Numerical Methods for Partial Differential Equations

WILEY

RESEARCH ARTICLE

A Finite Element Method for a Nonlinear Magnetostatic Problem in Terms of Scalar Potentials

Rafael Muñoz-Sola[1] | Pilar Salgado[1,2]

[1]Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, Santiago de Compostela, Spain | [2]CITMAga, Galician Centre for Mathematical Research and Technology, Universidade de Santiago de Compostela, Santiago de Compostela, Spain

Correspondence: Rafael Muñoz-Sola (rafael.munoz@usc.es)

Received: 22 May 2024 | **Revised:** 20 December 2024 | **Accepted:** 2 February 2025

Funding: This work was supported by FEDER, Ministerio de Economía, Industria y Competitividad-AEI research projects PID2021-122625OB-I00 and MTM2017-86459-R, by Xunta de Galicia (Spain) research project GRC GI-1563 ED431C 2021/15.

Keywords: finite element method | monotone operator | nonlinear magnetostatic problem | scalar potentials

ABSTRACT

The aim of this paper is to perform the analysis of a numerical method based on scalar potentials for solving a nonlinear magnetostatic problem in a three-dimensional bounded domain containing prescribed currents and magnetic materials. The method discretizes a well-known formulation of this problem based on two scalar potentials: the total potential, defined in magnetic materials, and the reduced potential, defined in dielectric media and in non-magnetic conductors carrying currents. The topology of the magnetic materials is not assumed to be trivial, which leads to a multivalued potential. The resulting nonlinear variational problem is proved to be well posed and is discretized by means of standard piecewise linear finite elements. A convergence result without regularity assumptions on the solutions is proved in both the linear and nonlinear cases. Moreover, optimal error estimates are proved for smooth functions. Numerical results for an analytical test are reported to assess the performance of the method in the case of the continuous solution being smooth.

1 | Introduction

The goal of this paper is to analyze a finite element method for solving a nonlinear magnetostatic problem in terms of scalar potentials in a three-dimensional bounded domain. Namely, we will consider a formulation based on the combination of two different potentials, the so-called *reduced scalar potential* and *total scalar potential* (see [1, 2]). In this approach, the total potential is defined in magnetic materials without current sources, while the reduced potential is defined in dielectric media and in non-magnetic materials carrying currents.

This formulation was introduced in [3] for two-dimensional domains and extended to three-dimensional domains in [4]. For the linear magnetostatic problem, this formulation, and a finite element method to compute its solutions in the three-dimensional case have been analyzed in [5], where error estimates were obtained under a regularity assumption of the solution. But a general convergence result for a solution having just the regularity provided by the existence theorem was not obtained. Moreover, to the author's knowledge, the contributions found in the literature for the 3D nonlinear magnetostatic problem are limited to magnetic vector potential formulations and focused on the mathematical analysis of the continuous model; see, for instance [6], and [7].

In this work we will address two main tasks. On the one hand, we will extend the numerical analysis presented in [5] to the

[Correction added on June 2, 2025, after first online publication: The copyright line was changed.]

nonlinear case and add permanent magnets to the magnetic domain; namely, we will study the existence and uniqueness of solution of the nonlinear problem and prove optimal-order estimates provided that the nonlinearity is Lipschitz continuous and strongly monotone and the regularity assumption on the solution made in [5] holds true. On the other hand, a relevant contribution of the paper consists in deriving a convergence result under weaker assumptions on the nonlinearity than those cited above and without any regularity assumption on the solution. Basically, this convergence result relies on the Galerkin method for a nonlinear elliptic problem defined by a monotone operator when the finite-dimensional subspaces are not nested. This approach has been used in [8], in the context of other nonlinear elliptic problems, in order to prove convergence without regularity assumptions. To prove the convergence result, we prove a density property (cf. Lemma 8). As a by-product, we complete the study of the linear case with a general convergence result that does not need any additional regularity of the solution of the continuous problem.

The outline of the paper is as follows. In Section 2, we introduce the nonlinear magnetostatic problem and the geometrical framework for our analysis. We recall the Biot-Savart law and introduce the reduced scalar potential. In Section 3, we briefly justify that the nonlinear magnetostatic problem is well posed by using the reduced scalar potential formulation. In Section 4, we deduce the differential equations of the total potential/reduced potential formulation on a bounded domain. In Section 5, we obtain a weak formulation of the problem, we prove that it is well-posed and that its solution satisfies the classical magnetostatic equations in a weak form. In Section 6 we introduce a discretization based on standard finite elements, with the interface constraint between both potentials being imposed as an essential condition. This section also contains the derivation of the optimal order error estimates, the general convergence theorem, and the density result. In Section 7, we report some numerical results that confirm some of the convergence properties of the method. Finally, Appendix A.1 contains an auxiliary proposition related to an isotropic magnetic law.

2 | The Magnetostatic Problem in Terms of One Scalar Potential

The classical magnetostatic model is obtained by neglecting the time derivatives in Maxwell equations. Given a divergence-free stationary source current density $\boldsymbol{J}$, the magnetic field $\boldsymbol{H}$, and the magnetic induction $\boldsymbol{B}$ satisfies the following equations:

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \tag{1}$$

$$\operatorname{div} \boldsymbol{B} = 0 \tag{2}$$

and a constitutive law relating $\boldsymbol{H}$ and $\boldsymbol{B}$ to be defined below. We consider a bounded three-dimensional domain Ω composed of non-magnetic conductors, a magnetic core, permanent magnets, and air. This domain is assumed to be simply connected with a Lipschitz-continuous connected boundary Γ. We denote by Ω_{M} an open subset of Ω containing all the magnetic materials, by Ω_{pm} the open subset of Ω occupied by the permanent magnets, and by Ω_{mc} the open subset of Ω occupied by the magnetic core. We assume that $\Omega_{\mathrm{M}} = \Omega_{pm} \cup \Omega_{mc} \cup E$, where $E \subset \partial\Omega_{pm} \cap \partial\Omega_{mc}$ is a measure zero set. (Notice that $\partial\Omega_{\mathrm{M}} \subset \partial\Omega_{pm} \cup \partial\Omega_{mc}$). We assume that:

- The conductors are not magnetic materials; that is, they satisfy the magnetic law for linear isotropic materials with magnetic permeability $\mu = \mu_0$ ($\mu_0 > 0$ being the permeability of vacuum).
- The magnetic core is a non-conducting ferromagnetic material (possibly non-isotropic) without hysteresis.
- The permanent magnets satisfy a linear magnetic law with a remanent flux density $\boldsymbol{B}_r$, and they are non-conducting.

We write the constitutive relations linking vector fields $\boldsymbol{H}$ and $\boldsymbol{B}$ in a uniform form as follows:

$$\boldsymbol{B} = \widetilde{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{H}) \quad \text{in } \Omega \tag{3}$$

Here and thereafter, the function $\widetilde{\mathcal{B}} : \Omega \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ is defined by

$$\widetilde{\mathcal{B}}(\boldsymbol{x}, \xi) = \begin{cases} \mu_0 \xi & \text{if } \boldsymbol{x} \in \Omega \setminus \overline{\Omega}_{\mathrm{M}}, \\ \mu(\boldsymbol{x})\xi + \boldsymbol{B}_r(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega_{pm}, \\ \mathcal{B}(\boldsymbol{x}, \xi) & \text{if } \boldsymbol{x} \in \Omega_{mc} \end{cases} \tag{4}$$

where $\mu : \Omega_{pm} \mapsto \mathrm{M}_{3\times 3}(\mathbb{R})$ is a measurable function satisfying

$$|\mu_{ij}(\boldsymbol{x})| \leq \mu_{\max} \qquad for\ i, j = 1, 2, 3, \quad a.e.\ \boldsymbol{x} \in \Omega_{pm} \tag{5}$$

$$\mu_{\min} |\xi|^2 \leq \sum_{i=1}^{3} \sum_{j=1}^{3} \mu_{ij}(\boldsymbol{x}) \xi_i \xi_j \quad \forall \xi \in \mathbb{R}^3 \quad a.e.\ \boldsymbol{x} \in \Omega_{pm} \tag{6}$$

with constants $\mu_{\max} > 0$ and $\mu_{\min} > 0$, and $\mathcal{B}$ is a function, $\mathcal{B} : \Omega_{mc} \times \mathbb{R}^3 \mapsto \mathbb{R}^3$. The assumptions about this function will be specified later.

Throughout the article we use boldface letters to denote vector fields and variables, as well as vector-valued operators. We also use standard notation for Sobolev spaces and norms.

Let $\boldsymbol{J} \in \mathrm{L}^2(\Omega)^3$ such that $\operatorname{div}\boldsymbol{J} = 0$ in Ω and let $\boldsymbol{B}_r \in \mathrm{L}^2(\Omega_{pm})^3$.

We assume that Ω_{M} has p connected components, denoted by $\Omega_{\mathrm{M},i}$. Domains $\Omega_{\mathrm{M},i}$ can be multiply connected. We further assume that all the $\Omega_{\mathrm{M},i}$ are Lipschitz domains, each boundary $\partial\Omega_{\mathrm{M},i}$ is connected, and

$$\overline{\Omega}_{\mathrm{M},i} \cap \overline{\Omega}_{\mathrm{M},j} = \emptyset, \quad 1 \leq i, j \leq p,\ i \neq j \tag{7}$$

Hence $\partial\Omega_{\mathrm{M}} = \bigcup_{i=1}^{p} \partial\Omega_{\mathrm{M},i}$. Note that $\Omega_{\mathrm{M},i}$ do not need to be included in Ω_{pm} or in Ω_{mc}, so the case where permanent magnets are in contact with the ferromagnetic core is accepted in our framework.

We denote by Ω_{J} the open subset of Ω occupied by the conductors so that $\overline{\Omega}_{\mathrm{J}}$ contains the support of the current source $\boldsymbol{J}$ in $\overline{\Omega}$. We assume that $\Omega_{\mathrm{M}} \cap \Omega_{\mathrm{J}} = \emptyset$, so that $\boldsymbol{J}|_{\Omega_{\mathrm{M}}} = \boldsymbol{0}$.

We further assume that $\overline{\Omega}_{\mathrm{M}} \subset \Omega$, which in practice is not restrictive at all. However, concerning Ω_{J}, it is useful not to impose a

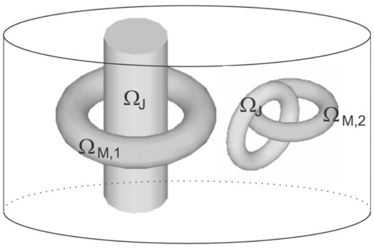

FIGURE 1 | Sketch of the domain Ω.

similar restriction in order to allow for eventual currents through the boundary of Ω. Our analysis covers problems in which domain Ω contains all the source currents (*closed circuits*), and also problems in which there is a current flow through a part $\Gamma_J := \partial\Omega_J \cap \Gamma$ of the boundary (*open circuits*); see, for instance, Figure 1, which combines both of these situations and includes several connected components for Ω_M and Ω_J. In any case, since the source current is divergence-free, the net current flux through the boundary of Ω has to vanish.

In order to state the magnetostatic problem in the bounded domain Ω, we have to add adequate boundary conditions to Equations (1)–(3). We consider the following one:

$$\boldsymbol{B} \cdot \boldsymbol{n} = g \qquad \text{on } \Gamma \tag{8}$$

where g is a given data function and $\boldsymbol{n}$ the outward unit normal vector to Γ. Notice that, by virtue of (2), the data g must have zero mean, that is, $\int_\Gamma g = 0$.

Therefore, the magnetostatic problem we are going to analyze, written in terms of the magnetic field, consists of finding $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$ satisfying

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \qquad \text{in } \Omega \tag{9}$$

$$\operatorname{div} \widetilde{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{H}) = 0 \qquad \text{in } \Omega \tag{10}$$

$$\mu_0 \boldsymbol{H} \cdot \boldsymbol{n} = g \qquad \text{on } \Gamma \tag{11}$$

We make the following assumptions concerning function $\mathcal{B}$:

B.1: $\mathcal{B} : \Omega_{mc} \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ satisfies the Caratheodory conditions
  i. $\mathcal{B}(\cdot, \boldsymbol{\xi}) : \Omega_{mc} \mapsto \mathbb{R}^3$ is measurable for each $\boldsymbol{\xi} \in \mathbb{R}^3$, and
  ii. $\mathcal{B}(\boldsymbol{x}, \cdot) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is continuous for a.e. $\boldsymbol{x} \in \Omega_{mc}$.

B.2: There exists a constant $c_1 > 0$ and a function $k_1 \in \mathrm{L}^2(\Omega_{mc})$ such that

$$|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi})| \le c_1 |\boldsymbol{\xi}| + k_1(\boldsymbol{x}) \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3 \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{12}$$

B.3: The following inequality holds:

$$(\mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0 \quad \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3, \quad \boldsymbol{\xi} \neq \boldsymbol{\eta} \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{13}$$

B.4: There exists a constant $c_2 > 0$ and a function $k_2 \in \mathrm{L}^1(\Omega_{mc})$ such that

$$\mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \ge c_2 |\boldsymbol{\xi}|^2 - k_2(\boldsymbol{x}) \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3 \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{14}$$

Some of our results will require the following assumptions:

B.5: There exists a constant $L > 0$ such that

$$|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\eta})| \le L |\boldsymbol{\xi} - \boldsymbol{\eta}| \quad \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3 \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{15}$$

B.6: There exists a constant $\omega > 0$ such that

$$(\mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \ge \omega |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \quad \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3 \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{16}$$

Remark 1. Note that (i) B.1 and B.2 imply $\mathcal{B}(\cdot, \mathbf{0}) \in \mathrm{L}^2(\Omega_{mc})^3$, (ii) B.1, B.5, and B.6 together with $\mathcal{B}(\cdot, \mathbf{0}) \in \mathrm{L}^2(\Omega_{mc})^3$ imply B.2, B.3, and B.4.

We consider now the case where the magnetic law of the magnetic core is isotropic, namely, the function $\mathcal{B}$ has the form

$$\mathcal{B}(\boldsymbol{x}, \boldsymbol{\xi}) = \mu(\boldsymbol{x}, |\boldsymbol{\xi}|)\boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3 \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{17}$$

where the function $\mu : \Omega_{mc} \times [0, \infty) \mapsto [0, \infty)$. Note that $\mathcal{B}(\boldsymbol{x}, \mathbf{0}) = \mathbf{0}$. Now, let us define the function $b : \Omega_{mc} \times [0, \infty) \mapsto [0, \infty)$ by

$$b(\boldsymbol{x}, s) = s\mu(\boldsymbol{x}, s) \quad \forall s \in [0, \infty) \qquad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{18}$$

The relationships between the properties of the function $\mathcal{B}$ and the ones of b are summarized in Proposition 1 (included in Appendix A.1).

