

Definition 4.4. Let (B, φ_B) be a left weak H-module algebra. For $n \geq 1$, with $Reg_{\varphi_B}(H^n, B)$ we will denote the set of morphisms $\sigma : H^n \to B$ such that there exists a morphism $\sigma^{-1} : H^n \to B$ (the convolution inverse of σ) satisfying the equalities $\sigma * \sigma^{-1} = \sigma^{-1} * \sigma = u_n^{\varphi_B}$, $\sigma * \sigma^{-1} * \sigma = \sigma$ and $\sigma^{-1} * \sigma * \sigma^{-1} = \sigma^{-1}$.

Note that, for $n = 1$, we recover the group $Reg_{\varphi_B}(H, B)$ introduced in Definition 3.1. For any n, $Reg_{\varphi_B}(H^n, B)$ is a group with unit element $u_n^{\varphi_B}$ because by (18) we know that $u_n^{\varphi_B} * u_n^{\varphi_B} = u_n^{\varphi_B}$. Also, if B is commutative and H is cocommutative, we have that $Reg_{\varphi_B}(H^{\otimes n}, B)$ is an abelian group.

We denote by $Reg_{\varphi_B}(H_L, B)$ the set of morphisms $g : H_L \to B$ such that there exists a morphism $g^{-1} : H_L \to B$ (the convolution inverse of g) satisfying $g*g^{-1} = g^{-1}*g = u_0^{\varphi_B}$, $g * g^{-1} * g = g$ and $g^{-1} * g * g^{-1} = g^{-1}$, where $u_0^{\varphi_B} = u_1^{\varphi_B} \circ i_H^L$. Then by (15) we have $u_1^{\varphi_B} = u_0^{\varphi_B} \circ p_H^L$.

Definition 4.5. For an algebra B we define the center of B as a subobject $Z(B)$ of B with a monomorphism $z_B : Z(B) \to B$ satisfying the identity

$$\mu \circ (B \otimes z_B) = \mu^{op} \circ (B \otimes z_B) \tag{84}$$

and such that, if $f : A \to B$ is a morphism such that $\mu\circ(B\otimes f) = \mu^{op}\circ(B\otimes f)$, there exists an unique morphism $f' : A \to Z(B)$ satisfying $z_B \circ f' = f$. As a consequence, we obtain that $Z(B)$ is a commutative algebra, where $\eta_{Z(B)}$ is the unique morphism satisfying $z_B \circ \eta_{Z(B)} = \eta$ and $\mu_{Z(B)}$ is the unique morphism satisfying $z_B \circ \mu_{Z(B)} = \mu \circ (z_B \otimes z_B)$.

For example, if C is a closed category with equalizers and α_B and β_B are the unit and the counit, respectively, of the C-adjunction $B \otimes - \dashv [B, -] : \mathsf{C} \to \mathsf{C}$, the center of B can be obtained by the equalizer of $\vartheta_B = [B, \mu] \circ \alpha_B(B)$ and $\theta_B = [B, \mu^{op}] \circ \alpha_B(B)$. Then in the category of modules over a commutative ring the center is an equalizer object. Finally, note that by (84), composing with the symmetry isomorphism we obtain $\mu \circ (z_B \otimes B) = \mu^{op} \circ (z_B \otimes B)$.

Example 4.6. Assume that H is cocommutative and let (B, φ_B) be a left weak H-module algebra. Then, $\Pi_H^L = \overline{\Pi}_H^L$ and by (13) and (14) we have that $\mu^{op} \circ (u_1^{\varphi_B} \otimes B) = \mu \circ (u_1^{\varphi_B} \otimes B)$. Then, $u_1^{\varphi_B}$ factors through $Z(B)$. Therefore, there exists an unique morphism $v_1^{\varphi_B} : H \to Z(B)$ such that $z_B \circ v_1^{\varphi_B} = u_1^{\varphi_B}$. Then, taking into account the equality (18), we obtain $\mu^{op} \circ (u_n^{\varphi_B} \otimes B) = \mu \circ (u_n^{\varphi_B} \otimes B)$ and, as a consequence, $u_n^{\varphi_B}$ factors through $Z(B)$. Therefore, there exists an unique morphism $v_n^{\varphi_B} : H^2 \to Z(B)$ such that $z_B \circ v_n^{\varphi_B} = u_n^{\varphi_B}$.

Remark 4.7. Let $\omega : H^n \to B$ be a morphism. Then, ω factors through the center of B iff $\overline{\omega} = \overline{\omega}^{op}$. Therefore, if H is cocommutative and (B, φ_B) is a left weak H-module algebra, $\overline{u_n^{\varphi_B}} = \overline{u_n^{\varphi_B}}^{op}$ for all $n \geq 1$. Also, if ω factors through the center of B, then by items (i) and (iv) of Proposition 4.2, we have $\omega * \tau = \tau * \omega$ for all $\tau : H^n \to B$.

In the rest of this section we will assume that H is a cocommutative weak Hopf algebra and (B, φ_B) is a left weak H-module algebra.

Proposition 4.8. *A morphism $\sigma : H^2 \to B$ satisfies the twisted condition (38) iff*

$$\overline{\sigma}^{op} \wedge \varphi_B^2 = \overline{\sigma} \wedge (\varphi_B \circ (\mu \otimes B)) \tag{85}$$

holds.

Proof. The proof follows from the following facts: By definition of F_σ, we have that $\mu \circ (B \otimes \varphi_B) \circ (F_\sigma \otimes B) = \overline{\sigma} \wedge (\varphi_B \circ (\mu \otimes B))$. On the other hand, by cocommutativity of δ and naturality of c, we have $\mu \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B}) = \overline{\sigma}^{op} \wedge \varphi_B^2$. □

Proposition 4.9. *Assume that there exists $\sigma \in Reg_{\varphi_B}(H^2, B)$ satisfying the twisted condition (38). Then, φ_B is φ_B-invertible.*

Proof. Let h_σ and $h_{\sigma^{-1}}$ be the morphisms defined by $h_\sigma = \sigma \circ (H \otimes \lambda) \circ \delta$, $h_{\sigma^{-1}} = \sigma^{-1} \circ (H \otimes \lambda) \circ \delta$. Then, $h_\sigma \in Reg_{\varphi_B}(H, B)$ and $h_\sigma^{-1} = h_{\sigma^{-1}}$. Indeed, first note that

$h_\sigma * h_{\sigma^{-1}} = (\sigma * \sigma^{-1}) \circ (H \otimes \lambda) \circ \delta$ (by coassociativity and cocommutativity of δ, naturality of c and (10))
$= u_1^{\varphi_B} \circ \Pi_H^L$ (by (11) and the fact that $\sigma \in Reg_{\varphi_B}(H^2, B)$)
$= u_1^{\varphi_B}$ (by (15))

and similarly, $h_{\sigma^{-1}} * h_\sigma = u_1^{\varphi_B}$. Also, by the coassociativity and the cocommutativity of δ, the naturality of c, (10) and $\sigma \in Reg_{\varphi_B}(H^2, B)$ we have that $h_\sigma * h_{\sigma^{-1}} * h_\sigma = (\sigma * \sigma^{-1} * \sigma) \circ (H \otimes \lambda) \circ \delta = h_\sigma$. Similarly we obtain that $h_{\sigma^{-1}} * h_\sigma * h_{\sigma^{-1}} = h_{\sigma^{-1}}$ holds.