Remark 2. In [9], a constitutive law for ferromagnetic materials of the form

$$\boldsymbol{H} = \nu(|\boldsymbol{B}|)\boldsymbol{B} \tag{19}$$

is considered, where the function $\nu : [0, \infty) \mapsto [0, \infty)$ is assumed to satisfy:

i. ν is continuous.
ii. $0 < \underline{\nu} \le \nu(s) \le \overline{\nu} \quad \forall s \in [0, \infty)$.
iii. $s \mapsto s\nu(s)$ is strictly increasing.
iv. $s \mapsto s\nu(s)$ is Lipschitz continuous.

Let $a : [0, \infty) \mapsto [0, \infty)$ be defined by $a(s) = s\nu(s)$. Assumptions (i)–(iii) imply that function a is a one-to-one correspondence. Let $b : [0, \infty) \mapsto [0, \infty)$ be its inverse. Function b is continuous and strictly increasing, $b(0) = 0$, and $\frac{r}{\overline{\nu}} \le b(r) \le \frac{r}{\underline{\nu}} \quad \forall r \in [0, \infty)$. Moreover, constitutive law (19) can be rewritten as

$$\boldsymbol{B} = \mu(|\boldsymbol{H}|)\boldsymbol{H} \tag{20}$$

where $\mu(r) = \frac{b(r)}{r}$ for $r > 0$ (and $\mu(0)$ can be assigned any value in $[0, \infty)$). Hence function b fulfills conditions (i)–(iv) of Proposition 1. Therefore, this case is included in our framework.

In order to obtain a potential formulation of the nonlinear problem, we proceed as in [5]. First, we build an extension of the current density to $\mathbb{R}^3$ in order to introduce a vector potential.

Remark 3. Let $\boldsymbol{H}(\mathrm{div}^0,\Omega) = \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^3; \mathrm{div}\,\boldsymbol{v} = 0\}$ and $\boldsymbol{H}(\mathrm{div}^0,\mathbb{R}^3) = \{\boldsymbol{v} \in \mathrm{L}^2(\mathbb{R}^3)^3; \mathrm{div}\,\boldsymbol{v} = 0\}$. We define the extension operator $\boldsymbol{E} : \boldsymbol{H}(\mathrm{div}^0,\Omega) \mapsto \boldsymbol{H}(\mathrm{div}^0,\mathbb{R}^3)$ by $\boldsymbol{E}\boldsymbol{J} := \widehat{\boldsymbol{J}}$, where $\widehat{\boldsymbol{J}}$ is constructed from $\boldsymbol{J}$ as in [5]: if $\boldsymbol{J}\cdot\boldsymbol{n} = 0$ on Γ, $\widehat{\boldsymbol{J}}$ is the extension of $\boldsymbol{J}$ by zero to the whole $\mathbb{R}^3$; if $\boldsymbol{J}\cdot\boldsymbol{n}$ does not vanish on the whole Γ, function $\widehat{\boldsymbol{J}}$ will be an extension of $\boldsymbol{J}$ constructed by introducing an auxiliary problem defined in a larger bounded domain denoted by $\widehat{\Omega}$ such that $\widehat{\Omega} \supset \overline{\Omega}$. We refer the reader to [5] to see the detailed definition of $\widehat{\boldsymbol{J}}$.

Then operator $\boldsymbol{E}$ is linear and continuous.

Now, for the vector field $\widehat{\boldsymbol{J}}$, we define a vector potential $\boldsymbol{T}$ as follows:

$$\boldsymbol{T}(\boldsymbol{x}) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \widehat{\boldsymbol{J}}(\boldsymbol{y}) \times \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{y}, \qquad \boldsymbol{x} \in \mathbb{R}^3$$

So defined, $\boldsymbol{T}$ satisfies (see, for instance, [5], Lemma 2.1)

$$\mathbf{curl}\,\boldsymbol{T} = \boldsymbol{J} \qquad \text{in } \Omega \tag{21}$$

$$\mathrm{div}\,\boldsymbol{T} = 0 \qquad \text{in } \Omega \tag{22}$$

Moreover, $\boldsymbol{T}|_\Omega \in \mathrm{H}^1(\Omega)^3$, with

$$\|\boldsymbol{T}\|_{1,\Omega} \le C\|\boldsymbol{J}\|_{0,\Omega} \tag{23}$$

where C is a strictly positive constant independent of $\boldsymbol{J}$.

The vector potential $\boldsymbol{T}$ will be used to obtain a potential formulation of nonlinear problem (9)–(11). For this purpose, let us notice that $\mathbf{curl}(\boldsymbol{H}-\boldsymbol{T}) = \mathbf{0}$, because of (9) and (21). Therefore, since Ω is a bounded, simply connected domain with Lipschitz boundary, there exists a scalar potential ϕ^{R} such that

$$\boldsymbol{H} = \boldsymbol{T} - \mathbf{grad}\,\phi^{\mathrm{R}} \qquad \text{in } \Omega \tag{24}$$

Scalar field ϕ^{R} is known as *reduced scalar potential*.

3 | Well-Posedness of the Nonlinear Magnetostatic Problem

In order to study the existence and uniqueness of the nonlinear magnetostatic problem (9)–(11), we first rewrite it as an equivalent problem in terms of ϕ^{R} as follows:

Find $\phi^{\mathrm{R}} \in \mathrm{H}^1(\Omega)/\mathbb{R}$ such that

$$\mathrm{div}\,\widetilde{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{T} - \mathbf{grad}\,\phi^{\mathrm{R}}) = 0 \qquad \text{in } \Omega \tag{25}$$

$$\mu_0(\boldsymbol{T} - \mathbf{grad}\,\phi^{\mathrm{R}})\cdot\boldsymbol{n} = g \qquad \text{on } \Gamma \tag{26}$$

Let $V = \mathrm{H}^1(\Omega)/\mathbb{R}$ be endowed with the norm $\|\psi^{\mathrm{R}}\|_V = \|\mathbf{grad}\,\psi^{\mathrm{R}}\|_{0,\Omega}$ and let V' be its topological dual space. Let us define the nonlinear operator $A : V \mapsto V'$ by

$$\begin{aligned}\langle A(\phi^{\mathrm{R}}), \psi^{\mathrm{R}}\rangle = \mu_0 &\int_{\Omega\setminus\overline{\Omega}_{\mathrm{M}}} \mathbf{grad}\,\phi^{\mathrm{R}}\cdot\mathbf{grad}\,\psi^{\mathrm{R}} \\ &+ \int_{\Omega_{pm}} \mu(\boldsymbol{x})\mathbf{grad}\,\phi^{\mathrm{R}}\cdot\mathbf{grad}\,\psi^{\mathrm{R}} \\ &- \int_{\Omega_{mc}} \mathcal{B}(\boldsymbol{x}, \boldsymbol{T} - \mathbf{grad}\,\phi^{\mathrm{R}})\cdot\mathbf{grad}\,\psi^{\mathrm{R}}\end{aligned} \tag{27}$$

Here $\langle\cdot,\cdot\rangle$ stands for the standard duality pairing between V' and V. Let $\ell : V \mapsto \mathbb{R}$ be the linear form defined by

$$\begin{aligned}\langle \ell, \psi^{\mathrm{R}}\rangle = \mu_0 &\int_{\Omega\setminus\overline{\Omega}_{\mathrm{M}}} \boldsymbol{T}\cdot\mathbf{grad}\,\psi^{\mathrm{R}} \\ &+ \int_{\Omega_{pm}} (\mu(\boldsymbol{x})\boldsymbol{T} + \boldsymbol{B}_r)\cdot\mathbf{grad}\,\psi^{\mathrm{R}} - \langle g, \psi^{\mathrm{R}}\rangle_\Gamma\end{aligned} \tag{28}$$

Notation $\langle\cdot,\cdot\rangle_\Gamma$ refers to the standard duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and its dual space $\mathrm{H}^{-1/2}(\Gamma)$. Data g is only assumed to belong to $\mathrm{H}^{-1/2}(\Gamma)$ and to satisfy $\langle g, 1\rangle_\Gamma = 0$, so that $\langle g, \psi^{\mathrm{R}}\rangle_\Gamma$ is well-defined for $\psi^{\mathrm{R}} \in \mathrm{H}^1(\Omega_{\mathrm{R}})/\mathbb{R}$. Notice that the linear form ℓ is well-defined and continuous.

It is easy to check that the weak formulation of problem (25)–(26) reads:

Problem P1. *Find $\phi^{\mathrm{R}} \in V$, such that $A(\phi^{\mathrm{R}}) = \ell$.*

Theorem 1. *Let us assume that function $\mathcal{B}$ satisfies assumptions B.1–B.4. Then problem* **P1** *has a unique solution.*

Proof. It can be easily checked that operator A is continuous, bounded, coercive, and strictly monotone. The results follow from monotone operator theory (see, for instance, [10], Chapter 3, or [11], Chapter II). □

Remark 4. In fact, problem **P1** is equivalent to problem (25)–(26), which in turn is equivalent to (9)–(11). Hence, Theorem 1 ensures that the nonlinear magnetostatic problem (9)–(11) has a unique solution.

As we noticed in the introduction, the existence and uniqueness of solution of the magnetostatic problem in terms of the magnetic vector potential have also been studied in [6], Section 3 and in [7].

4 | The Reduced/Total Potential Formulation of the Nonlinear Magnetostatic Problem

In the linear case, although rewriting the nonlinear magnetostatic problem in terms of ϕ^{R} allows for an important saving in computational effort, this alternative is not recommended in the literature because its numerical approximation leads to the so called *cancellation error*. The origin of this error is that, often, the two terms $\boldsymbol{T}$ and $\mathbf{grad}\,\phi^{\mathrm{R}}$ in (24) are almost of the same size and opposite direction within the magnetic materials, which leads to large cancellation errors when computing the magnetic field $\boldsymbol{H}$ (see, for instance, [1, 12, 13]). An alternative proposed by several authors consists in introducing an additional scalar potential in the domain Ω_{M} containing the magnetic materials (see [1, 3]).

To do this, we follow again the lines of [5]. For the sake of completeness, we include the main steps, and moreover, we introduce in the description the case in which Ω_{M} is not connected. Let $\Omega_{\mathrm{R}} := \Omega \setminus \overline{\Omega}_{\mathrm{M}}$ and recall that $\mu = \mu_0$ (constant) in Ω_{R}. Also, let $\Gamma_{\mathrm{I}} := \partial\Omega_{\mathrm{R}} \cap \partial\Omega_{\mathrm{M}}$ be the interface between Ω_{R} and Ω_{M}, and $\boldsymbol{\nu}$ the unit normal vector to Γ_{I} pointing outwards Ω_{M}. Notice that $\Gamma_{\mathrm{I}} = \partial\Omega_{\mathrm{M}}$, because we have chosen $\Omega \supset \overline{\Omega}_{\mathrm{M}}$. We assume that Ω_{R} is connected. We recall that the connected components of

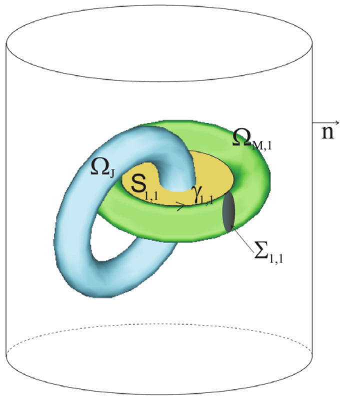

FIGURE 2 | Cut surface and geometry description.

Ω_M are domains $\Omega_{M,i}$, $i = 1, \dots, p$. In general, these domains are connected but not simply connected. We assume that for each domain $\Omega_{M,i}$, $i = 1, \dots, p$, there exists a finite number of open connected surfaces (so-called "cuts") $\Sigma_{i,j}$, $j = 1, \dots, J_i$, such that:

- $\Sigma_{i,j} \subset \Omega_{M,i}$ and $\partial\Sigma_{i,j} \subset \partial\Omega_{M,i}$,
- $\overline{\Sigma}_{i,j} \cap \overline{\Sigma}_{i,k} = \emptyset$, for $j \neq k$,
- the open set $\widetilde{\Omega}_{M,i} := \Omega_{M,i} \setminus \bigcup_{j=1}^{J_i} \Sigma_{i,j}$ is pseudo-Lipschitz and simply connected (see [14]).

We also assume that for each of these cuts $\Sigma_{i,j}$, there exists a simple closed curve $\gamma_{i,j} \subset \partial\Omega_{M,i}$, crossing $\overline{\Sigma}_{i,j}$ once and only once, and such that $\gamma_{i,j}$ is the boundary of an open surface $S_{i,j} \subset \Omega_R$; see Figure 2 for $J_1 = 1$.

Let $i \in \{1, \dots, p\}$ be fixed and $\widetilde{\Omega}^j_{M,i} := \Omega_{M,i} \setminus \Sigma_{i,j}$, $j = 1, \dots, J_i$. We choose a unit normal vector $\boldsymbol{n}_{i,j}$ on $\Sigma_{i,j}$ and denote its two faces by $\Sigma^-_{i,j}$ and $\Sigma^+_{i,j}$, with $\boldsymbol{n}_{i,j}$ being the 'outer' normal to $\widetilde{\Omega}^j_{M,i}$ along $\Sigma^+_{i,j}$.

For any function $\widetilde{\psi} \in \mathrm{H}^1(\widetilde{\Omega}_{M,i})$, we denote by

$$[\,[\widetilde{\psi}]\,]_{\Sigma_{i,j}} := \widetilde{\psi}|_{\Sigma^-_{i,j}} - \widetilde{\psi}|_{\Sigma^+_{i,j}}$$

the jump of $\widetilde{\psi}$ through $\Sigma_{i,j}$ along $\boldsymbol{n}_{i,j}$. The gradient of $\widetilde{\psi}$ in $\mathcal{D}'(\widetilde{\Omega}_{M,i})$ (i.e., in the sense of distributions in $\widetilde{\Omega}_{M,i}$) can be extended to $\mathrm{L}^2(\Omega_{M,i})^3$ and will be denoted by $\widetilde{\mathbf{grad}}\ \widetilde{\psi}$.

Let Θ_i be the subspace of $\mathrm{H}^1(\widetilde{\Omega}_{M,i})$ defined by

$$\Theta_i = \left\{ \widetilde{\psi} \in \mathrm{H}^1(\widetilde{\Omega}_{M,i}) \ : \ [\,[\widetilde{\psi}]\,]_{\Sigma_{i,j}} = \text{constant},\ j = 1, \dots, J_i \right\}$$

The kernel of the operator $\mathbf{curl} : \ \mathrm{H}(\mathbf{curl}, \Omega_{M,i}) \to \mathrm{L}^2(\Omega_{M,i})^3$ is given by (see [14])

$$\mathrm{Ker}(\mathbf{curl}) = \widetilde{\mathbf{grad}}\ \Theta_i \tag{29}$$

Now we are in a position to introduce the additional scalar potential that will be used to solve the nonlinear magnetostatic problem (9)–(11). Since $\boldsymbol{J}$ vanishes in $\Omega_M = \cup_{i=1}^p \Omega_{M,i}$, by virtue of (29), the solution of this problem can be written in these subdomains as follows:

$$\boldsymbol{H}|_{\Omega_{M,i}} = -\widetilde{\mathbf{grad}}\ \widetilde{\phi}_i\,, \qquad \widetilde{\phi}_i \in \Theta_i\,, \quad i = 1, \dots, p \tag{30}$$

Let us introduce the space $\Theta = \prod_{i=1}^p \Theta_i$ and the function $\widetilde{\phi} = (\widetilde{\phi}_1, \dots \widetilde{\phi}_p)$. We notice that every function $\widetilde{\psi} \in \Theta$ can be identified with the scalar function defined in $\widetilde{\Omega}_M := \bigcup_{i=1}^p \widetilde{\Omega}_{M,i}$ whose restriction to each $\Omega_{M,i}$ is $\widetilde{\psi}_i$. If $\widetilde{\psi} \in \Theta$, we denote by $\widetilde{\mathbf{grad}}\ \widetilde{\psi} \in \mathrm{L}^2(\Omega_M)$ the vector-valued function whose restriction to each $\Omega_{M,i}$ is $\widetilde{\mathbf{grad}}\ \widetilde{\psi}_i$. Because of assumption (7), we have for all integer $k \geq 0$ the isomorphism $\mathrm{H}^k(\widetilde{\Omega}_M) \equiv \prod_{i=1}^p \mathrm{H}^k(\widetilde{\Omega}_{M,i})$. Scalar function $\widetilde{\phi}$ is known as the *total scalar potential*. Then, Equation (30) can be rewritten as

$$\boldsymbol{H}|_{\Omega_M} = -\widetilde{\mathbf{grad}}\ \widetilde{\phi}$$

Our next step is to write the nonlinear magnetostatic problem in terms of total and reduced scalar potentials, $\widetilde{\phi} \in \Theta$ and $\phi^R \in \mathrm{H}^1(\Omega_R)$. We have

$$\boldsymbol{H} = \begin{cases} -\widetilde{\mathbf{grad}}\ \widetilde{\phi}, & \text{in } \Omega_M, \\ \boldsymbol{T} - \mathbf{grad}\ \phi^R, & \text{in } \Omega_R \end{cases} \tag{31}$$

Boundary and interface conditions must be written in terms of these potentials, as well. With this purpose, let us notice that for $\boldsymbol{J} \in \mathrm{L}^2(\Omega)^3$ we have from (9) that $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$ and hence $\boldsymbol{H} \times \boldsymbol{v}$ cannot jump across Γ_I. Analogously, $\boldsymbol{B} \cdot \boldsymbol{v}$ does not jump across Γ_I either, because of (10). Consequently, we have to find $\widetilde{\phi} \in \Theta$ and $\phi^R \in \mathrm{H}^1(\Omega_R)$, satisfying the following equations:

$$-\operatorname{div}\left(\mu_0\, \mathbf{grad}\ \phi^R\right) = 0 \qquad \text{in } \Omega_R \tag{32}$$

$$\operatorname{div}\left(\widetilde{\mathcal{B}}(\boldsymbol{x}, -\widetilde{\mathbf{grad}}\ \widetilde{\phi})\right) = 0 \qquad \text{in } \Omega_M \tag{33}$$

$$-\mu_0\, \mathbf{grad}\ \phi^R \cdot \boldsymbol{n} = g - \mu_0\, \boldsymbol{T} \cdot \boldsymbol{n} \qquad \text{on } \Gamma \tag{34}$$

$$\mu_0\, \mathbf{grad}\ \phi^R \cdot \boldsymbol{v} + (\boldsymbol{B}_r - \mu(\boldsymbol{x})\, \widetilde{\mathbf{grad}}\ \widetilde{\phi}) \cdot \boldsymbol{v} = \mu_0\, \boldsymbol{T} \cdot \boldsymbol{v} \qquad \text{on } \Gamma_I \cap \partial\Omega_{pm} \tag{35}$$

$$\mu_0\, \mathbf{grad}\ \phi^R \cdot \boldsymbol{v} + \mathcal{B}(\boldsymbol{x}, -\widetilde{\mathbf{grad}}\ \widetilde{\phi}) \cdot \boldsymbol{v} = \mu_0\, \boldsymbol{T} \cdot \boldsymbol{v} \qquad \text{on } \Gamma_I \cap \partial\Omega_{mc} \tag{36}$$

$$\mathbf{grad}\ \phi^R \times \boldsymbol{v} - \widetilde{\mathbf{grad}}\ \widetilde{\phi} \times \boldsymbol{v} = \boldsymbol{T} \times \boldsymbol{v} \qquad \text{on } \Gamma_I \tag{37}$$

(We recall that $\Gamma_I = \partial\Omega_M \subset \partial\Omega_{pm} \cup \partial\Omega_{mc}$).

5 | Well-Posedness of the Reduced/Total Potential Formulation of the Nonlinear Magnetostatic Problem

The aim of this section is to obtain a weak formulation of the magnetostatic problem in terms of total and reduced scalar potentials and to prove its well-posedness.

In order to obtain the weak formulation, let us multiply Equations (32) and (33) by a sufficiently smooth test function ψ.

Then we integrate by parts and use (34)–(37). Thus, we obtain

$$
\begin{aligned}
&\mu_0 \int_{\Omega_R} \mathbf{grad}\, \phi^R \cdot \mathbf{grad}\, \psi + \int_{\Omega_{pm}} \mu(\boldsymbol{x})\, \widetilde{\mathbf{grad}}\, \widetilde{\phi} \cdot \mathbf{grad}\, \psi \\
&- \int_{\Omega_{mc}} \mathcal{B}(\boldsymbol{x}, -\widetilde{\mathbf{grad}}\, \widetilde{\phi}) \cdot \mathbf{grad}\, \psi = -\langle g, \psi \rangle_\Gamma + \int_\Gamma \mu_0 \boldsymbol{T} \cdot \boldsymbol{n}\, \psi \\
&- \int_{\Gamma_I} \mu_0 \boldsymbol{T} \cdot \boldsymbol{\nu}\, \psi + \int_{\Omega_{pm}} \boldsymbol{B}_r \cdot \mathbf{grad}\, \psi
\end{aligned}
\tag{38}
$$

For the subsequent analysis, Equation (37) will be imposed as an essential condition. Let us introduce the space

$$
\mathcal{X} := \Theta/\mathbb{R}^p \times \mathrm{H}^1(\Omega_R)/\mathbb{R} \equiv \prod_{i=1}^{p} [\Theta_i/\mathbb{R}] \times \mathrm{H}^1(\Omega_R)/\mathbb{R}
$$

endowed with the norm

$$
\left\| \left(\widetilde{\psi}, \psi^R \right) \right\|_{\mathcal{X}} := \left(\left\| \widetilde{\mathbf{grad}}\, \widetilde{\psi} \right\|_{0,\Omega_M}^2 + \left\| \mathbf{grad}\, \psi^R \right\|_{0,\Omega_R}^2 \right)^{1/2}
$$

the closed linear manifold

$$
\mathcal{V}(\boldsymbol{T}) := \left\{ \left(\widetilde{\psi}, \psi^R \right) \in \mathcal{X} :\ \mathbf{grad}\, \psi^R \times \boldsymbol{\nu} - \widetilde{\mathbf{grad}}\, \widetilde{\psi} \times \boldsymbol{\nu} = \boldsymbol{T} \times \boldsymbol{\nu} \text{ on } \Gamma_I \right\}
$$

and the closed subspace

$$
\mathcal{V}(\mathbf{0}) := \left\{ \left(\widetilde{\psi}, \psi^R \right) \in \mathcal{X} :\ \mathbf{grad}\, \psi^R \times \boldsymbol{\nu} - \widetilde{\mathbf{grad}}\, \widetilde{\psi} \times \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma_I \right\}
$$

The following two lemmas are slight generalizations of Lemmas 3.1 and 3.2 proved in [5], where it is assumed Ω_M is connected. The proof mimics that of [5], working with all the connected components $\Omega_{M,i}$ instead of Ω_M.

Lemma 1. $\mathcal{V}(\boldsymbol{T}) \cap \left[\mathrm{H}^2(\widetilde{\Omega}_M)/\mathbb{R}^p \times \mathrm{H}^2(\Omega_R)/\mathbb{R} \right] \equiv \mathcal{V}(\boldsymbol{T}) \cap \left[\prod_{i=1}^p (\mathrm{H}^2(\widetilde{\Omega}_{M,i})/\mathbb{R}) \times \mathrm{H}^2(\Omega_R)/\mathbb{R} \right] \neq \emptyset$

Lemma 2. *$\left(\widetilde{\psi}, \psi^R \right) \in \mathcal{V}(\mathbf{0})$ if and only if $\exists \psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ such that $\widetilde{\psi} = \psi|_{\Omega_M}$ and $\psi^R = \psi|_{\Omega_R}$*

The last lemma allows us to write the variational formulation of the magnetostatic problem in terms of the total and the reduced scalar potentials as follows:

Problem P2. *Find $\left(\widetilde{\phi}, \phi^R \right) \in \mathcal{V}(\boldsymbol{T})$, such that*

$$
\begin{aligned}
&\mu_0 \int_{\Omega_R} \mathbf{grad}\, \phi^R \cdot \mathbf{grad}\, \psi^R + \int_{\Omega_{pm}} \mu(\boldsymbol{x})\, \widetilde{\mathbf{grad}}\, \widetilde{\phi} \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi} \\
&\quad - \int_{\Omega_{mc}} \mathcal{B}(\boldsymbol{x}, -\widetilde{\mathbf{grad}}\, \widetilde{\phi}) \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi} \\
&= -\langle g, \psi^R \rangle_\Gamma + \int_\Gamma \mu_0 \boldsymbol{T} \cdot \boldsymbol{n}\, \psi^R - \int_{\Gamma_I} \mu_0 \boldsymbol{T} \cdot \boldsymbol{\nu}\, \psi^R \\
&\quad + \int_{\Omega_{pm}} \boldsymbol{B}_r \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi} \qquad \forall \left(\widetilde{\psi}, \psi^R \right) \in \mathcal{V}(\mathbf{0})
\end{aligned}
\tag{39}
$$

In order to analyze problem **P2** and to study its numerical approximation, we introduce the nonlinear operator $A : \mathcal{X} \mapsto \mathcal{X}'$, where $\mathcal{X}'$ stands for the topological dual space of $\mathcal{X}$, by means of

$$
\begin{aligned}
&\left\langle A\left(\widetilde{\phi}, \phi^R \right), \left(\widetilde{\psi}, \psi^R \right) \right\rangle \\
&= \mu_0 \int_{\Omega_R} \mathbf{grad}\, \phi^R \cdot \mathbf{grad}\, \psi^R + \int_{\Omega_{pm}} \mu(\boldsymbol{x})\, \widetilde{\mathbf{grad}}\, \widetilde{\phi} \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi} \\
&\quad - \int_{\Omega_{mc}} \mathcal{B}(\boldsymbol{x}, -\widetilde{\mathbf{grad}}\, \widetilde{\phi}) \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi} \quad \forall \left(\widetilde{\phi}, \phi^R \right), \left(\widetilde{\psi}, \psi^R \right) \in \mathcal{X}
\end{aligned}
\tag{40}
$$

We introduce the linear form $\ell : \mathcal{V}(\mathbf{0}) \mapsto \mathbb{R}$ defined by

$$
\begin{aligned}
\left\langle \ell, \left(\widetilde{\psi}, \psi^R \right) \right\rangle &= -\langle g, \psi^R \rangle_\Gamma + \int_\Gamma \mu_0 \boldsymbol{T} \cdot \boldsymbol{n}\, \psi^R - \int_{\Gamma_I} \mu_0 \boldsymbol{T} \cdot \boldsymbol{\nu}\, \psi^R \\
&\quad + \int_{\Omega_{pm}} \boldsymbol{B}_r \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi} \quad \forall \left(\widetilde{\psi}, \psi^R \right) \in \mathcal{V}(\mathbf{0})
\end{aligned}
\tag{41}
$$

Notice that the linear form ℓ is well defined and continuous, that is, $\ell \in (\mathcal{V}(\mathbf{0}))'$.

Remark 5. Following the terminology of [10], we say that operator $A : \mathcal{X} \mapsto \mathcal{X}'$ satisfies property (S) if

$$
\left.
\begin{array}{l}
\left(\widetilde{\psi}_n, \psi_n^R \right) \rightharpoonup \left(\widetilde{\psi}, \psi^R \right) \textit{ weakly in } \mathcal{X} \\
\lim_{n \to \infty} \langle A\left(\widetilde{\psi}_n, \psi_n^R \right) - A\left(\widetilde{\psi}, \psi^R \right), \\
\quad \left(\widetilde{\psi}_n - \widetilde{\psi}, \psi_n^R - \psi^R \right) \rangle = 0
\end{array}
\right\} \Rightarrow \left(\widetilde{\psi}_n, \psi_n^R \right) \to \left(\widetilde{\psi}, \psi^R \right) \textit{ strongly in } \mathcal{X}
\tag{42}
$$

Lemma 3. *Let us assume that function $\mathcal{B}$ satisfies assumptions B.1–B.4. Then operator $A : \mathcal{X} \mapsto \mathcal{X}'$ is continuous, strictly monotone and coercive, and satisfies property (S) and the estimate*

$$
\| A\left(\widetilde{\phi}, \phi^R \right) \|_{\mathcal{X}'} \le C_1 \| \left(\widetilde{\phi}, \phi^R \right) \|_{\mathcal{X}} + C_2 \quad \forall \left(\widetilde{\phi}, \phi^R \right) \in \mathcal{X}
\tag{43}
$$

where $C_1 = C_1(\mu_0, \mu_{\max}, c_1)$ and $C_2 = \sqrt{2} \| k_1 \|_{0,\Omega_{mc}}$ are constants.