Now, let φ_σ be the morphism defined by $\varphi_\sigma := \mu \circ (\mu \otimes B) \circ (h_\sigma \otimes B \otimes h_{\sigma^{-1}}) \circ (H \otimes c) \circ (\delta \otimes H)$. Then, φ_σ is φ_B-invertible with inverse defined by $\varphi_\sigma^\dagger = \mu \circ (\mu \otimes B) \circ (h_{\sigma^{-1}} \otimes B \otimes h_\sigma) \circ (H \otimes c) \circ (\delta \otimes H)$. Indeed:

$\varphi_\sigma \wedge \varphi_\sigma^\dagger$
$= \mu \circ (u_1^{\varphi_B} \otimes (\mu^{op} \circ (u_1^{\varphi_B} \otimes B))) \circ (\delta \otimes B)$ (by coassociativity and cocommutativity of δ, naturality of c, associativity of μ and $h_\sigma \in Reg_{\varphi_B}(H, B)$)
$= \mu \circ ((u_1^{\varphi_B} * u_1^{\varphi_B}) \otimes B)$ (by the factorization of $u_1^{\varphi_B}$ through the center of B)
$= \mu \circ (u_1^{\varphi_B} \otimes B)$ (by (18)).

On the other hand, let be the morphism $\varphi_B \wedge (\varphi_B \circ (\lambda \otimes B))$. For this morphism we have the following:

$\overline{h_\sigma}^{op} \wedge (\varphi_B \wedge (\varphi_B \circ (\lambda \otimes B)))$
$= \mu \circ (\varphi_B \otimes \sigma) \circ (H \otimes c \otimes H) \circ (\delta \otimes (c \circ (H \otimes \varphi_B) \circ ((\delta \circ \lambda) \otimes B))) \circ (\delta \otimes B)$ (by coassociativity and cocommutativity of δ, naturality of c and (10))
$= \mu \circ (B \otimes \varphi_B) \circ (F_\sigma \otimes B) \circ (((H \otimes \lambda) \circ \delta) \otimes B)$ (by cocommutativity of δ, naturality of c and (38))

$= \mu \circ (B \otimes \varphi_B) \circ (((h_\sigma \otimes \Pi_H^L) \circ \delta) \otimes B)$ (by cocommutativity of δ, the naturality of c, (10) and (11))
$= \mu \circ (B \otimes \mu) \circ (((h_\sigma \otimes u_1^{\varphi_B}) \circ \delta) \otimes B)$ (by (13))
$= \overline{h_\sigma}$ (by associativity of μ and $h_\sigma \in Reg_{\varphi_B}(H, B)$)

and, as a consequence,

$$\varphi_\sigma = \varphi_B \wedge (\varphi_B \circ (\lambda \otimes B)) \tag{86}$$

holds. Indeed, on the one hand:

$\overline{h_{\sigma^{-1}}}^{op} \wedge (\overline{h_\sigma}^{op} \wedge (\varphi_B \wedge (\varphi_B \circ (\lambda \otimes B))))$
$= \overline{h_\sigma * h_{\sigma^{-1}}}^{op} \wedge (\varphi_B \wedge (\varphi_B \circ (\lambda \otimes B)))$ (by associativity of $\wedge$ and Proposition 4.2(iii))
$= \overline{u_1^{\varphi B}}^{op} \wedge (\varphi_B \wedge (\varphi_B \circ (\lambda \otimes B)))$ (by $h_\sigma \in Reg_{\varphi_B}(H, B)$)
$= \varphi_B \wedge (\varphi_B \circ (\lambda \otimes B))$ (by associativity of $\wedge$ and Proposition 4.2(vi))

and, on the other hand, by the cocommutativity of δ, the naturality of c and the associativity of μ, $\overline{h_{\sigma^{-1}}}^{op} \wedge \overline{h_\sigma} = \varphi_\sigma$ holds. Finally, define the morphism $\varphi_B^\dagger$ by $\varphi_B^\dagger := (\varphi_B \circ (\lambda \otimes B)) \wedge \varphi_\sigma^\dagger$. By (86) we have $\varphi_B \wedge \varphi_B^\dagger = \varphi_\sigma \wedge \varphi_\sigma^\dagger = \mu \circ (u_1^{\varphi_B} \otimes B)$ and then φ_B is φ_B-invertible with inverse $\varphi_B^\dagger$. □

Proposition 4.10. *If φ_B is φ_B-invertible, φ_B^n is φ_B^n-invertible.*

Proof. By assumption the assertion is true for $n = 1$. Then we will proceed by induction. Define $\varphi_B^{n\dagger}$ by $\varphi_B^{n\dagger} := \varphi_B^{(n-1)\dagger} \circ (H^{(n-1)} \otimes \varphi_B^\dagger) \circ (c_{H,H^{\otimes(n-1)}} \otimes B)$. Then,

$\varphi_B^{\otimes n} \wedge \varphi_B^{\otimes n\dagger}$
$= \varphi_B \circ (H \otimes (\mu \circ (u_{n-1}^{\varphi_B} \otimes B))) \circ (H \otimes H^{n-1} \otimes \varphi_B^\dagger) \circ (H \otimes c \otimes B) \circ (\delta \otimes H^{n-1} \otimes B)$ (by naturality of c and the induction hypothesis)
$= \varphi_B \circ (H \otimes \mu) \circ (H \otimes \varphi_B^\dagger \otimes B) \circ (\delta \otimes (c \circ (u_{n-1}^{\varphi_B} \otimes B)))$ (by the factorization of $u_{n-1}^{\varphi_B}$ through the center of B and the naturality of c)
$= \mu \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c \otimes B) \circ (\delta \otimes \varphi_B^\dagger \otimes B) \circ (\delta \otimes (c \circ (u_{n-1}^{\varphi_B} \otimes B)))$ (by Definition 1.3(b1))
$= \mu \circ ((\mu \circ (u_1^{\varphi_B} \otimes B)) \otimes \varphi_B) \circ (H \otimes c \otimes B) \circ (\delta \otimes (c \circ (u_{n-1}^{\varphi_B} \otimes B)))$ (by naturality of c, coassociativity and cocommutativity of δ and the φ_B-invertivility of φ_B)
$= \mu \circ (B \otimes (\mu \circ (u_1^{\varphi_B} \otimes \varphi_B) \circ (\delta \otimes B))) \circ (c \otimes B) \circ (H \otimes (c \circ (u_{n-1}^{\varphi_B} \otimes B)))$ (by naturality of c, associativity of μ and the factorization of $u_1^{\varphi_B}$ through the center of B)
$= \mu \circ (B \otimes \varphi_B) \circ (c \otimes B) \circ (H \otimes (c \circ (u_{n-1}^{\varphi_B} \otimes B)))$ (by (26))
$= \mu \circ (u_n^{\varphi_B} \otimes B)$ (by naturality of c and the factorization of $u_n^{\varphi_B}$ through the center of B)

and, therefore, φ_B^n is φ_B^n-invertible. □

Proposition 4.11. *Assume that φ_B is φ_B-invertible. Then, a morphism $\omega \in Reg_{\varphi_B}(H^n, B)$ satisfies*

$$\overline{\omega} \wedge \varphi_B^n = \overline{\omega}^{op} \wedge \varphi_B^n \tag{87}$$

iff it factors through the center of B.

Proof. Assume that (87) holds. Then, by the associativity of μ and $\omega \in Reg_{\varphi_B}(H^n, B)$, we have $\overline{\omega} \wedge \varphi_B^n \wedge \varphi_B^{n\dagger} = \overline{\omega} \wedge (\mu \circ (u_n^{\varphi_B} \otimes B)) = \mu \circ ((\omega * u_n^{\varphi_B}) \otimes B) = \overline{\omega}$.