Proof. A straightforward computation yields an estimate (43) (with $C_1 = \max(\mu_0, 3\mu_{\max}, c_1\sqrt{2})$). The proof of the other properties of operator A follows essentially the same lines as the proof of [10], Theorem 3.3.42. □

Remark 6. From (40) and (41), we can rewrite problem **P2** as follows:

Find $\left(\widetilde{\phi}, \phi^R \right) \in \mathcal{V}(\boldsymbol{T})$, such that

$$
\left\langle A\left(\widetilde{\phi}, \phi^R \right), \left(\widetilde{\psi}, \psi^R \right) \right\rangle = \left\langle \ell, \left(\widetilde{\psi}, \psi^R \right) \right\rangle \quad \forall \left(\widetilde{\psi}, \psi^R \right) \in \mathcal{V}(\mathbf{0})
\tag{44}
$$

Theorem 2. *If function $\mathcal{B}$ satisfies assumptions B.1–B.4, then problem **P2** has a unique solution.*

Proof. From Lemma 1, $\mathcal{V}(\boldsymbol{T}) \neq \emptyset$. Let $\left(\widetilde{\psi}_1, \psi_1^R \right)$ be a fixed element of $\mathcal{V}(\boldsymbol{T})$. By representing every element $\left(\widetilde{\psi}, \psi^R \right) \in \mathcal{V}(\boldsymbol{T})$ in the form

$$
\left(\widetilde{\psi}, \psi^R \right) = \left(\widetilde{\psi}_1, \psi_1^R \right) + \left(\widetilde{\psi}_0, \psi_0^R \right), \quad \left(\widetilde{\psi}_0, \psi_0^R \right) \in \mathcal{V}(\mathbf{0})
$$

we can rewrite problem (44) as

Find $\left(\widetilde{\phi}_0, \phi_0^{\mathrm{R}}\right) \in \mathcal{V}(\mathbf{0})$, *such that*

$$\langle A_0\left(\widetilde{\phi}_0, \phi_0^{\mathrm{R}}\right), (\widetilde{\psi}, \psi^{\mathrm{R}})\rangle = \langle \ell, (\widetilde{\psi}, \psi^{\mathrm{R}})\rangle \quad \forall(\widetilde{\psi}, \psi^{\mathrm{R}}) \in \mathcal{V}(\mathbf{0}) \tag{45}$$

where $A_0 : \mathcal{V}(\mathbf{0}) \mapsto \mathcal{V}(\mathbf{0})'$ is the operator defined by

$$\langle A_0(\widetilde{\psi}_0, \psi_0^{\mathrm{R}}), (\widetilde{\psi}, \psi^{\mathrm{R}})\rangle = \langle A(\widetilde{\psi}_1 + \widetilde{\psi}_0, \psi_1^{\mathrm{R}} + \psi_0^{\mathrm{R}}), (\widetilde{\psi}, \psi^{\mathrm{R}})\rangle \quad \forall(\widetilde{\psi}_0, \psi_0^{\mathrm{R}}), (\widetilde{\psi}, \psi^{\mathrm{R}}) \in \mathcal{V}(\mathbf{0}) \tag{46}$$

We know from Lemma 3 that operator $A : \mathcal{X} \mapsto \mathcal{X}'$ is continuous, strictly monotone, coercive, and satisfies (43). Hence, A_0 is bounded, continuous, strictly monotone and coercive. Now, the existence and uniqueness of a solution follow again from monotone operator theory (see, for instance [10], Chapter 3, or [11], Chapter II). □

Once the total and the reduced scalar potentials are known, the magnetic field $\boldsymbol{H}$ can be readily computed by means of (31). In the following theorem, we show that this vector field actually satisfies the equations of the nonlinear magnetostatic problem (9)–(11). The proof is very similar to that of [5], Theorem 3.4.

Theorem 3. *Let* $\left(\widetilde{\phi}, \phi^{\mathrm{R}}\right)$ *be the solution of Problem* **P2** *and* $\boldsymbol{H}$ *be defined by (31); namely,* $\boldsymbol{H} := -\widetilde{\mathbf{grad}}\, \widetilde{\phi}$ *in* Ω_{M} *and* $\boldsymbol{H} := \boldsymbol{T} - \mathbf{grad}\, \phi^{\mathrm{R}}$ *in* Ω_{R}. *Then*

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \Omega \tag{47}$$

$$\operatorname{div} \widetilde{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{H}) = 0 \quad \text{in } \Omega \tag{48}$$

$$\mu_0 \boldsymbol{H} \cdot \boldsymbol{n} = g \quad \text{on } \Gamma \tag{49}$$

We have only needed assumptions B.1 to B.4 in order to establish the existence and uniqueness of the solution of the total/reduced potential formulation of the nonlinear magnetostatic problem. However, some of our results concerning the finite element approximation will require the stronger assumptions on function $\mathcal{B}$ stated in the following lemma.

Lemma 4. *Let us assume that function* $\mathcal{B}$ *satisfies assumptions B.1, B.5, B.6, and* $\mathcal{B}(\cdot, 0) \in \mathrm{L}^2(\Omega_{mc})^3$. *Then operator* $A : \mathcal{X} \mapsto \mathcal{X}'$ *is Lipschitz continuous and strongly monotone. More specifically, the following inequalities hold true for all* $\left(\widetilde{\psi}_1, \psi_1^{\mathrm{R}}\right), \left(\widetilde{\psi}_2, \psi_2^{\mathrm{R}}\right) \in \mathcal{X}$

$$\|A\left(\widetilde{\psi}_1, \psi_1^{\mathrm{R}}\right) - A\left(\widetilde{\psi}_2, \psi_2^{\mathrm{R}}\right)\|_{\mathcal{X}'} \le \tilde{L}\|\left(\widetilde{\psi}_1 - \widetilde{\psi}_2, \psi_1^{\mathrm{R}} - \psi_2^{\mathrm{R}}\right)\|_{\mathcal{X}} \tag{50}$$

$$\left\langle A\left(\widetilde{\psi}_1, \psi_1^{\mathrm{R}}\right) - A\left(\widetilde{\psi}_2, \psi_2^{\mathrm{R}}\right), \left(\widetilde{\psi}_1 - \widetilde{\psi}_2, \psi_1^{\mathrm{R}} - \psi_2^{\mathrm{R}}\right)\right\rangle \ge \tilde{\omega}\|\left(\widetilde{\psi}_1 - \widetilde{\psi}_2, \psi_1^{\mathrm{R}} - \psi_2^{\mathrm{R}}\right)\|_{\mathcal{X}}^2 \tag{51}$$

where $\tilde{L} = \max(\mu_0, 3\mu_{\max}, L)$ *and* $\tilde{\omega} = \min(\mu_0, \mu_{\min}, \omega) > 0$.

The proof of the lemma just involves straightforward computations.

Remark 7. In the previous section, we focused on a domain that includes linear and non-linear soft ferromagnetic materials, that is, materials modeled by their first law of magnetization, which is a curve B-H passing through the origin; linear permanent magnets, that is, hard magnetic materials modeled by a linear curve B-H that does not pass through the origin due to the presence of a remanent flux density in the direction of magnetization. It is worth mentioning that the study of permanent magnets modeled by a non-linear curve would also fit into this framework by introducing its constitutive law with a more general function $\mathcal{B}$, similar to that introduced for ferromagnetic materials with appropriate assumptions.

6 | Finite Element Discretization

In this section we introduce a discretization of Problem **P2**, we prove both an error estimate under suitable regularity assumptions on the solution and a convergence result that does not need any regularity assumption.

In what follows, we assume that Ω and subdomains $\Omega_{\mathrm{M},i}$, $i = 1, \dots, p$, and Ω_{R} are Lipschitz polyhedra. We assume also that, for each $i = 1, \dots, p$, one of the three following cases hold: (i) $\Omega_{\mathrm{M},i} \subset \Omega_{mc}$, (ii) $\Omega_{\mathrm{M},i} \subset \Omega_{pm}$, (iii) if $\Omega_{\mathrm{M},i} \cap \Omega_{mc} \neq \emptyset$ and $\Omega_{\mathrm{M},i} \cap \Omega_{pm} \neq \emptyset$, then both $\Omega_{\mathrm{M},i} \cap \Omega_{mc}$ and $\Omega_{\mathrm{M},i} \cap \Omega_{pm}$ are Lipschitz polyhedra.

We consider a family $\{\mathcal{T}_h\}$ of regular tetrahedral meshes of Ω, such that each element $K \in \mathcal{T}_h$ is contained either in some $\overline{\Omega}_{\mathrm{M},i}$ or in $\overline{\Omega}_{\mathrm{R}}$ (h stands as usual for the mesh size). Moreover, we assume that if $\Omega_{\mathrm{M},i} \cap \Omega_{mc} \neq \emptyset$ and $\Omega_{\mathrm{M},i} \cap \Omega_{pm} \neq \emptyset$, then each element $K \in \mathcal{T}_h$ contained in $\overline{\Omega}_{\mathrm{M},i}$ is contained either in $\overline{\Omega}_{\mathrm{M},i} \cap \overline{\Omega}_{pm}$ or in $\overline{\Omega}_{\mathrm{M},i} \cap \overline{\Omega}_{mc}$. We define $\mathcal{T}_h^{\Omega_{\mathrm{R}}} := \left\{K \in \mathcal{T}_h : K \subset \overline{\Omega}_{\mathrm{R}}\right\}$ and $\mathcal{T}_h^{\Omega_{\mathrm{M},i}} := \left\{K \in \mathcal{T}_h : K \subset \overline{\Omega}_{\mathrm{M},i}\right\}$.

We assume that the cut surfaces $\Sigma_{i,j}$ are polyhedral and that the meshes are compatible with them, in the sense that each $\overline{\Sigma}_{i,j}$ is a union of faces of tetrahedra for each mesh $\mathcal{T}_h$. Therefore, $\mathcal{T}_h^{\Omega_{\mathrm{M},i}}$ can also be seen as a mesh of $\widetilde{\Omega}_{\mathrm{M},i}$.

Let us introduce the following finite element spaces:

$$\mathcal{L}_h(\widetilde{\Omega}_{\mathrm{M},i}) := \left\{\widetilde{\psi}_h \in C(\widetilde{\Omega}_{\mathrm{M},i}) : \widetilde{\psi}_h|_K \in \mathcal{P}_1(K)\ \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{M},i}}\right\}, \quad i = 1, \dots, p,$$

$$\Theta_{i,h} := \left\{\widetilde{\psi}_h \in \mathcal{L}_h(\widetilde{\Omega}_{\mathrm{M},i}) : [\![\widetilde{\psi}]\!]_{\Sigma_{i,j}} = \text{constant},\ j = 1, \dots, J_i\right\}, \quad i = 1, \dots, p,$$

$$\Theta_h := \prod_{i=1}^{p} \Theta_{i,h},$$

$$\mathcal{L}_h(\Omega_{\mathrm{R}}) := \left\{\psi_h^{\mathrm{R}} \in C(\Omega_{\mathrm{R}}) : \psi_h^{\mathrm{R}}|_K \in \mathcal{P}_1(K)\ \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{R}}}\right\},$$

$$\mathcal{X}_h := \Theta_h/\mathbb{R}^p \times \mathcal{L}_h(\Omega_{\mathrm{R}})/\mathbb{R} \equiv \left(\prod_{i=1}^{p}[\Theta_{i,h}/\mathbb{R}]\right) \times \mathcal{L}_h(\Omega_{\mathrm{R}})/\mathbb{R}$$

Analogous definitions to that of $\mathcal{L}_h(\Omega_{\mathrm{R}})$ hold for $\mathcal{L}_h(\Omega_{\mathrm{M}})$, $\mathcal{L}_h(\Omega)$, etc.

We also have to introduce suitable approximations of $\boldsymbol{T}$ to be used on Γ_{I} and Γ for the discrete problem. Here we follow once again [5]. Notice that only the normal component of $\boldsymbol{T}$ is needed on Γ, whereas the tangential and normal components are needed on Γ_{I}. We will use the Nédélec interpolant of $\boldsymbol{T}$ for the former and its Raviart–Thomas interpolant for the latter. In what follows, we remind the definition and some basic properties of these interpolants (see, for instance, [15], Chapter 5).

The lowest-order Nédélec edge finite element space is defined as follows:

$$\mathcal{N}_h(\Omega) := \left\{ \boldsymbol{G}_h \in \mathrm{H}(\mathbf{curl}, \Omega) : \; \boldsymbol{G}_h|_K \in \mathcal{N}(K) \; \forall K \in \mathcal{T}_h \right\}$$

where $\mathcal{N}(K) := \{\boldsymbol{G}_h \in \mathcal{P}_1(K)^3 : \; \boldsymbol{G}_h(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b}, \; \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3, \; \boldsymbol{x} \in K\}$. If $\boldsymbol{G}$ is smooth enough, then its Nédélec interpolant $\boldsymbol{G}_{\mathrm{N}}$ is well defined by

$$\boldsymbol{G}_{\mathrm{N}} \in \mathcal{N}_h(\Omega) : \quad \int_{\ell} \boldsymbol{G}_{\mathrm{N}} \cdot \boldsymbol{t}_{\ell} = \int_{\ell} \boldsymbol{G} \cdot \boldsymbol{t}_{\ell} \quad \forall \ell \text{ edge of } \mathcal{T}_h$$

where $\boldsymbol{t}_{\ell}$ denotes a unit vector tangent to ℓ. The results of [14] allow extending the above definition to the space $\mathrm{H}^r(\mathbf{curl}, \Omega) := \{\boldsymbol{G} \in \mathrm{H}^r(\Omega) : \; \mathbf{curl}\, \boldsymbol{G} \in \mathrm{H}^r(\Omega)\}$ with $r > \frac{1}{2}$ (see also [16]).

On the other hand, the lowest-order Raviart-Thomas finite element space is defined as follows:

$$\mathcal{R}_h(\Omega) := \left\{ \boldsymbol{G}_h \in \mathrm{H}(\mathrm{div}, \Omega) : \; \boldsymbol{G}_h|_K \in \mathcal{R}(K) \; \forall K \in \mathcal{T}_h \right\}$$

where $\mathcal{R}(K) := \{\boldsymbol{G}_h \in \mathcal{P}_1(K)^3 : \; \boldsymbol{G}_h(\boldsymbol{x}) = a\boldsymbol{x} + \boldsymbol{b}; \; a \in \mathbb{R}, \; \boldsymbol{b} \in \mathbb{R}^3, \; \boldsymbol{x} \in K\}$. For $\boldsymbol{G}$ sufficiently smooth, its Raviart-Thomas interpolant $\boldsymbol{G}_{\mathrm{RT}}$ is defined by

$$\boldsymbol{G}_{\mathrm{RT}} \in \mathcal{R}_h(\Omega) : \quad \int_T \boldsymbol{G}_{\mathrm{RT}} \cdot \boldsymbol{n}_T = \int_T \boldsymbol{G} \cdot \boldsymbol{n}_T \quad \forall T \text{ face of } \mathcal{T}_h$$

where $\boldsymbol{n}_T$ denotes a unit vector normal to T. The following error estimate holds for all $\boldsymbol{G} \in \mathrm{H}^1(\Omega)^3$:

$$\left\| \boldsymbol{G} - \boldsymbol{G}_{\mathrm{RT}} \right\|_{0,\Omega} \leq Ch \|\boldsymbol{G}\|_{1,\Omega} \tag{52}$$

Here and thereafter, C denotes a strictly positive constant, not necessarily the same at each occurrence, but always independent of h.