On the other hand,

$\overline{\omega}^{op} \wedge \varphi_B^n \wedge \varphi_B^{n\dagger} = \overline{\omega}^{op} \wedge (\mu \circ (u_n^{\varphi_B} \otimes B))$ (by the φ_B^n-invertivility)
$= \mu^{op} \circ (\omega \otimes (\mu \circ c \circ (u_n^{\varphi_B} \otimes B))) \circ (\delta_{H^n} \otimes B)$ (by the factorization of $u_n^{\varphi_B}$ through the center of B)
$= \overline{\omega}^{op}$ (by naturality of c, associativity of μ, cocommutativity of δ and $\omega \in Reg_{\varphi_B}(H^{\otimes n}, B)$).

Therefore, $\overline{\omega} = \overline{\omega}^{op}$ and, as a consequence, ω factors through the center of B.

Conversely, if ω factors through the center of B, by Remark 4.7, we have that $\overline{\omega} = \overline{\omega}^{op}$ and then (87) holds trivially. □

Proposition 4.12. *Assume that φ_B is φ_B-invertible. Then, if $\omega \in Reg_{\varphi_B}(H^n, B)$ satisfies (87), ω^{-1} also satisfies (87). As a consequence, ω^{-1} factors through the center of B.*

Proof. By the equalities of Proposition 4.2, Remark 4.7 and Proposition 4.11, the following equalities:

$$\overline{\omega} \wedge \overline{\omega^{-1}} = \overline{\omega * \omega^{-1}} = \overline{u_n^{\varphi_B}} = \overline{u_n^{\varphi_B}}^{op} = \overline{\omega^{-1} * \omega}^{op} = \overline{\omega}^{op} \wedge \overline{\omega^{-1}}^{op} = \overline{\omega} \wedge \overline{\omega^{-1}}^{op}$$

hold. Then, we have that

$$\overline{\omega^{-1}} \wedge \varphi_B^{\otimes n} = \overline{\omega^{-1}} \wedge \overline{\omega} \wedge \overline{\omega^{-1}}^{op} \wedge \varphi_B^{\otimes n} = \overline{u_n^{\varphi_B}} \wedge \overline{\omega^{-1}}^{op} \wedge \varphi_B^{\otimes n}$$
$$= \overline{\omega^{-1}}^{op} \wedge \overline{u_n^{\varphi_B}} \wedge \varphi_B^{\otimes n} = \overline{\omega^{-1}}^{op} \wedge \varphi_B^{\otimes n}.$$

Therefore, ω^{-1} satisfies (87) and, by the previous proposition, ω^{-1} factors through the center of B. □

Proposition 4.13. *If there exists $\sigma \in Reg_{\varphi_B}(H^2, B)$ satisfying the twisted condition (38), the action φ_B induces a left H-module algebra structure on the center of B, where the action $\varphi_{Z(B)}$ is the factorization of $\varphi_B \circ (H \otimes z_B) : H \otimes Z(B) \to B$ through the center of B.*

Proof. First note that, by (84), Definition 1.3(b1), the cocommutativity of δ and the naturality of c, the identity

$$\mu\circ(\varphi_B\otimes\varphi_B)\circ(H\otimes c\otimes B)\circ(\delta_B\otimes z_B\otimes B)=\mu^{op}\circ(\varphi_B\otimes\varphi_B)\circ(H\otimes c\otimes B)\circ(\delta_B\otimes z_B\otimes B) \quad (88)$$

holds. Then, on the one hand, by the associativity of μ and (26)

$$\mu\circ(\varphi_B\otimes(\mu\circ(u_1^{\varphi_B}\otimes B)))\circ(H\otimes c\otimes B)\circ(\delta\otimes z_B\otimes B)=\mu\circ((\varphi_B\circ(H\otimes z_B))\otimes B)$$

holds and, on the other hand,

$\mu\circ(\varphi_B\otimes(\mu\circ(u_1^{\varphi_B}\otimes B)))\circ(H\otimes c\otimes B)\circ(\delta\otimes z_B\otimes B)$
$=\mu\circ(\varphi_B\otimes(\varphi_B\wedge\varphi_B^{\dagger}))\circ(H\otimes c\otimes B)\circ(\delta\otimes z_B\otimes B)$ (by φ_B-invertivility of φ_B)
$=\mu\circ c\circ(\varphi_B\otimes\varphi_B)\circ(H\otimes c\otimes\varphi_B^{\dagger})\circ(\delta\otimes c\otimes B)\circ(\delta\otimes z_B\otimes B)$ (by coassociativity of δ, naturality of c and (88))
$=\mu\circ(\mu\otimes B)\circ(B\otimes c)\circ((c\circ(\varphi_B\otimes u_1^{\varphi_B})\circ(H\otimes c)\circ(\delta\otimes z_B))\otimes B)$ (by the φ_B-invertivility of φ_B and naturality of c)
$=\mu\circ(B\otimes\mu)\circ(c\otimes B)\circ(B\otimes c)\circ(((u_1^{\varphi_B}\otimes\varphi_B)\circ(\delta\otimes z_B))\otimes B)$ (by naturality of c, cocommutativity of δ, (84) and associativity of μ)
$=\mu^{op}\circ((\varphi_B\circ(H\otimes z_B))\otimes B)$ (by naturality of c, (26) and the factorization of $u_1^{\varphi_B}$ through the center of B).

Therefore, as a consequence of the previous equalities, we have that there exists a unique morphism $\varphi_{Z(B)}:H\otimes Z(B)\to Z(B)$ such that $z_B\circ\varphi_{Z(B)}=\varphi_B\circ(H\otimes z_B)$. Using this last equality, it is an easy exercise to prove that $(Z(B),\varphi_{Z(B)})$ is a left H-module algebra and the details are left to the reader. □

Remark 4.14. Note that, under the conditions of the previous proposition, the equality

$$z_B\circ u_1^{\varphi_{Z(B)}}=u_1^{\varphi_B} \quad (89)$$

holds.

4.15. By [3, Theorem 3.1] we know that, if $\sigma\in Reg_{\varphi_A}(H^2,B)$ satisfies the twisted condition (38), (B,φ_B) is a left H-module algebra iff the morphism σ factorizes through the center of B. Moreover, by [3, Corollary 3.1], (B,φ_B) is a left H-module algebra iff the morphism $u_2^{\varphi_B}$ satisfies the twisted condition (38).

Proposition 4.16. *Let $\sigma\in Reg_{\varphi_B}(H^2,B)$ satisfying the twisted condition (38). Then, $\alpha\in Reg_{\varphi_B}(H^2,B)$ satisfies the twisted condition (38) iff there exists $\tau\in Reg_{\varphi_{Z(B)}}(H^2,Z(B))$ such that*

$$\alpha=(z_B\circ\tau)*\sigma. \quad (90)$$

Proof. Suppose that α satisfies the twisted condition (38). Then $\sigma * \alpha^{-1}$ factors through the center of B. Indeed, following Proposition 4.11, to prove it we will see that $\overline{\sigma * \alpha^{-1}} \wedge \varphi_B^2 = \overline{\sigma * \alpha^{-1}}^{op} \wedge \varphi_B^2$. First, note that $\varphi_B \circ (\mu \otimes B) = \overline{\sigma^{-1}} \wedge \overline{\sigma}^{op} \wedge \varphi_B^2$ because

$$\begin{aligned}
&\varphi_B \circ (\mu \otimes B) = \overline{u_2^{\varphi_B}} \wedge (\varphi_B \circ (\mu \otimes B)) \text{ (by Definition 1.1(a1) and (26))}\\
&= \overline{\sigma^{-1}} \wedge \overline{\sigma} \wedge (\varphi_B \circ (\mu \otimes B)) \text{ (by } \sigma \in Reg_{\varphi_B}(H^2, B) \text{ and Proposition 4.2(i))}\\
&= \overline{\sigma^{-1}} \wedge \overline{\sigma}^{op} \wedge \varphi_B^2 \text{ (by (85))}.
\end{aligned}$$

Thus, for α we have the same identity and then

$$\overline{\sigma^{-1}} \wedge \overline{\sigma}^{op} \wedge \varphi_B^2 = \overline{\alpha^{-1}} \wedge \overline{\alpha}^{op} \wedge \varphi_B^2 \tag{91}$$

holds. As a consequence,

$$\overline{\sigma}^{op} \wedge \varphi_B^2 = \overline{\alpha}^{op} \wedge \overline{\sigma * \alpha^{-1}} \wedge \varphi_B^2 \tag{92}$$

also holds since

$$\begin{aligned}
&\overline{\sigma}^{op} \wedge \varphi_B^2 = \overline{u_2^{\varphi_B}} \wedge \overline{\sigma}^{op} \wedge \varphi_B^2 \text{ (by Proposition 4.2(v)–(iv))}\\
&= \overline{\sigma} \wedge \overline{\sigma^{-1}} \wedge \overline{\sigma}^{op} \wedge \varphi_B^2 \text{ (by } \sigma \in Reg_{\varphi_B}(H^2, B) \text{ and Proposition 4.2(i))}\\
&= \overline{\sigma} \wedge \overline{\alpha^{-1}} \wedge \overline{\alpha}^{op} \wedge \varphi_B^2 \text{ (by (91))}\\
&= \overline{\alpha}^{op} \wedge \overline{\sigma * \alpha^{-1}} \wedge \varphi_B^2 \text{ (by items (i) and (iv) of Proposition 4.2)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\overline{\sigma * \alpha^{-1}}^{op} \wedge \varphi_B^2 = \overline{\alpha^{-1}}^{op} \wedge \overline{\sigma}^{op} \wedge \varphi_B^2 \text{ (by Proposition 4.2(iii))}\\
&= \overline{\alpha^{-1}}^{op} \wedge \overline{\alpha}^{op} \wedge \overline{\sigma * \alpha^{-1}} \wedge \varphi_B^2 \text{ (by (92))}\\
&= \overline{u_2^{\varphi_B}}^{op} \wedge \overline{\sigma * \alpha^{-1}} \wedge \varphi_B^2 \text{ (by } \alpha \in Reg_{\varphi_B}(H^2, B) \text{ and Proposition 4.2(iii))}\\
&= \overline{u_2^{\varphi_B}} \wedge \overline{\sigma * \alpha^{-1}} \wedge \varphi_B^2 \text{ (by Remark 4.7)}\\
&= \overline{\sigma * \alpha^{-1}} \wedge \overline{u_2^{\varphi_B}} \wedge \varphi_B^2 \text{ (by Remark 4.7 and Proposition 4.2(iv))}\\
&= \overline{\sigma * \alpha^{-1}} \wedge \varphi_B^2 \text{ (by Proposition 4.2(v))}
\end{aligned}$$

and this implies that $\sigma * \alpha^{-1}$ factors through the center of B. Then, by Proposition 4.12, the morphism $(\sigma * \alpha^{-1})^{-1} = \alpha * \sigma^{-1}$ also factors through the center of B. If τ is the factorization, we have that $z_B \circ \tau = \alpha * \sigma^{-1}$. Then, (90) holds.

Conversely, if (90) holds for $\tau \in Reg_{\varphi_{Z(B)}}(H^2, Z(B))$, we have that

$$\begin{aligned}
&\overline{\alpha}^{op} \wedge \varphi_B^2 = \overline{(z_B \circ \tau) * \sigma}^{op} \wedge \varphi_B^2 \text{ (by (90))}\\
&= \overline{\sigma}^{op} \wedge \overline{z_B \circ \tau}^{op} \wedge \varphi_B^2 \text{ (by Proposition 4.2(iii))}\\
&= \overline{\sigma}^{op} \wedge \overline{z_B \circ \tau} \wedge \varphi_B^2 \text{ (by the factorization through the center of } B)\\
&= \overline{z_B \circ \tau} \wedge \overline{\sigma}^{op} \wedge \varphi_B^2 \text{ (by Proposition 4.2(iv))}\\
&= \overline{z_B \circ \tau} \wedge \overline{\sigma} \wedge (\varphi_B \circ (\mu \otimes B)) \text{ (by (85))}
\end{aligned}$$

$$= \overline{(z_B \circ \tau) \circ \sigma} \wedge (\varphi_B \circ (\mu \otimes B)) \text{ (by Proposition 4.2(i))}$$

and, therefore, α satisfies the twisted condition. □

5. Cohomological obstructions in a weak setting

In the beginning of this section we review the basic facts about the Sweedler cohomology in a weak setting. This cohomology was introduced in [2] as a generalization of the classical Sweedler cohomology for Hopf algebras [17].

Let H be a cocommutative weak Hopf algebra and let (B, φ_B) be a left weak H-module algebra. The groups $Reg_{\varphi_B}(H_L, B)$ and $Reg_{\varphi_B}(H^n, B)$, introduced in the previous section, are the objects of the corresponding cosimplicial complex. Following [2] we define the coface operators as $\partial_{0,i} : Reg_{\varphi_B}(H_L, B) \to Reg_{\varphi_B}(H, B)$, $i \in \{0, 1\}$, where $\partial_{0,0}(g) = \varphi_B \circ (H \otimes (g \circ p_L \circ \Pi_H^R)) \circ \delta$, $\partial_{0,1}(g) = g \circ p_L$, and $\partial_{k,i} : Reg_{\varphi_B}(H^k, B) \to Reg_{\varphi_B}(H^{k+1}, B)$, $k \geq 1$, $i \in \{0, 1, \cdots, k+1\}$

$$\partial_{k,i}(\sigma) = \begin{cases} \varphi_B \circ (H \otimes \sigma), & i = 0 \\ \sigma \circ (H^{i-1} \otimes \mu \otimes H^{k-i}), & i \in \{1, \cdots, k\} \\ \sigma \circ (H^{\otimes(k-1)} \otimes (\mu \circ (H \otimes \Pi_H^L))), & i = k+1. \end{cases}$$

On the other hand, we define the codegeneracy operators by $s_{1,0} : Reg_{\varphi_B}(H, B) \to Reg_{\varphi_B}(H_L, B)$, by $s_{1,0}(h) = h \circ i_L$, and $s_{k+1,i} : Reg_{\varphi_B}(H^{k+1}, B) \to Reg_{\varphi_B}(H^k, B)$, $k \geq 1$, $i \in \{0, 1, \cdots, k\}$, $s_{k+1,i}(\sigma) = \sigma \circ (H^i \otimes \eta \otimes H^{k-i})$. Taking into account the codegeneracy operators, we define the groups

$$Reg^+_{\varphi_B}(H^{k+1}, B) = \bigcap_{i=0}^{k} Ker(s_{k+1,i}),$$

$$Reg^+_{\varphi_B}(H_L, B) = \{g \in Reg_{\varphi_B}(H_L, B) \; ; \; g \circ \eta_{H_L} = \eta\}.$$

Note that $Reg^+_{\varphi_B}(H^2, B)$ is the subgroup of $Reg_{\varphi_B}(H^2, B)$ formed by the elements satisfying the normal condition and $Reg^+_{\varphi_B}(H^3, B) = \{\sigma \in Reg_{\varphi_B}(H^3, B) \; ; \; \sigma \circ (\eta \otimes H^2) = \sigma \circ (H \otimes \eta \otimes H) = \sigma \circ (H^2 \otimes \eta) = u_2^{\varphi_B}\}$.