One basic property that will be used in the sequel is that the Raviart-Thomas interpolant of a divergence-free field is divergence-free:

$$\mathrm{div}\, \boldsymbol{G} = 0 \quad \text{in } \Omega \qquad \Rightarrow \qquad \mathrm{div}\, \boldsymbol{G}_{\mathrm{RT}} = 0 \quad \text{in } \Omega \tag{53}$$

Next, we introduce the following finite-dimensional approximations of $\mathcal{V}(\boldsymbol{T})$ and $\mathcal{V}(\mathbf{0})$:

$$\begin{aligned} \mathcal{V}_h(\boldsymbol{T}) := \Big\{ \left(\widetilde{\psi}_h, \psi_h^{\mathrm{R}}\right) \in \mathcal{X}_h : \; & \mathbf{grad}\, \psi_h^{\mathrm{R}} \times \boldsymbol{\nu} - \widetilde{\mathbf{grad}}\, \widetilde{\psi}_h \times \boldsymbol{\nu} \\ & = \boldsymbol{T}_{\mathrm{N}} \times \boldsymbol{\nu} \text{ on } \Gamma_{\mathrm{I}} \Big\}, \\ \mathcal{V}_h(\mathbf{0}) := \Big\{ \left(\widetilde{\psi}_h, \psi_h^{\mathrm{R}}\right) \in \mathcal{X}_h : \; & \mathbf{grad}\, \psi_h^{\mathrm{R}} \times \boldsymbol{\nu} - \widetilde{\mathbf{grad}}\, \widetilde{\psi}_h \times \boldsymbol{\nu} \\ & = \mathbf{0} \text{ on } \Gamma_{\mathrm{I}} \Big\} \end{aligned}$$

Let us remark that the Nédélec interpolant $\boldsymbol{T}_{\mathrm{N}}$ in the definition of $\mathcal{V}_h(\boldsymbol{T})$ is well defined, because $\boldsymbol{T} \in \mathrm{H}^1(\Omega_{\mathrm{M}})$ and $\mathbf{curl}\, \boldsymbol{T} = \mathbf{0}$ in Ω_{M} (because of Equation (21) and $\boldsymbol{J}|_{\Omega_{\mathrm{M}}} = \mathbf{0}$).

Thus we arrive at the following discrete version of Problem **P2**:

Problem DP2. *Find* $\left(\widetilde{\phi}_h, \phi_h^{\mathrm{R}}\right) \in \mathcal{V}_h(\boldsymbol{T})$ *such that*

$$\begin{aligned} & \mu_0 \int_{\Omega_{\mathrm{R}}} \mathbf{grad}\, \phi_h^{\mathrm{R}} \cdot \mathbf{grad}\, \psi_h^{\mathrm{R}} + \int_{\Omega_{pm}} \mu(\boldsymbol{x})\, \widetilde{\mathbf{grad}}\, \widetilde{\phi}_h \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi}_h \\ & \quad - \int_{\Omega_{mc}} \mathcal{B}(\boldsymbol{x}, -\widetilde{\mathbf{grad}}\, \widetilde{\phi}_h) \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi}_h \\ & = -\langle g, \psi_h^{\mathrm{R}} \rangle_{\Gamma} + \int_{\Gamma} \mu_0 \boldsymbol{T}_{\mathrm{RT}} \cdot \boldsymbol{n}\, \psi_h^{\mathrm{R}} \\ & \quad - \int_{\Gamma_{\mathrm{I}}} \mu_0 \boldsymbol{T}_{\mathrm{RT}} \cdot \boldsymbol{\nu}\, \psi_h^{\mathrm{R}} + \int_{\Omega_{pm}} \boldsymbol{B}_r \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi}_h \\ & \qquad \forall \left(\widetilde{\psi}_h, \psi_h^{\mathrm{R}}\right) \in \mathcal{V}_h(\mathbf{0}) \end{aligned} \tag{54}$$

Notice that the right-hand side above is well defined for $\psi_h^{\mathrm{R}} \in \mathcal{L}_h(\Omega_{\mathrm{R}})/\mathbb{R}$. Indeed, $\int_{\Gamma} \mu_0 \boldsymbol{T}_{\mathrm{RT}} \cdot \boldsymbol{n} - \int_{\Gamma_{\mathrm{I}}} \mu_0 \boldsymbol{T}_{\mathrm{RT}} \cdot \boldsymbol{\nu} = \int_{\Omega_{\mathrm{R}}} \mathrm{div}\left(\mu_0 \boldsymbol{T}_{\mathrm{RT}}\right) = 0$, because of (53), since $\mathrm{div}\, \boldsymbol{T} = 0$ in Ω_{R} and μ_0 is constant. On the other hand, g has been assumed to satisfy $\langle g, 1 \rangle_{\Gamma} = 0$. By introducing the linear form $\ell_h : \mathcal{V}_h(\mathbf{0}) \mapsto \mathbb{R}$ defined by

$$\begin{aligned} \langle \ell_h, (\widetilde{\psi}_h, \psi_h^{\mathrm{R}}) \rangle = & -\langle g, \psi_h^{\mathrm{R}} \rangle_{\Gamma} + \int_{\Gamma} \mu_0 \boldsymbol{T}_{\mathrm{RT}} \cdot \boldsymbol{n}\, \psi_h^{\mathrm{R}} \\ & - \int_{\Gamma_{\mathrm{I}}} \mu_0 \boldsymbol{T}_{\mathrm{RT}} \cdot \boldsymbol{\nu}\, \psi_h^{\mathrm{R}} + \int_{\Omega_{pm}} \boldsymbol{B}_r \cdot \widetilde{\mathbf{grad}}\, \widetilde{\psi}_h \\ & \forall \left(\widetilde{\psi}_h, \psi_h^{\mathrm{R}}\right) \in \mathcal{V}_h(\mathbf{0}) \end{aligned} \tag{55}$$

we can rewrite problem **DP2** in abstract form as follows:

Find $\left(\widetilde{\phi}_h, \phi_h^{\mathrm{R}}\right) \in \mathcal{V}_h(\boldsymbol{T})$, *such that*

$$\langle A\left(\widetilde{\phi}_h, \phi_h^{\mathrm{R}}\right), (\widetilde{\psi}_h, \psi_h^{\mathrm{R}}) \rangle = \langle \ell_h, (\widetilde{\psi}_h, \psi_h^{\mathrm{R}}) \rangle \quad \forall \left(\widetilde{\psi}_h, \psi_h^{\mathrm{R}}\right) \in \mathcal{V}_h(\mathbf{0}) \tag{56}$$

Next lemmas are slight generalizations of Lemmas 4.1 and 4.3 of [5], where it was assumed that Ω_{M} is connected. The proofs can be adapted to our case with minor modifications.

Lemma 5. $\mathcal{V}_h(\boldsymbol{T}) \neq \emptyset$.

Lemma 6. *Let* $\left(\widetilde{\phi}, \phi^{\mathrm{R}}\right) \in \mathcal{V}(\boldsymbol{T})$ *be such that* $\widetilde{\mathbf{grad}}\, \widetilde{\phi} \in \mathrm{H}^r(\Omega_{\mathrm{M}})$ *and* $\phi^{\mathrm{R}} \in \mathrm{H}^{1+r}(\Omega_{\mathrm{R}})/\mathbb{R}$, *with* $0 < r \leq 1$. *Then, there exists* $\left(\widetilde{\phi}_{\mathrm{I}}, \phi_{\mathrm{I}}^{\mathrm{R}}\right) \in \mathcal{V}_h(\boldsymbol{T})$ *such that*

$$\begin{aligned} & \left\| \left(\widetilde{\phi} - \widetilde{\phi}_{\mathrm{I}}, \phi^{\mathrm{R}} - \phi_{\mathrm{I}}^{\mathrm{R}} \right) \right\|_{\mathcal{X}} \\ & \leq Ch^r \left(\left\| \widetilde{\mathbf{grad}}\, \widetilde{\phi} \right\|_{r,\Omega_{\mathrm{M}}} + \left\| \mathbf{grad}\, \phi^{\mathrm{R}} \right\|_{r,\Omega_{\mathrm{R}}} + \|\boldsymbol{J}\|_{0,\Omega} \right) \end{aligned}$$

Using Lemma 5, the properties of operator A stated in Lemma 3 and monotone operator theory, we obtain the following result:

Theorem 4. *If function $\mathscr{B}$ satisfies assumptions B.1–B.4, then problem* **DP2** *has a unique solution.*

Remark 8. Concerning the implementation of Problem DP2, it is worth mentioning that there are several ways to implement the interface conditions involved in $\mathcal{V}_h(\mathbf{0})$ and $\mathcal{V}_h(\boldsymbol{T})$. A widely extended proposal is that found in papers from the engineering literature, based on eliminating the degrees of freedom of one of the scalars in terms of the degrees of freedom of the other one. This option can be found, for example, in [12, 17]; it consists of assuming that, at some reference point $\boldsymbol{P}$ on the interface $\Gamma_{\rm I}$, the two potentials coincide, $\psi_h^{\rm R}(\boldsymbol{P}) = \widetilde{\psi}_h(\boldsymbol{P})$, and integrating the condition linking both potentials on a curve $\boldsymbol{C}$ contained in $\Gamma_{\rm I}$ and formed by edges of the mesh, which goes from $\boldsymbol{P}$ to an end point A, to obtain

$$\psi_h^{\rm R}(A) - \widetilde{\psi}_h(A) = 0 \text{ for } \mathcal{V}_h(\mathbf{0})$$

and

$$\psi_h^{\rm R}(A) - \widetilde{\psi}_h(A) = \int_C \boldsymbol{T}_{\rm N} \cdot \boldsymbol{t} = \int_C \boldsymbol{T} \cdot \boldsymbol{t} \text{ for } \mathcal{V}_h(\boldsymbol{T})$$

Notice that the difference $\psi_h^{\rm R}(A) - \widetilde{\psi}_h(A)$ in the last expression is well defined because the last integral does not depend on the choice of the curve $\boldsymbol{C}$ due to the Ampere's law and the fact that no electric current flows through the interface $\Gamma_{\rm I}$. This is, for instance, the option implemented in the commercial software Flux3D used in Section 7 for obtaining the numerical results.

However, we can also find in the literature another proposal that avoids this elimination by introducing a scalar Lagrange multiplier to implement the interface condition in a weak sense; namely, this is the option proposed by one of the authors in the paper about the linear case [5]. This option would be, in principle, more costly in terms of degrees of freedom, but it only needs the usual data structures for the finite element method and its computer implementation is quite straightforward.

Next lemma gives an upper bound of the error due to the approximation of the continuous right-hand side ℓ by the discrete one, ℓ_h, measured in the norm of $(\mathcal{V}_h(\mathbf{0}))'$, the dual space of $\mathcal{V}_h(\mathbf{0})$. This last space is endowed with the norm of $\mathcal{X}$.

Lemma 7. *There exists a constant $C > 0$ such that*

$$\|\ell - \ell_h\|_{(\mathcal{V}_h(\mathbf{0}))'} := \sup_{0 \neq (\widetilde{\psi}_h, \psi_h^{\rm R}) \in \mathcal{V}_h(\mathbf{0})} \frac{\langle \ell - \ell_h, (\widetilde{\psi}_h, \psi_h^{\rm R}) \rangle}{\|(\widetilde{\psi}_h, \psi_h^{\rm R})\|_{\mathcal{X}}} \leq Ch \|\boldsymbol{J}\|_{0,\Omega_{\rm J}} \tag{57}$$

Proof. The proof is implicitly contained in the proof of Theorem 4.4 of [5]. We give it for the sake of completeness. From Equations (41) and (55), integrating by parts and using $\operatorname{div}\boldsymbol{T} = \operatorname{div}\boldsymbol{T}_{\rm RT} = 0$ in $\Omega_{\rm R}$ (because of (22) and (53)), we get

$$\langle \ell - \ell_h, (\widetilde{\psi}_h, \psi_h^{\rm R}) \rangle = \int_{\Omega_{\rm R}} \mu_0 (\boldsymbol{T} - \boldsymbol{T}_{\rm RT}) \cdot \mathbf{grad}\, \psi_h^{\rm R} \tag{58}$$

Hence

$$|\langle \ell - \ell_h, (\widetilde{\psi}_h, \psi_h^{\rm R}) \rangle| \leq \mu_0 \|\boldsymbol{T} - \boldsymbol{T}_{\rm RT}\|_{0,\Omega_{\rm R}} \|\mathbf{grad}\, \psi_h^{\rm R}\|_{0,\Omega_{\rm R}} \tag{59}$$

This, together with the estimate (52) applied to $\boldsymbol{T}$ and (23), gives (57). □

6.1 | Convergence Under Regularity Assumptions

The goal of this subsection is to obtain an optimal-order error estimate under a regularity assumption for the solution of the continuous problem. Concerning function $\mathscr{B}$, we shall require more restrictive assumptions than those needed to establish the well-posedness of the continuous problem.