If (A, φ_A) is a left H-module algebra, by [2], the groups $Reg_{\varphi_A}(H_L, A)$ and $Reg_{\varphi_A}(H^n, A)$, $n \geq, 1$ are the objects of a cosimplicial complex of groups with the previous coface and codegeneracy operators. In this case, $D^k_{\varphi_A} = \partial_{k,0} * \partial_{k,1}^{-1} * \cdots * \partial_{k,k+1}^{(-1)^{k+1}} : Reg_{\varphi_A}(H^k, A) \to Reg_{\varphi_A}(H^{k+1}, A)$ denote the coboundary morphisms of the cochain complex associated to the cosimplicial complex $Reg_{\varphi_A}(H^\bullet, A)$.

5.1. If $\sigma \in Reg_{\varphi_B}(H^2, B)$, by [12, Proposition 5.5], the morphism $E(\sigma) : H^3 \to B$ defined by $E(\sigma) = \sigma \otimes \varepsilon$ satisfies the following identities:

$$E(\sigma) * u_3^{\varphi_B} = u_3^{\varphi_B} * E(\sigma) = \partial_{2,3}(\sigma). \tag{93}$$

Then, using that $\partial_{2,3}$ is a group morphism, we have $u_3^{\varphi_B} = \partial_{2,3}(\sigma)^{-1} * u_3^{\varphi_B} * E(\sigma) = \partial_{2,3}(\sigma)^{-1} * E(\sigma)$. Therefore,

$$\partial_{2,3}(\sigma) = E(\sigma). \tag{94}$$

Similarly, $\partial_{2,3}(\sigma^{-1}) = E(\sigma^{-1}) * u_3^{\varphi_B} = u_3^{\varphi_B} * E(\sigma) = E(\sigma^{-1})$, where $E(\sigma^{-1}) = \sigma^{-1} \otimes \varepsilon$.

If (A, φ_A) is a left H-module algebra, by (93), the second coboundary morphism of the cosimplicial complex $Reg_{\varphi_B}(H^\bullet, A)$ is $D^2_{\varphi_A}(\sigma) = \partial_{2,0}(\sigma) * \partial_{2,1}(\sigma^{-1}) * \partial_{2,2}(\sigma) * E(\sigma^{-1})$. Moreover, if A is commutative, $(Reg_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})$ gives the Sweedler cohomology of H in (A, φ_A), where the kth group is defined by $\mathcal{H}^k_{\varphi_A}(H, A) = \dfrac{Ker(D^k_{\varphi_A})}{Im(D^{k-1}_{\varphi_A})}$ for $k \geq 1$. The normalized cochain subcomplex of A, denoted by $(Reg^+_{\varphi_A}(H^\bullet, A), D^{\bullet+}_{\varphi_A})$, is defined by the groups $Reg^+_{\varphi_A}(H^{k+1}, A)$, $Reg^+_{\varphi_A}(H_L, A)$ with $D^{k+}_{\varphi_A}$ the restriction of $D^k_{\varphi_A}$. We have that $(Reg^+_{\varphi_A}(H^\bullet, A), D^{\bullet+}_{\varphi_A})$, is a subcomplex of $(Reg_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})$ and the injection map induces an isomorphism of cohomology.

5.2. Let $\sigma, \tau \in Reg^+_{\varphi_B}(H^2, B)$ satisfying the twisted condition (38) and the 2-cocycle condition (39). Then by Theorem 3.8, $(B \otimes H, \mu_{B \otimes^\sigma_{\varphi_B} H})$ and $(B \otimes H, \mu_{B \otimes^\tau_{\varphi_B} H})$ are equivalent if, and only if, there exists $h \in Reg^+_{\varphi_B}(H, B)$ satisfying (83) and (78). Then, by [2, Corollary 4.8, Theorem 4.9], $(B \otimes H, \mu_{B \otimes^\sigma_{\varphi_B} H})$ and $(B \otimes H, \mu_{B \otimes^\tau_{\varphi_B} H})$ are equivalent iff there exists $h \in Reg^+_{\varphi_B}(H, B)$ such that the equalities (83) and

$$\sigma * \partial_{1,1}(h) = \partial_{1,0}(h) * \partial_{1,2}(h) * \tau, \tag{95}$$

hold. Note that the equality (83) is always true if B is commutative. Then, under these conditions, if (B, φ_B) is a left H-module algebra, the equivalence between two weak crossed products $(B \otimes H, \mu_{B \otimes^\sigma_{\varphi_B} H})$ and $(B \otimes H, \mu_{B \otimes^\tau_{\varphi_B} H})$ is determined by the existence of h in $Reg^+_{\varphi_B}(H, B)$ satisfying the equality (95). In this case (95) is equivalent to say that $\sigma * \tau^{-1} \in Im(D^{1+}_{\varphi_B})$, i.e., $[\sigma] = [\tau]$ in $\mathcal{H}^{2+}_{\varphi_B}(H, B)$.

5.3. Let $\sigma \in Reg_{\varphi_B}(H^2, B)$. Then, using the coface operators, it is an easy exercise to prove that σ satisfy the cocycle condition (39) iff

$$\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) = \partial_{2,3}(\sigma) * \partial_{2,1}(\sigma) \tag{96}$$

holds. Then, by (93), we have that σ satisfy the cocycle condition (39) iff σ satisfies the equality $\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) = E(\sigma) * \partial_{2,1}(\sigma)$.

Definition 5.4. Let $\sigma \in Reg_{\varphi_B}(H^2, B)$. We define the pre-obstruction of σ as the morphism $w_\sigma : H^3 \to B$, where $w_\sigma = \partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) * \partial_{2,1}(\sigma)^{-1} * \partial_{2,3}(\sigma)^{-1}$.

Then using that $\partial_{2,1}(\sigma)^{-1} = \partial_{2,1}(\sigma^{-1})$ and $\partial_{2,3}(\sigma)^{-1} = \partial_{2,3}(\sigma^{-1})$, by the previous considerations, we have that σ satisfies the cocycle condition (39) iff $w_\sigma = u_3^{\varphi_B}$. Also, note that by (94) for σ^{-1}, we have $w_\sigma = \partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) * \partial_{2,1}(\sigma^{-1}) * E(\sigma^{-1})$. On the other hand, it is easy to show that $\omega_\sigma = S_\sigma * R_\sigma$, where $S_\sigma = \partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) = \mu \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes F_\sigma)$ and $R_\sigma = \partial_{2,1}(\sigma^{-1}) * E(\sigma^{-1}) = \mu \circ (\sigma^{-1} \otimes B) \circ (H \otimes c) \circ (G_{\sigma^{-1}} \otimes H)$, are morphisms in $Reg_{\varphi_B}(H^3, B)$, where $S_\sigma^{-1} = \mu \circ (\sigma^{-1} \otimes \varphi_B) \circ (H \otimes c \otimes B) \circ (\delta \otimes G_{\sigma^{-1}})$ and $R_\sigma^{-1} = \mu \circ (B \otimes \sigma) \circ (F_\sigma \otimes H)$.