Theorem 5. *Let us assume that function $\mathscr{B}$ satisfies assumptions B.1, B.5, B.6, and $\mathscr{B}(\cdot, 0) \in \mathrm{L}^2(\Omega_{mc})^3$. If the solution $(\widetilde{\phi}, \phi^{\rm R})$ of Problem* **P2** *is such that $\boldsymbol{H}|_{\Omega_{\rm M}} = -\widetilde{\mathbf{grad}}\, \widetilde{\phi} \in \mathrm{H}^r(\Omega_{\rm M})^3$ and $\boldsymbol{H}|_{\Omega_{\rm R}} = \boldsymbol{T}|_{\Omega_{\rm R}} - \mathbf{grad}\, \phi^{\rm R} \in \mathrm{H}^r(\Omega_{\rm R})^3$, with $0 < r \leq 1$, then there exists a strictly positive constant C such that*

$$\left\| \left(\widetilde{\phi} - \widetilde{\phi}_h, \phi^{\rm R} - \phi_h^{\rm R} \right) \right\|_{\mathcal{X}} \leq Ch^r \left(\left\| \boldsymbol{H} \right\|_{r,\Omega_{\rm M}} + \left\| \boldsymbol{H} \right\|_{r,\Omega_{\rm R}} + \|\boldsymbol{J}\|_{0,\Omega} \right)$$

Proof. From Lemma 4, operator $A : \mathcal{X} \mapsto \mathcal{X}'$ satisfies properties (50) and (51). Using them together with standard arguments, it is easy to obtain the upper bound

$$\begin{aligned} &\|\left(\widetilde{\phi} - \widetilde{\phi}_h, \phi^{\rm R} - \phi_h^{\rm R} \right)\|_{\mathcal{X}} \\ &\leq \left(1 + \frac{\widetilde{L}}{\widetilde{\omega}} \right) \inf_{(\widetilde{\psi}_h, \psi_h^{\rm R}) \in \mathcal{V}_h(\boldsymbol{T})} \|\left(\widetilde{\phi} - \widetilde{\psi}_h, \phi^{\rm R} - \psi_h^{\rm R} \right)\|_{\mathcal{X}} \\ &\quad + \frac{1}{\widetilde{\omega}} \|\ell - \ell_h\|_{(\mathcal{V}_h(\mathbf{0}))'} \end{aligned} \tag{60}$$

As a consequence of the regularity assumption about $\boldsymbol{H}$ made in this theorem and the fact that $\boldsymbol{T}|_{\Omega} \in \mathrm{H}^1(\Omega)^3$, the solution $\left(\widetilde{\phi}, \phi^{\rm R}\right)$ of Problem **P2** satisfies the hypotheses of Lemma 6. Then, we can take $\left(\widetilde{\phi}_{\rm I}, \phi_{\rm I}^{\rm R}\right)$ as in that lemma in order to get an upper estimate of the first term of the right-hand side of (60). This estimate combined with (57) yields

$$\begin{aligned} &\|\left(\widetilde{\phi} - \widetilde{\phi}_h, \phi^{\rm R} - \phi_h^{\rm R} \right)\|_{\mathcal{X}} \\ &\leq Ch^r \left(\left\| \widetilde{\mathbf{grad}}\, \widetilde{\phi} \right\|_{r,\Omega_{\rm M}} + \left\| \mathbf{grad}\, \phi^{\rm R} \right\|_{r,\Omega_{\rm R}} + \|\boldsymbol{J}\|_{0,\Omega} \right) \end{aligned} \tag{61}$$

To conclude the proof, we use the fact that $\left\| \widetilde{\mathbf{grad}}\, \widetilde{\phi} \right\|_{r,\Omega_{\rm M}} + \left\| \mathbf{grad}\, \phi^{\rm R} \right\|_{r,\Omega_{\rm R}} = \|\boldsymbol{H}\|_{r,\Omega_{\rm M}} + \|\boldsymbol{H} - \boldsymbol{T}\|_{r,\Omega_{\rm R}} \leq \|\boldsymbol{H}\|_{r,\Omega_{\rm M}} + \|\boldsymbol{H}\|_{r,\Omega_{\rm R}} + C\|\boldsymbol{J}\|_{0,\Omega_{\rm J}}$. □

Remark 9. Notice that from the discrete scalar potentials $\widetilde{\phi}_h$ and $\phi_h^{\rm R}$ we can recover the discrete magnetic field $\boldsymbol{H}_h$ by using either $\boldsymbol{T}_{\rm RT}$ or $\boldsymbol{T}_{\rm N}$ in the domain $\Omega_{\rm R}$. In the first case, we can define

$$\boldsymbol{H}_h = \begin{cases} -\widetilde{\mathbf{grad}}\, \widetilde{\phi}_h, & \text{in } \Omega_{\rm M}, \\ \boldsymbol{T}_{\rm RT} - \mathbf{grad}\, \phi_h^{\rm R}, & \text{in } \Omega_{\rm R} \end{cases} \tag{62}$$

and under the assumptions of Theorem 5 there holds for $r = 1$:

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{\mathrm{L}^2(\Omega)^3} \leq Ch \left(\left\| \boldsymbol{H} \right\|_{1,\Omega_{\rm M}} + \left\| \boldsymbol{H} \right\|_{1,\Omega_{\rm R}} + \|\boldsymbol{J}\|_{0,\Omega} \right)$$

In the second case, in order to use the Nédélec interpolant for defining the magnetic field within Ω_R, we would need some regularity for the current density; namely, if $\boldsymbol{J} \in \mathrm{H}^1(\Omega_R)$, we could define

$$\boldsymbol{H}_h = \begin{cases} -\widetilde{\mathbf{grad}}\ \widetilde{\phi}_h, & \text{in } \Omega_M, \\ \boldsymbol{T}_N - \mathbf{grad}\ \phi_h^R, & \text{in } \Omega_R \end{cases} \tag{63}$$

and, owing to [15], Th. 5.41, we would have

$$\|\boldsymbol{T} - \boldsymbol{T}_N\|_{0,\Omega_R} \leq Ch\|\boldsymbol{J}\|_{1,\Omega_R}$$

Hence, recalling that $\boldsymbol{J}|_{\Omega_M} = \mathbf{0}$ and $\Omega_R := \Omega \setminus \overline{\Omega}_M$, under the assumptions of Theorem 5 there holds for $r = 1$

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{\mathrm{L}^2(\Omega)^3} \leq Ch\left(\left\|\boldsymbol{H}\right\|_{1,\Omega_M} + \left\|\boldsymbol{H}\right\|_{1,\Omega_R} + \|\boldsymbol{J}\|_{1,\Omega_R}\right)$$

6.2 | Convergence Without Regularity Assumptions

The goal of this subsection is to prove the convergence without any regularity assumption for the solution of the continuous problem. Concerning function $\mathcal{B}$, we shall just require the assumptions needed to establish the well-posedness of the continuous problem.

Although there are some similarities between some arguments of our proof and those of the proof of [8], Theorem 5.3.2, the main difference is that we do not require the operator A to satisfy a Lipschitz property. The price to be paid is that we cannot handle the two following kinds of variational crimes: (i) the approximation of the integrals involved in the definition of the operator A by means of numerical integration and (ii) the approximation of a curved domain by a sequence of polyhedral ones. These two variational crimes are considered in [8] for 2D problems. Concerning the first one, we remark that it is possible to compute exactly the integrals involved in the definition of A for $\left(\widetilde{\phi}_h, \phi_h^R\right), (\widetilde{\psi}_h, \psi_h^R) \in \mathcal{X}_h$ under the following additional assumptions:

i. For each $i = 1, \ldots, p$ such that $\Omega_{M,i} \cap \Omega_{mc} \neq \emptyset$, there exists a decomposition

$$\overline{\Omega}_{M,i} \cap \overline{\Omega}_{mc} = \bigcup_{j=1}^{N_i} \overline{\Omega}_{mc,i,j}, \qquad \Omega_{mc,i,j} \bigcap \Omega_{mc,i,k} = \emptyset \text{ for } j \neq k$$

where subdomains $\overline{\Omega}_{mc,i,j}$ are Lipschitz polyhedra and for each $j = 1, \ldots, N_i$ there exists a function $\mathcal{B}_{i,j} : \mathbb{R}^3 \mapsto \mathbb{R}^3$. such that

$$\mathcal{B}(\boldsymbol{x}, \xi) = \mathcal{B}_{i,j}(\xi) \quad \forall \xi \in \mathbb{R}^3 \quad a.e.\ \boldsymbol{x} \in \Omega_{mc,i,j}$$

ii. The triangulations are compatible with this decomposition in the following sense: if $K \in \mathcal{T}_h$ is contained in $\overline{\Omega}_{M,i} \cap \overline{\Omega}_{mc}$, then $K \subset \overline{\Omega}_{mc,i,j}$ for some $j = 1, \ldots N_i$.

iii. For each $i = 1, \ldots, p$ such that $\Omega_{M,i} \cap \Omega_{pm} \neq \emptyset$, there exists a decomposition

$$\overline{\Omega}_{M,i} \cap \overline{\Omega}_{pm} = \bigcup_{j=1}^{M_i} \overline{\Omega}_{pm,i,j}, \qquad \Omega_{pm,i,j} \bigcap \Omega_{pm,i,k} = \emptyset \text{ for } j \neq k$$

where subdomains $\overline{\Omega}_{pm,i,j}$ are Lipschitz polyhedra and for each $j = 1, \ldots, M_i$ there exists a constant matrix $\mu^{i,j} \in \mathrm{M}_{3\times 3}(\mathbb{R})$ such that

$$\mu(\boldsymbol{x}) = \mu^{i,j} \quad a.e.\ \boldsymbol{x} \in \Omega_{pm,i,j}$$

iv. The triangulations are compatible with this decomposition in the following sense: if $K \in \mathcal{T}_h$ is contained in $\overline{\Omega}_{M,i} \cap \overline{\Omega}_{pm}$, then $K \subset \overline{\Omega}_{pm,i,j}$ for some $j = 1, \ldots M_i$.

Notice that, due to the assumption (i), the third term of the left-hand side of (54) splits into a sum of integrals $\int_{\Omega_{mc,i,j}} \mathcal{B}_{i,j}(-\widetilde{\mathbf{grad}}\ \widetilde{\phi}_h) \cdot \widetilde{\mathbf{grad}}\ \widetilde{\psi}_h$. Due to hypothesis (ii), this integral decomposes in turn into a sum of the integrals $\int_K \mathcal{B}_{i,j}(-\widetilde{\mathbf{grad}}\ \widetilde{\phi}_h) \cdot \widetilde{\mathbf{grad}}\ \widetilde{\psi}_h$ where K runs over the set of the tetrahedra of the triangulation contained in $\overline{\Omega}_{mc,i,j}$. In each of these tetrahedra $\widetilde{\mathbf{grad}}\ \widetilde{\phi}_h$ is constant and therefore, even for a general $\mathcal{B}_{i,j}$, the integral can be calculated exactly.

First we establish a density result.

Lemma 8. *Let us denote* $\mathcal{X}_2 := \{(\widetilde{\phi}, \phi^R) \in \mathcal{X} : \widetilde{\mathbf{grad}}\ \widetilde{\phi} \in \mathrm{H}^1(\Omega_M),\ \phi^R \in \mathrm{H}^2(\Omega_R)/\mathbb{R}\}$. *The set* $\mathcal{V}(\boldsymbol{T}) \cap \mathcal{X}_2$ *is dense in* $\mathcal{V}(\boldsymbol{T})$.

Proof. Let $(\widetilde{\phi}, \phi^R)$ be an arbitrary element of $\mathcal{V}(\boldsymbol{T})$. As in [5], Theorem 3.4, the field $\boldsymbol{H}$ defined by (31), namely, $\boldsymbol{H} := -\widetilde{\mathbf{grad}}\ \widetilde{\phi}$ in Ω_M and $\boldsymbol{H} := \boldsymbol{T} - \mathbf{grad}\ \phi^R$ in Ω_R, satisfies $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$ and $\mathbf{curl}\ \boldsymbol{H} = \boldsymbol{J}$ in Ω. Since $\mathcal{D}(\overline{\Omega})^3$ is dense in $\mathrm{H}(\mathbf{curl}, \Omega)$ (see [18], Chapter I, Theorem 2.10), there exists a sequence $\boldsymbol{H}_n$ contained in $\mathcal{D}(\overline{\Omega})^3$ such that $\boldsymbol{H}_n \to \boldsymbol{H}$ strongly in $\mathrm{H}(\mathbf{curl}, \Omega)$.

We set $\boldsymbol{J}_n := \mathbf{curl}\ \boldsymbol{H}_n$, so $\boldsymbol{J}_n \to \boldsymbol{J}$ strongly in $\mathrm{L}^2(\Omega)^3$. We introduce $\hat{\boldsymbol{J}} = \boldsymbol{E}\boldsymbol{J}$ and $\hat{\boldsymbol{J}}_n = \boldsymbol{E}\boldsymbol{J}_n$, where $\boldsymbol{E}$ is the extension operator defined in Remark 3, and the field $\boldsymbol{T}_n$ is defined by

$$\boldsymbol{T}_n(\boldsymbol{x}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \hat{\boldsymbol{J}}_n(\boldsymbol{y}) \times \frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^3}\, \mathrm{d}\boldsymbol{y}\,, \qquad \boldsymbol{x} \in \mathbb{R}^3$$

The support of $\hat{\boldsymbol{J}}_n$ is contained in a fixed compact set for all n ($\overline{\Omega}$ or $\hat{\Omega}$), so $\boldsymbol{T}_n \in \mathrm{H}^1(\Omega)^3$ and $\boldsymbol{T}_n \to \boldsymbol{T}$ strongly in $\mathrm{H}^1(\Omega)^3$. Besides

$$\mathbf{curl}\ \boldsymbol{T}_n = \boldsymbol{J}_n \qquad \text{in } \Omega \tag{64}$$

Now, we define $\boldsymbol{G}_n := \boldsymbol{H}_n + \boldsymbol{T} - \boldsymbol{T}_n$. We have:

$$\boldsymbol{G}_n \in \mathrm{H}^1(\Omega)^3 \tag{65}$$

$$\boldsymbol{G}_n \to \boldsymbol{H} \qquad \text{strongly in } \mathrm{H}(\mathbf{curl}, \Omega) \tag{66}$$

$$\mathbf{curl}\ \boldsymbol{G}_n = \boldsymbol{J} \qquad \text{in } \Omega \tag{67}$$

$$\mathbf{curl}(\boldsymbol{G}_n - \boldsymbol{T}) = \mathbf{curl}(\boldsymbol{H}_n - \boldsymbol{T}_n) = \mathbf{0} \text{ in } \Omega \tag{68}$$

Since the domain Ω is bounded, Lipschitz and simply connected, there exists $\phi_n^R \in \mathrm{H}^1(\Omega)/\mathbb{R}$ such that

$$\boldsymbol{T} - \mathbf{grad}\ \phi_n^R = \boldsymbol{G}_n \qquad \text{in } \Omega \tag{69}$$

As $\boldsymbol{T} - \boldsymbol{G}_n \in \mathrm{H}^1(\Omega)^3$, we have $\phi_n^R \in \mathrm{H}^2(\Omega)/\mathbb{R}$. On the other hand, from (67), $\boldsymbol{J}|_{\Omega_M} = \mathbf{0}$ and (29), we deduce (by working in each connected component $\Omega_{M,i}$ of Ω_M) that there exists $\widetilde{\phi}_n \in \Theta/\mathbb{R}^p$ such that

$$-\widetilde{\mathbf{grad}}\,\widetilde{\phi}_n = \boldsymbol{G}_n \qquad \text{in } \Omega_M \tag{70}$$

As $\boldsymbol{G}_n \in \mathrm{H}^1(\Omega_M)^3$, we have $(\widetilde{\phi}_n, \phi_n^R) \in \mathcal{X}_2$. Now, from (65), (69), and (70) we deduce that

$$(\boldsymbol{T} - \mathbf{grad}\,\phi_n^R) \times \boldsymbol{\nu} = -\widetilde{\mathbf{grad}}\,\widetilde{\phi}_n \times \boldsymbol{\nu} \text{ in } \Gamma_I$$

and hence $(\widetilde{\phi}_n, \phi_n^R) \in \mathcal{V}(\boldsymbol{T})$. Finally, convergence (66) together with (70), (69), and (31) imply

$$\widetilde{\mathbf{grad}}\,\widetilde{\phi}_n = -\boldsymbol{G}_n \to -\boldsymbol{H} = \widetilde{\mathbf{grad}}\,\widetilde{\phi} \text{ strongly in } \mathrm{L}^2(\Omega_M)^3$$

and

$$\mathbf{grad}\,\phi_n^R = \boldsymbol{T} - \boldsymbol{G}_n \to \boldsymbol{T} - \boldsymbol{H} = \mathbf{grad}\,\phi^R \text{ strongly in } \mathrm{L}^2(\Omega_R)^3$$

Therefore $(\widetilde{\phi}_n, \phi_n^R) \to (\widetilde{\phi}, \phi^R)$ strongly in $\mathcal{X}$. □

Remark 10. Lemmas 6 and 8 imply that, given an arbitrary $(\widetilde{\xi}, \xi^R) \in \mathcal{V}(\boldsymbol{T})$, for all $h > 0$ small enough (say $h < h_0$), there exist $(\widetilde{\xi}_{Ah}, \xi_{Ah}^R) \in \mathcal{V}_h(\boldsymbol{T})$ such that

$$\lim_{h\to 0+} \left\| \left(\widetilde{\xi} - \widetilde{\xi}_{Ah}, \xi^R - \xi_{Ah}^R\right) \right\|_{\mathcal{X}} = 0 \tag{71}$$

Note that we can approximate every $(\widetilde{\psi}, \psi^R) \in \mathcal{V}(\mathbf{0})$ by $(\widetilde{\psi}_{Ah}, \psi_{Ah}^R) \in \mathcal{V}_h(\mathbf{0})$ in an analogous way.