Proposition 5.5. *Let $\sigma \in Reg_{\varphi_B}(H^2, B)$. Then $\partial_{3,4}(\omega_\sigma) = \omega_\sigma \otimes \varepsilon$.*

Proof. Using that $\omega_\sigma = S_\sigma * R_\sigma$, if we prove that $\partial_{3,4}(S_\sigma) = S_\sigma \otimes \varepsilon$, $\partial_{3,4}(R_\sigma) = R_\sigma \otimes \varepsilon$ hold, we obtain the desired identity because $\partial_{3,4}(\omega_\sigma) = \partial_{3,4}(S_\sigma * R_\sigma) = \partial_{3,4}(S_\sigma) * \partial_{3,4}(R_\sigma) = (S_\sigma \otimes \varepsilon) * (R_\sigma \otimes \varepsilon) = (S_\sigma * R_\sigma) \otimes \varepsilon = \omega_\sigma \otimes \varepsilon$. The identity $\partial_{3,4}(S_\sigma) = S_\sigma \otimes \varepsilon$, follows from [2, Proposition 2.6](i), Definition 1.1(a1), the naturality of c, the associativity of μ and (94). Finally, $\partial_{3,4}(R_\sigma) = R_\sigma \otimes \varepsilon$ follows from the naturality of c and (94) for σ^{-1}. □

Proposition 5.6. *Let $\sigma \in Reg_{\varphi_B}(H^2, B)$ satisfying (38). Then, ω_σ factors through the center of B.*

Proof. We will use Proposition 4.11 to obtain that ω_σ factors through the center of B. To prove that $\overline{\omega_\sigma} \wedge \varphi_B^3 = \overline{\omega_\sigma}^{op} \wedge \varphi_B^3$ we first see

$$\overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)}^{op} \wedge \varphi_B^3 = \overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)} \wedge (\varphi_B \circ (m_H^3 \otimes B)) \tag{97}$$

and

$$\overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)}^{op} \wedge \varphi_B^3 = \overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)} \wedge (\varphi_B \circ (m_H^3 \otimes B)). \tag{98}$$

Indeed:

$\overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)}^{op} \wedge \varphi_B^3 = \overline{\partial_{2,2}(\sigma)}^{op} \wedge \overline{\partial_{2,0}(\sigma)}^{op} \wedge \varphi_B^3$ (by Proposition 4.2(iii))
$= \overline{\partial_{2,2}(\sigma)}^{op} \wedge (\varphi_B \circ (H \otimes (\overline{\sigma}^{op} \wedge \varphi_B^2)))$ (by naturality of c, Definition 1.3(b1) and cocommutativity of δ)
$= \overline{\partial_{2,2}(\sigma)}^{op} \wedge (\varphi_B \circ (H \otimes (\overline{\sigma} \wedge (\varphi_B \circ (\mu \otimes B)))))$ (by (85))
$= \overline{\partial_{2,2}(\sigma)}^{op} \wedge \overline{\partial_{2,0}(\sigma)} \wedge (\varphi_B \circ (H \otimes (\varphi_B \circ (\mu \otimes B))))$ (by Proposition 4.2(vii))
$= \overline{\partial_{2,0}(\sigma)} \wedge \overline{\partial_{2,2}(\sigma)}^{op} \wedge (\varphi_B^2 \circ (H \otimes \mu \otimes B))$ (by Proposition 4.2(iv))
$= \overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)} \wedge (\varphi_B \circ (m_H^3 \otimes B))$ (by Definition 1.1(a1), naturality of c, (85) and Proposition 4.2(i))

and

$\overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)}^{op} \wedge \varphi_B^3 = \overline{\partial_{2,1}(\sigma)}^{op} \wedge \overline{\partial_{2,3}(\sigma)}^{op} \wedge \varphi_B^3$ (by Proposition 4.2(iii))
$= \overline{\partial_{2,1}(\sigma)}^{op} \wedge ((\overline{\sigma}^{op} \wedge \varphi_B^2) \circ (H^2 \otimes \varphi_B))$ (by naturality of c, counit properties and (94))
$= \overline{\partial_{2,1}(\sigma)}^{op} \wedge ((\overline{\sigma} \wedge (\varphi_B \circ (\mu \otimes B))) \circ (H^2 \otimes \varphi_B))$ (by (85))

$= \overline{\partial_{2,3}(\sigma)} \wedge \overline{\partial_{2,1}(\sigma)}^{op} \wedge (\varphi_B \circ (\mu \otimes \varphi_B))$ (by naturality of c, counit properties and Proposition 4.2(iv))
$= \overline{\partial_{2,3}(\sigma)} \wedge \overline{\partial_{2,1}(\sigma)} \wedge (\varphi_B \circ (m_H^3 \otimes B))$ (by naturality of c, Definition 1.1(a1) and (85))
$= \overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)} \wedge (\varphi_B \circ (m_H^3 \otimes B))$ (by Proposition 4.2(i)).

Also,

$\overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)} \wedge \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge (\varphi_B \circ (m_H^3 \otimes B))$
$= \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge \overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)} \wedge (\varphi_B \circ (m_H^3 \otimes B))$ (by Proposition 4.2(iv))
$= \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge \overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)}^{op} \wedge \varphi_B^3$ (by (98))
$= \overline{\partial_{2,3}(\sigma) * \partial_{2,1}(\sigma) * \partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge \varphi_B^3$ (by Proposition 4.2(iii))
$= \varphi_B^3$ (by the property of group morphism for $\partial_{2,1}$ and $\partial_{2,3}$ and Proposition 4.2(v))

and, as a consequence, the following identity holds:

$$\overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge (\varphi_B \circ (m_H^3 \otimes B)) = \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})} \wedge \varphi_B^3. \tag{99}$$

Therefore,

$\overline{\omega_\sigma}^{op} \wedge \varphi_B^3 = \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge \overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)}^{op} \wedge \varphi_B^3$ (by Proposition 4.2(iii))
$= \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge \overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)} \wedge (\varphi_B \circ (m_H^3 \otimes B))$ (by (97))
$= \overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)} \wedge \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})}^{op} \wedge (\varphi_B \circ (m_H^3 \otimes B))$ (by Proposition 4.2(iv))
$= \overline{\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)} \wedge \overline{\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1})} \wedge \varphi_B^3$ (by (99))
$= \overline{\omega_\sigma} \wedge \varphi_B^3$ (by Proposition 4.2(i)) □

Definition 5.7. Let $\sigma \in Reg_{\varphi_B}(H^2, B)$ satisfying (38). The obstruction of σ is defined as the unique morphism $\theta_\sigma : H^3 \to Z(B)$ such that $z_B \circ \theta_\sigma = \omega_\sigma$, where ω_σ is the pre-obstruction of σ.

Note that, by the previous proposition, we can assure that θ_σ exists. Also, $\theta_\sigma \in Reg_{\varphi_{Z(B)}}(H^3, Z(B))$.