Now we state the main result of this section.

Theorem 6. *Let us assume that function $\mathscr{B}$ satisfies assumptions B.1–B.4. Then*

$$\lim_{h\to 0+} \left\| \left(\widetilde{\phi} - \widetilde{\phi}_h, \phi^R - \phi_h^R\right) \right\|_{\mathcal{X}} = 0$$

Proof. We first prove that $(\widetilde{\phi}_h, \phi_h^R)$ is bounded in $\mathcal{X}$ for $h \in (0, h_0)$. To see this, we pick an element $(\widetilde{\xi}, \xi^R) \in \mathcal{V}(\boldsymbol{T})$ and we consider its approximation $(\widetilde{\xi}_{Ah}, \xi_{Ah}^R) \in \mathcal{V}_h(\boldsymbol{T})$ given by Remark 10. Owing to (57) and (71), there exists constants $C_3 > 0$ and $C_4 > 0$ such that for all $h \in (0, h_0)$

$$\|\ell_h\|_{V'_{0h}} \leq C_3 \tag{72}$$

$$\|(\widetilde{\xi}_{Ah}, \xi_{Ah}^R)\|_{\mathcal{X}} \leq C_4 \tag{73}$$

By taking $(\widetilde{\phi}_h - \widetilde{\xi}_{Ah}, \phi_h^R - \xi_{Ah}^R) \in \mathcal{V}_h(\mathbf{0})$ as a test function in (56), we obtain

$$\begin{aligned}&\langle A((\widetilde{\phi}_h, \phi_h^R), (\widetilde{\phi}_h - \widetilde{\xi}_{Ah}, \phi_h^R - \xi_{Ah}^R)\rangle \\ &= \langle \ell_h, (\widetilde{\phi}_h - \widetilde{\xi}_{Ah}, \phi_h^R - \xi_{Ah}^R)\rangle\end{aligned}$$

This equation, combined with (43), (72), and (73), gives

$$\begin{aligned}&\langle A(\widetilde{\phi}_h, \phi_h^R), (\widetilde{\phi}_h, \phi_h^R)\rangle \\ &\leq C_4(C_1\|(\widetilde{\phi}_h, \phi_h^R)\|_{\mathcal{X}} + C_2) + C_3(\|(\widetilde{\phi}_h, \phi_h^R)\| + C_4)\end{aligned}$$

Now, the coerciveness of A implies that $\|(\widetilde{\phi}_h, \phi_h^R)\|_{\mathcal{X}}$ is bounded. From this fact and estimate (43), we have that $A(\widetilde{\phi}_h, \phi_h^R)$ is bounded in $\mathcal{X}'$. Since $\mathcal{X}$ and $\mathcal{X}'$ are both reflexive and separable ([19], Chapter III), there exists a sequence $h_n \to 0^+$, $(\widetilde{\varphi}, \varphi^R) \in \mathcal{X}$ and $g \in \mathcal{X}'$ such that

$$(\widetilde{\phi}_{h_n}, \phi_{h_n}^R) \rightharpoonup (\widetilde{\varphi}, \varphi^R) \text{ weakly in } \mathcal{X} \tag{74}$$

$$A(\widetilde{\phi}_{h_n}, \phi_{h_n}^R) \rightharpoonup g \text{ weakly in } \mathcal{X}' \tag{75}$$

In virtue of (71) and (74), we have that

$$(\widetilde{\phi}_{h_n} - \widetilde{\xi}_{Ah_n}, \phi_{h_n}^R - \xi_{Ah_n}^R) \rightharpoonup (\widetilde{\varphi} - \widetilde{\xi}, \varphi^R - \xi^R) \text{ weakly in } \mathcal{X} \tag{76}$$

Since $(\widetilde{\phi}_{h_n} - \widetilde{\xi}_{Ah_n}, \phi_{h_n}^R - \xi_{Ah_n}^R) \in \mathcal{V}(\mathbf{0})$, this linear subspace is closed and $(\widetilde{\xi}, \xi^R) \in \mathcal{V}(\boldsymbol{T})$, we have that $(\widetilde{\varphi}, \varphi^R) \in \mathcal{V}(\boldsymbol{T})$.

Now we prove that $g|_{\mathcal{V}(\mathbf{0})} = \ell$. Let $(\widetilde{\psi}, \psi^R)$ be an arbitrary element of $\mathcal{V}(\mathbf{0})$ and let $(\widetilde{\psi}_{Ah_n}, \psi_{Ah_n}^R) \in \mathcal{V}_{h_n}(\mathbf{0})$ be its approximation according to Remark 10. Since $(\widetilde{\phi}_{h_n}, \phi_{h_n}^R)$ is a solution of the discrete problem (56) (with $h = h_n$), we have

$$\begin{aligned}\langle A(\widetilde{\phi}_{h_n}, \phi_{h_n}^R), (\widetilde{\psi}_{Ah_n}, \psi_{Ah_n}^R)\rangle &= \langle \ell_{h_n}, (\widetilde{\psi}_{Ah_n}, \psi_{Ah_n}^R)\rangle \\ &= \langle \ell, (\widetilde{\psi}_{Ah_n}, \psi_{Ah_n}^R)\rangle + \langle \ell_{h_n} - \ell, (\widetilde{\psi}_{Ah_n}, \psi_{Ah_n}^R)\rangle\end{aligned} \tag{77}$$

Since the sequence $(\widetilde{\psi}_{Ah_n}, \psi_{Ah_n}^R)$ is bounded in $\mathcal{X}$, the absolute value of the last term is bounded by $C\|\ell - \ell_{h_n}\|_{(\mathcal{V}_{h_n}(\mathbf{0}))'}$ and then it converges to zero because of (57). Passing to the limit as $n \to \infty$ in (77), we obtain

$$\langle g, (\widetilde{\psi}, \psi^R)\rangle = \langle \ell, (\widetilde{\psi}, \psi^R)\rangle \quad \forall (\widetilde{\psi}, \psi^R) \in \mathcal{V}(\mathbf{0})$$

that is $g|_{\mathcal{V}(\mathbf{0})} = \ell$.

Now we prove the strong convergence of $(\widetilde{\phi}_{h_n}, \phi_{h_n}^R)$ towards $(\widetilde{\varphi}, \varphi^R)$ in $\mathcal{X}$. Using again that $(\widetilde{\phi}_{h_n}, \phi_{h_n}^R)$ is a solution of the discrete problem (56) (with $h = h_n$), we can write

$$\begin{aligned}&\langle A(\widetilde{\phi}_{h_n}, \phi_{h_n}^R), (\widetilde{\phi}_{h_n} - \widetilde{\xi}_{Ah_n}, \phi_{h_n}^R - \xi_{Ah_n}^R)\rangle \\ &= \langle \ell_{h_n}, (\widetilde{\phi}_{h_n} - \widetilde{\xi}_{Ah_n}, \phi_{h_n}^R - \xi_{Ah_n}^R)\rangle \\ &= \langle \ell, (\widetilde{\phi}_{h_n} - \widetilde{\xi}_{Ah_n}, \phi_{h_n}^R - \xi_{Ah_n}^R)\rangle \\ &\quad + \langle \ell_{h_n} - \ell, (\widetilde{\phi}_{h_n} - \widetilde{\xi}_{Ah_n}, \phi_{h_n}^R - \xi_{Ah_n}^R)\rangle\end{aligned} \tag{78}$$

By the same argument as before, the last term converges to zero. Using (76), we obtain

$$\begin{aligned}&\lim_{n\to\infty} \langle A(\widetilde{\phi}_{h_n}, \phi_{h_n}^R), (\widetilde{\phi}_{h_n} - \widetilde{\xi}_{Ah_n}, \phi_{h_n}^R - \xi_{Ah_n}^R)\rangle \\ &= \langle \ell, (\widetilde{\varphi} - \widetilde{\xi}, \varphi^R - \xi^R)\rangle\end{aligned}$$

Because of (75) and (71), we also have

$$\lim_{n\to\infty} \langle A(\widetilde{\phi}_{h_n}, \phi_{h_n}^R), (\widetilde{\xi}_{Ah_n}, \xi_{Ah_n}^R)\rangle = \langle g, (\widetilde{\xi}, \xi^R)\rangle$$

From the last two equations, taking into account that $g|_{\mathcal{V}(\mathbf{0})} = \ell$, we deduce that

$$\begin{aligned}&\lim_{n\to\infty} \langle A(\widetilde{\phi}_{h_n}, \phi_{h_n}^R), (\widetilde{\phi}_{h_n}, \phi_{h_n}^R)\rangle = \langle g, (\widetilde{\xi}, \xi^R)\rangle \\ &\quad + \langle \ell, (\widetilde{\varphi} - \widetilde{\xi}, \varphi^R - \xi^R)\rangle = \langle g, (\widetilde{\varphi}, \varphi^R)\rangle\end{aligned} \tag{79}$$

This, together with (74) and (75), implies

$$\begin{aligned}
&\lim_{n\to\infty}\langle A(\widetilde{\phi}_{h_n},\phi^{\mathrm{R}}_{h_n})-A(\widetilde{\varphi},\varphi^{\mathrm{R}}),(\widetilde{\phi}_{h_n}-\widetilde{\varphi},\phi^{\mathrm{R}}_{h_n}-\varphi^{\mathrm{R}})\rangle\\
&\quad=\lim_{n\to\infty}\langle A(\widetilde{\phi}_{h_n},\phi^{\mathrm{R}}_{h_n}),(\widetilde{\phi}_{h_n},\phi^{\mathrm{R}}_{h_n})\rangle\\
&\qquad-\lim_{n\to\infty}\langle A(\widetilde{\phi}_{h_n},\phi^{\mathrm{R}}_{h_n}),(\widetilde{\varphi},\varphi^{\mathrm{R}})\rangle\\
&\qquad-\lim_{n\to\infty}\langle A(\widetilde{\varphi},\varphi^{\mathrm{R}}),(\widetilde{\phi}_{h_n}-\widetilde{\varphi},\phi^{\mathrm{R}}_{h_n}-\varphi^{\mathrm{R}})\rangle\\
&\quad=\langle g,(\widetilde{\varphi},\varphi^{\mathrm{R}})\rangle-\langle g,(\widetilde{\varphi},\varphi^{\mathrm{R}})\rangle=0
\end{aligned} \tag{80}$$

Since we also have the weak convergence (74) and A fulfills property (S), we deduce that

$$(\widetilde{\phi}_{h_n},\phi^{\mathrm{R}}_{h_n})\to(\widetilde{\varphi},\varphi^{\mathrm{R}}) \text{ strongly in } \mathcal{X} \tag{81}$$

Now, the continuity of A implies that $A(\widetilde{\phi}_{h_n},\phi^{\mathrm{R}}_{h_n})\to A(\widetilde{\varphi},\varphi^{\mathrm{R}})$ strongly in $\mathcal{X}'$ and, owing to (75), $A(\widetilde{\varphi},\varphi^{\mathrm{R}})=g$. We have $A(\widetilde{\varphi},\varphi^{\mathrm{R}})|_{\mathcal{V}(\mathbf{0})}=g|_{\mathcal{V}(\mathbf{0})}=\ell$ in $(\mathcal{V}(\mathbf{0}))'$ and $(\widetilde{\varphi},\varphi^{\mathrm{R}})\in\mathcal{V}(\boldsymbol{T})$, namely, $(\widetilde{\varphi},\varphi^{\mathrm{R}})$ is a solution of problem P2. By the uniqueness of the solution to this problem, we have $(\widetilde{\varphi},\varphi^{\mathrm{R}})=(\widetilde{\phi},\phi^{\mathrm{R}})$. Using the uniqueness of the weak accumulation point $(\widetilde{\varphi},\varphi^{\mathrm{R}})$ and a standard argument, we conclude that $(\widetilde{\phi}_h,\phi^{\mathrm{R}}_h)\to(\widetilde{\phi},\phi^{\mathrm{R}})$ strongly in $\mathcal{X}$ as $h\to 0^+$ (without extracting a sequence). Therefore, we get the result. □

Remark 11. As a by-product of Lemma 8, an analogous result of convergence for the linear magnetostatic problem analyzed in [5] holds true without any regularity assumption of the solution of the continuous problem. This can be easily proved by using the results of [5] together with Lemma 8 and a standard argument.

7 | Numerical Results. Test With known Analytical Solution

The objective of this section is to present some numerical results that confirm the convergence properties proved in the previous sections in the case where the solution of the continuous problem is smooth. To attain this goal, we will solve a magnetostatic problem that has an analytical solution and which will be built from the analytical test presented in [5] for a linear case. Indeed, we will complete the example presented in that paper by adding nonlinear magnetic materials and permanent magnets in order to fit in the framework studied in the present paper.