Theorem 5.8. *Let σ be as in Definition 5.7. Then, ω_σ is a 3-cocycle, i.e., the equality $\partial_{3,0}(\omega_\sigma) * \partial_{3,2}(\omega_\sigma) * \partial_{3,4}(\omega_\sigma) = \partial_{3,1}(\omega_\sigma) * \partial_{3,3}(\omega_\sigma)$ holds.*

Proof. In order to prove the theorem we will see some equalities. First of all observe that by the definition of the pre-obstruction ω_σ we have:

$$\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) = \omega_\sigma * \partial_{2,3}(\sigma) * \partial_{2,1}(\sigma). \tag{100}$$

Now using that $\partial_{3,2}$ is group morphism we have that

$$\partial_{3,2}(\partial_{2,0}(\sigma)) * \partial_{3,2}(\partial_{2,2}(\sigma)) = \partial_{3,2}(\omega_\sigma) * \partial_{3,2}(\partial_{2,3}(\sigma)) * \partial_{3,2}(\partial_{2,1}(\sigma)). \tag{101}$$

But observe that, by the associativity of μ and (94), we have

$$\partial_{3,2}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,2}(\sigma)), \tag{102}$$

$$\partial_{3,3}(\partial_{2,2}(\sigma)) = \partial_{3,2}(\partial_{2,2}(\sigma)). \tag{103}$$

Then, as a consequence of (101), (102) and $\partial_{3,2}(\partial_{2,0}(\sigma)) = \partial_{3,0}(\partial_{2,1}(\sigma))$ we obtain

$$\partial_{3,0}(\partial_{2,1}(\sigma)) * \partial_{3,2}(\partial_{2,2}(\sigma)) = \partial_{3,2}(\omega_\sigma) * \partial_{3,4}(\partial_{2,2}(\sigma)) * \partial_{3,2}(\partial_{2,1}(\sigma)). \tag{104}$$

On the other hand, by (94), we have

$$\partial_{3,0}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,0}(\sigma)). \tag{105}$$

Also,

$$\partial_{3,3}(\partial_{2,1}(\sigma^{-1})) * \partial_{3,3}(\partial_{2,3}(\sigma^{-1})) = \partial_{3,1}(\partial_{2,2}(\sigma^{-1})) * \partial_{3,4}(\partial_{2,3}(\sigma^{-1})) \tag{106}$$

holds, because

$\partial_{3,3}(\partial_{2,1}(\sigma^{-1})) * \partial_{3,3}(\partial_{2,3}(\sigma^{-1})) = \partial_{3,3}(\partial_{2,1}(\sigma^{-1}) * \partial_{2,3}(\sigma^{-1}))$ (because $\partial_{3,3}$ is a group morphism)
$= \mu \circ (\sigma^{-1} \otimes B) \circ (H \otimes c) \circ (G_{\sigma^{-1}} \otimes \mu)$ (by (94) for σ^{-1}, counit properties and naturality of c)
$= \partial_{3,1}(\partial_{2,2}(\sigma^{-1})) * \partial_{3,4}(\partial_{2,3}(\sigma^{-1}))$ (by Definition 1.1 (a1), naturality of c and (9)).

Then, as a consequence of (106), we have the identity $\partial_{3,3}(\partial_{2,3}(\sigma)) * \partial_{3,3}(\partial_{2,1}(\sigma)) = \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,2}(\sigma))$ and, using that $\partial_{3,3}$ is a group morphism, $\partial_{3,3}(\partial_{2,0}(\sigma)) = \partial_{3,0}(\partial_{2,2}(\sigma))$ and (103) we can assure that

$$\partial_{3,0}(\partial_{2,2}(\sigma)) * \partial_{3,2}(\partial_{2,2}(\sigma)) = \partial_{3,3}(\omega_\sigma) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,2}(\sigma)) \tag{107}$$

holds. Moreover,

$$\partial_{3,0}(\partial_{2,0}(\sigma)) * \partial_{3,4}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,0}(\sigma)) \tag{108}$$

holds because

$\partial_{3,0}(\partial_{2,0}(\sigma)) * \partial_{3,4}(\partial_{2,3}(\sigma))$
$= \mu \circ (B \otimes \sigma) \otimes (P_{\varphi_B} \otimes H) \circ (H \otimes (P_{\varphi_B} \circ (H \otimes \sigma)))$ (by (9), naturality of c and (31))
$= \mu \circ (B \otimes \varphi_B) \circ (F_\sigma \otimes ((\varepsilon \otimes B) \circ G_\sigma))$ (by (38) and (32))
$= \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,0}(\sigma))$ (by (94), naturality of c and (9))

and by, (94) and the associativity of μ, we obtain the equalities

$$\partial_{3,1}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,1}(\sigma)), \quad \partial_{3,1}(\partial_{2,1}(\sigma)) = \partial_{3,2}(\partial_{2,1}(\sigma)). \tag{109}$$

Finally, observe that, as ω_σ factors through the center of B, for all $i \in \{1,2,3,4\}$ and $\tau \in Reg_{\varphi_B}(H^4, B)$, we have

$$\tau * \partial_{3,i}(\omega_\sigma) = \partial_{3,i}(\omega_\sigma) * \tau. \tag{110}$$

Therefore, we conclude the proof by cancellation because in one hand

$\partial_{3,0}(\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)) * \partial_{3,2}(\partial_{2,2}(\sigma))$
$= \partial_{3,0}(\omega_\sigma) * \partial_{3,0}(\partial_{2,3}(\sigma)) * \partial_{3,0}(\partial_{2,1}(\sigma)) * \partial_{3,2}(\partial_{2,2}(\sigma))$ ($\partial_{3,0}$ is a group morphism and (100))
$= \partial_{3,0}(\omega_\sigma) * \partial_{3,0}(\partial_{2,3}(\sigma)) * \partial_{3,2}(\omega_\sigma) * \partial_{3,4}(\partial_{2,2}(\sigma)) * \partial_{3,2}(\partial_{2,1}(\sigma))$ (by (104))
$= \partial_{3,0}(\omega_\sigma) * \partial_{3,2}(\omega_\sigma) * \partial_{3,0}(\partial_{2,3}(\sigma)) * \partial_{3,4}(\partial_{2,2}(\sigma)) * \partial_{3,2}(\partial_{2,1}(\sigma))$ (by (110))
$= \partial_{3,0}(\omega_\sigma) * \partial_{3,2}(\omega_\sigma) * \partial_{3,4}(\partial_{2,0}(\sigma)) * \partial_{3,4}(\partial_{2,2}(\sigma)) * \partial_{3,2}(\partial_{2,1}(\sigma))$ (by (105))
$= \partial_{3,0}(\omega_\sigma) * \partial_{3,2}(\omega_\sigma) * \partial_{3,4}(\omega_\sigma) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,4}(\partial_{2,1}(\sigma)) * \partial_{3,2}(\partial_{2,1}(\sigma))$ (by the condition of group morphism for $\partial_{3,4}$ and (100)),

and on the other hand

$\partial_{3,0}(\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma)) * \partial_{3,2}(\partial_{2,2}(\sigma))$
$= \partial_{3,0}(\partial_{2,0}(\sigma)) * \partial_{3,3}(\omega_\sigma) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,2}(\sigma))$ (by (107) and because $\partial_{3,0}$ is a group morphism)
$= \partial_{3,3}(\omega_\sigma) * \partial_{3,0}(\partial_{2,0}(\sigma)) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,2}(\sigma))$ (by (110))
$= \partial_{3,3}(\omega_\sigma) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,0}(\sigma)) * \partial_{3,1}(\partial_{2,2}(\sigma))$ (by (108))
$= \partial_{3,3}(\omega_\sigma) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\omega_\sigma) * \partial_{3,1}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,1}(\sigma))$ ($\partial_{3,1}$ is a group morphism and by (100))
$= \partial_{3,1}(\omega_\sigma) * \partial_{3,3}(\omega_\sigma) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,3}(\sigma)) * \partial_{3,1}(\partial_{2,1}(\sigma))$ (by (110))
$= \partial_{3,1}(\omega_\sigma) * \partial_{3,3}(\omega_\sigma) * \partial_{3,4}(\partial_{2,3}(\sigma)) * \partial_{3,4}(\partial_{2,1}(\sigma)) * \partial_{3,2}(\partial_{2,1}(\sigma))$ (by (109)). □

Theorem 5.9. *Let σ be as in Definition 5.7. Then, $\theta_\sigma \in Im(D^2_{\varphi_{Z(B)}})$ iff there exists $\alpha \in Reg_{\varphi_B}(H^2, B)$ that satisfies the twisted condition (38) and the cocycle condition (39).*

Proof. If $\theta_\sigma \in Im(D^2_{\varphi_{Z(B)}})$, there exists $\tau \in Reg_{\varphi_{Z(B)}}(H^2, Z(B))$ such that $D^2_{\varphi_{Z(B)}}(\tau) = \theta_\sigma$. Then, $z_B \circ D^2_{\varphi_{Z(B)}}(\tau) = \omega_\sigma$. By Proposition 4.16, the morphism $\alpha = (z_B \circ \tau^{-1}) * \sigma$ satisfies the twisted condition (38) and belongs to $Reg_{\varphi_B}(H^2, B)$. On the other hand, by the properties of $\partial_{i,j}$, τ and $D^2_{\varphi_B}$, we have that $\omega_\alpha = D^2_{\varphi_B}(z_B \circ \tau^{-1}) * \omega_\sigma = z_B \circ D^2_{\varphi_{Z(B)}}(\tau^{-1}) * \omega_\sigma = \omega_\sigma^{-1} * \omega_\sigma = u_3^{\varphi_B}$ and, as a consequence, α satisfies the cocycle condition (39).

Conversely, assume that there exists $\alpha \in Reg_{\varphi_B}(H^2, B)$ that satisfies the twisted condition (38) and the cocycle condition (39). Then, by Proposition 4.16, there exists $\tau \in Reg_{\varphi_{Z(B)}}(H^2, Z(B))$ such that (90) holds, i.e., $\alpha = (z_B \circ \tau) * \sigma$. As a consequence, $\sigma = (z_B \circ \tau)^{-1} * \alpha$ and $\theta_\sigma \in Im(D^2_{\varphi_{Z(B)}})$ since $\omega_\sigma = D^2_{\varphi_B}(z_B \circ \tau^{-1}) * \partial_{2,0}(\alpha) * \partial_{2,2}(\alpha) * \partial_{2,1}(\alpha^{-1}) * \partial_{2,3}(\alpha^{-1}) \overset{(96)}{=} D^2_{\varphi_B}(z_B \circ \tau^{-1}) = z_B \circ D^2_{\varphi_{Z(B)}}(\tau^{-1})$. □

Proposition 5.10. *Let $\sigma \in Reg^+_{\varphi_B}(H^2, B)$ be as in Definition 5.7. A morphism α in $Reg^+_{\varphi_B}(H^2, B)$ satisfies the twisted condition (38) iff there exists $\tau \in Reg^+_{\varphi_{Z(B)}}(H^2, Z(B))$ satisfying (90).*

Proof. First note that, if H is cocommutative, (D, φ_D) is a left weak H-module algebra and $\beta \in Reg_{\varphi_D}(H^2, D)$, using that $\overline{\Pi}^L_H = \Pi^L_H$ and (45) we obtain that $\beta \circ (\eta \otimes H) = \beta \circ (\Pi^L_H \otimes H) \circ \delta$ holds. Also, (46), holds for β and therefore β satisfies the normal condition (47), i.e., $\beta \in Reg^+_{\varphi_D}(H^2, D)$ iff

$$\beta \circ (\Pi^L_H \otimes H) \circ \delta = \beta \circ (H \otimes \Pi^R_H) \circ \delta = u_1^{\varphi_D}. \tag{111}$$

Let $\alpha \in Reg^+_{\varphi_B}(H^2, B)$ satisfying (38). By Proposition 4.16 there exists $\tau \in Reg_{\varphi_{Z(B)}}(H^2, Z(B))$ satisfying (90). Then, $z_B \circ \tau = \alpha * \sigma^{-1}$ and τ satisfies the normal condition (47) because, in one hand, by the naturality of c, [2, Proposition 2.6](i), the cocommutativity of δ, (111) for α and (89) we have $z_B \circ \tau \circ (\Pi^L_H \otimes H) \circ \delta = z_B \circ u_2^{\varphi_{Z(B)}}$ and, on the other hand, using the same arguments we have $z_B \circ \tau \circ (H \otimes \Pi^R_H) \circ \delta = z_B \circ u_2^{\varphi_{Z(B)}}$.

Conversely, if there exists $\tau \in Reg^+_{\varphi_{Z(B)}}(H^2, Z(B))$ satisfying (90), by the previous arguments, we obtain that $\alpha \circ (\Pi^L_H \otimes H) \circ \delta = (z_B \circ \tau \circ (\Pi^L_H \otimes H) \circ \delta) * (\sigma^{-1} \circ (\Pi^L_H \otimes H) \circ \delta) = (z_B \circ u_2^{\varphi_{Z(B)}}) * u_2^{\varphi_B} = u_2^{\varphi_B}$ and similarly $\alpha \circ (H \otimes \Pi^R_H) \circ \delta = u_2^{\varphi_B}$. Therefore, $\alpha \in Reg^+_{\varphi_B}(H^2, B)$. □

Remark 5.11. Let $\sigma, \beta \in Reg^+_{\varphi_B}(H^2, B)$ be as in Definition 5.7. Let θ_σ, θ_β be the corresponding obstructions of σ and β. Then, by the previous proposition, it is easy to show that $[\theta_\sigma] = [\theta_\beta]$ in $\mathcal{H}^{3+}_{\varphi_{Z(B)}}(H, Z(B))$, i.e., θ_σ and θ_β are cohomologous.

Corollary 5.12. *Let $\sigma \in Reg^+_{\varphi_B}(H^2, B)$ be as in Definition 5.7. Then, $\theta_\sigma \in Im(D^{2+}_{\varphi_{Z(B)}})$ iff there exists $\alpha \in Reg^+_{\varphi_B}(H^2, B)$ that satisfies the twisted condition (38) and the cocycle condition (39).*

Proof. The result is a direct consequence of Theorem 5.9 and Proposition 5.10. □

Corollary 5.13. *Let $\sigma \in Reg^+_{\varphi_B}(H^2, B)$ be as in Definition 5.7. Then, $[\theta_\sigma] = 0$ in $\mathcal{H}^{3+}_{\varphi_{Z(B)}}(H, Z(B))$ iff there exists a morphism $\alpha \in Reg^+_{\varphi_B}(H^2, B)$ that satisfies the twisted condition (38), the cocycle condition (39) and the normal condition (47).*

Proof. The proof follows by the previous corollary and Corollary 1.18. □

As a consequence of this corollary, we can assure that the obstruction vanishes iff there exists a weak crossed product with preunit $\nabla_{BH}^{\varphi_B} \circ (\eta \otimes \eta)$ and normalized with respect to $\nabla_{BH}^{\varphi_B}$. Equivalently, by [12, Theorem 6.17, Corollary 6.18], this is equivalent to say that B admits a H-cleft extension (see also [3, Proposition 3.5]).

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