Let us consider a magnetostatic problem defined in a cylindrical domain Ω that corresponds to a section of height L of two infinite coaxial conductors, Ω_{J}^1 and Ω_{J}^2, separated by a magnetic material, Ω_{mc}, and a permanent magnet, Ω_{pm}, (see Figure 3).

Let us consider a cylindrical coordinate system (ρ,θ,z); the unit vectors in main directions are denoted by $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_z$. The z–axis coincides with the common axis of the cylinder. Let us assume a current flowing through each of the two conductors, which is axial and uniformly distributed with the same current

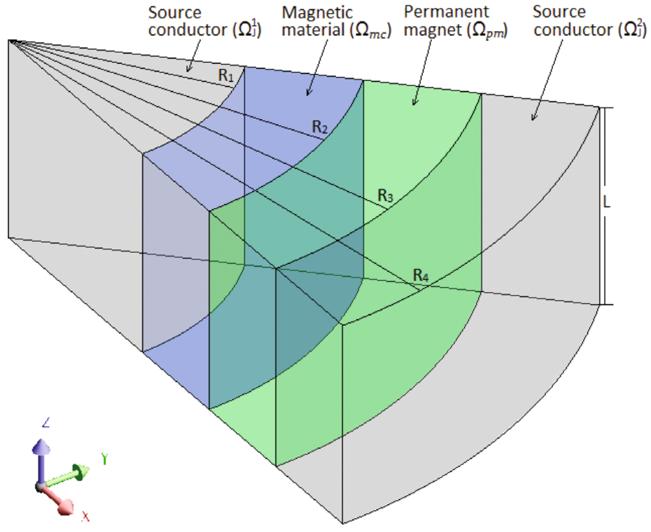

FIGURE 3 | Sketch of the domain Ω. Section corresponding to 1/8 of the full cylinder.

intensity, I, and opposite direction; namely,

$$\boldsymbol{J}=J_z(\rho)\,\boldsymbol{e}_z=\begin{cases}\frac{\mathrm{I}}{\pi\mathrm{R}_1^2}\,\boldsymbol{e}_z & \text{in } \Omega_{\mathrm{J}}^1,\\ \mathbf{0} & \text{in } \Omega_{mc}\cup\Omega_{pm},\\ -\frac{\mathrm{I}}{\pi(\mathrm{R}_4^2-\mathrm{R}_3^2)}\,\boldsymbol{e}_z & \text{in } \Omega_{\mathrm{J}}^2\end{cases}$$

The constitutive law considered is a linear law for the material of the conductors, $\boldsymbol{B}=\mu\,\boldsymbol{H}$; a nonlinear law for the material of the magnetic region Ω_{mc}, $\boldsymbol{B}=\mu(|\boldsymbol{H}|)\,\boldsymbol{H}$; and a linear law for the permanent magnet, $\boldsymbol{B}=\mu\,\boldsymbol{H}+\boldsymbol{B}_r$ with $\boldsymbol{B}_r=B_r\,\boldsymbol{e}_\theta$ and $B_r\in\mathbb{R}$.

By using this source and constitutive laws, the solution of the magnetostatic problem

$$\begin{cases}\mathbf{curl}\,\boldsymbol{H}=\boldsymbol{J} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{B}=0 & \text{in } \Omega,\\ \boldsymbol{B}\cdot\boldsymbol{n}=0 & \text{in } \partial\Omega\end{cases}$$

is given by

$$\boldsymbol{H}=H_\theta(\rho)\boldsymbol{e}_\theta=\begin{cases}\frac{\rho\mathrm{I}}{2\pi\mathrm{R}_1^2}\,\boldsymbol{e}_\theta & \text{in } \Omega_{\mathrm{J}}^1,\\ \frac{\mathrm{I}}{2\pi\rho}\,\boldsymbol{e}_\theta & \text{in } \Omega_{mc}\cup\Omega_{pm},\\ \left\{-\frac{\rho\mathrm{I}}{2\pi(\mathrm{R}_4^2-\mathrm{R}_3^2)}+\frac{1}{\rho}\left[\frac{\mathrm{I}}{2\pi}+\frac{\mathrm{R}_3^2\mathrm{I}}{2\pi(\mathrm{R}_4^2-\mathrm{R}_3^2)}\right]\right\}\boldsymbol{e}_\theta & \text{in } \Omega_{\mathrm{J}}^2\end{cases}$$

To obtain the numerical solution of the problem, we will use the following geometrical data: $\mathrm{R}_1=0.5$ m, $\mathrm{R}_2=0.75$ m, $\mathrm{R}_3=1$ m, $\mathrm{R}_4=1.25$ m, and $\mathrm{L}=0.5$ m. The current intensity that crosses the conducting domains is $\mathrm{I}=70000$ A.

The relative magnetic permeability of the material of the conducting domains is $\mu_r=1$. The constitutive nonlinear law in the material of the magnetic region is written like $\boldsymbol{B}=\mu(|\boldsymbol{H}|)\boldsymbol{H}$, being

$\mu(|H_\theta|)H_\theta = B_\theta(\rho) = \mathcal{B}(H_\theta(\rho))$; $\mathcal{B}(H_\theta)$ is given by the following curve

$$\mathcal{B}(H_\theta) = \mu_0\, H_\theta + \frac{2\, J_s}{\pi}\ \arctan\left(\frac{\pi\,(\mu_r - 1)\,\mu_0\, H_\theta}{2\, J_s}\right)$$

where we consider $\mu_0 = 4\pi \times 10^{-7}\ H\ m^{-1}$, $\mu_r = 5000$, and $J_s = 1.75$ T. On the other hand, the material corresponding to the permanent magnet is characterized by $\mu = \mu_r\ \mu_0$ with $\mu_r = 1.05$ and $\boldsymbol{B}_r = 1.3$ T.

The magnetostatic problem has been approximated by using the commercial software *Flux3D* (see [20]), which implements a procedure similar to the one described in this article and is based on the total scalar potential and the reduced scalar potential. Thus, we recall that the decomposition of the magnetic field $\boldsymbol{H}$ in each subdomain is as follows

$$\boldsymbol{H} = \begin{cases} \boldsymbol{T} - \mathbf{grad}\,\phi^{\mathrm{R}} & \text{in } \Omega_{\mathrm{J}}^1 \cup \Omega_{\mathrm{J}}^2, \\ \widetilde{\mathbf{grad}}\,\widetilde{\phi} & \text{in } \Omega_{mc} \cup \Omega_{pm} \end{cases}$$

This software approximates the scalar potentials by using Lagrange finite elements of order one or two; mainly, we have chosen the first-order approximation for our computations. To analyze the convergence, we have exploited the cylindrical symmetry and solved the problem in a section that corresponds to 1/8 of the domain Ω. Note that for solving the problem in this section, it is necessary to add suitable boundary conditions on the symmetry planes, where we impose $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}$, which is written as follows:

$$-\widetilde{\mathbf{grad}}\,\widetilde{\phi} \times \boldsymbol{n} = \boldsymbol{0} \text{ on the intersection of the symmetry planes with } \Omega_{\mathrm{M}}, \text{ and} \tag{82}$$

$$\boldsymbol{T} \times \boldsymbol{n} - \mathbf{grad}\,\phi^{\mathrm{R}} \times \boldsymbol{n} = \boldsymbol{0} \text{ on the intersection of the symmetry planes with } \Omega_{\mathrm{R}} \tag{83}$$

We have started with the coarse mesh shown in Figure 4 and refined it sequentially to compute the percentage error between the numerical solution and the analytical one. This error is computed in terms of $\boldsymbol{H}$, that is, $100\|\boldsymbol{H} - \boldsymbol{H}_h\|_{\mathrm{L}^2(\Omega)}/\|\boldsymbol{H}\|_{\mathrm{L}^2(\Omega)}$ and is reported in the Table 1.

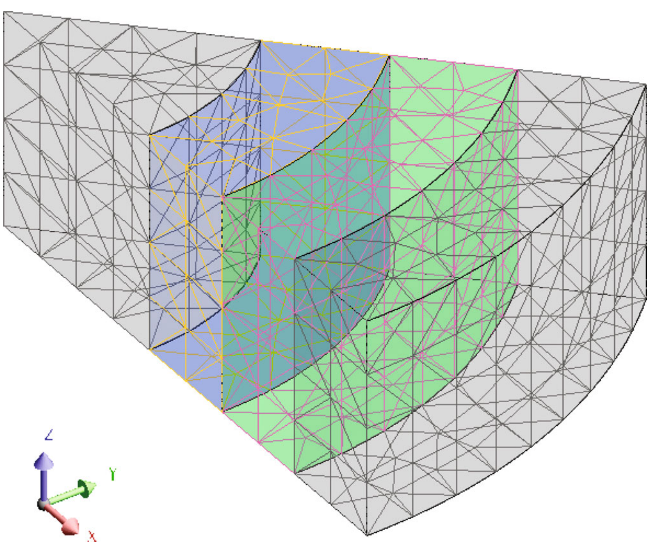

FIGURE 4 | Initial mesh for the analytical test (mesh size h).

TABLE 1 | Percentage errors (%) for the magnetic field vs. the mesh size h.

Mesh-size	d.o.f.	$100\frac{\|\boldsymbol{H}-\boldsymbol{H}_h\|_{\mathrm{L}^2(\Omega)^3}}{\|\boldsymbol{H}\|_{\mathrm{L}^2(\Omega)^3}}$
h	521	5.8919
$h/2$	3120	3.1171
$h/4$	19423	1.6389
$h/8$	114462	0.9117

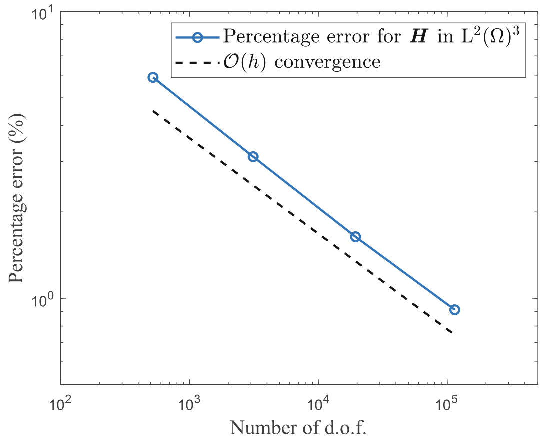

FIGURE 5 | Percentage error for $\boldsymbol{H}$ versus number of d.o.f (*log-log* scale).

The percentage error of the magnetic field calculated in norm $\mathrm{L}^2(\Omega)$ versus the number of degrees of freedom is represented in Figure 5 in the *log-log* scale. It can be observed a linear convergence that agrees with the order predicted in the theory under enough regularity assumptions.

The nonlinear problem is solved using a Newton-Raphson method. By using a tolerance of 10^{-4} in the relative error, the number of iterations required to reach convergence is between 10 and 12 for all the examples given in Table 1.

Acknowledgments

We would like to thank Aida Martínez Amado for helping us to obtain the numerical results by using Flux3D.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

1. C. Magele, H. Stogner, and K. Preis, "Comparison of Different Finite Element Formulations for 3d Magnetostatic Problems," *IEEE Transactions on Magnetics* 24, no. 1 (1988): 31–34.

2. K. Preis, I. Bardi, O. Biro, et al., "Numerical Analysis of 3d Magnetostatic Fields," *IEEE Transactions on Magnetics* 27, no. 5 (1991): 3798–3803.

3. J. Simkin and C. W. Trowbridge, "On the Use of the Total Scalar Potential on the Numerical Solution of Fields Problems in Electromagnetics," *International Journal for Numerical Methods in Engineering* 14, no. 3 (1979): 423–440. https://onlinelibrary.wiley.com/doi/abs/10.1002/nme.1620140308.

4. J. Simkin and C. W. Trowbridge, "Three-Dimensional Nonlinear Electromagnetic Field Computations, Using Scalar Potentials," *IEE Proceedings B - Electric Power Applications* 127, no. 6 (1980): 368–374.

5. A. Bermúdez, R. Rodríguez, and P. Salgado, "A Finite Element Method for the Magnetostatic Problem in Terms of Scalar Potentials," *SIAM Journal on Numerical Analysis* 46, no. 3 (2008): 1338–1363, https://doi.org/10.1137/06067568X.

6. F. Bachinger, U. Langer, and J. Schöberl, "Numerical Analysis of Nonlinear Multiharmonic Eddy Currents Problems," 2004.SFB-Report No. 2004-01, Johannes Kepler University Linz, SFB Numerical and Symbolic Scientific Computing.

7. B. Heise, M. Kuhn, and U. Langer, "A Mixed Variational Formulation for 3d Linear and Nonlinear Magnetostatics in the Space H_0(curl)∩ H(div)," *Hungarian Electronic Journal of Sciences. Applied and Numerical Mathematics Section* 1 (2001), https://heja.szif.hu/ANM/ANM-981030-A/anm981030a/index.html.

8. M. Feistauer and A. Ženíšek, "Compactness Method in the Finite Element Theory of Nonlinear Elliptic Problems," *Numerische Mathematik* 52, no. 2 (1988): 147–163, https://doi.org/10.1007/BF01398687.

9. F. Bachinger, U. Langer, and J. Schöberl, "Numerical Analysis of Nonlinear Multiharmonic Eddy Current Problems," *Numerische Mathematik* 100, no. 4 (2005): 593–616, https://doi.org/10.1007/s00211-005-0597-2.

10. J. Necas, *Introduction to the Theory of Nonlinear Elliptic Equations* (John Wiley & Sons, 1986).

11. R. E. Showalter, "Monotone Operators in Banach Space and Nonlinear Partial Differential Equations," *Mathematical Surveys and Monographs* 49 (American Mathematical Society, 1997).

12. S. Balac and G. Caloz, "The Reduced Scalar Potential in Regions With Permeable Materials: Reasons for Loss of Accuracy and Cancellation," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields* 20, no. 4 (2007): 163–180. https://onlinelibrary.wiley.com/doi/abs/10.1002/jnm.638.

13. I. Mayergoyz, M. Chari, and J. D'Angelo, "A New Scalar Potential Formulation for Three-Dimensional Magnetostatic Problems," *IEEE Transactions on Magnetics* 23, no. 6 (1987): 3889–3894.

14. C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, "Vector Potentials in Three-Dimensional Non-smooth Domains," *Mathematicsl Methods in the Applied Sciences* 21, no. 9 (1998): 823–864, https://doi.org/10.1002/(SICI)1099-1476(199806)21:9<823::AID-MMA976>3.0.CO;2-B.

15. P. Monk, "*Finite Element Methods for Maxwell's Equations*," (Oxford University Press, 2003), https://doi.org/10.1093/acprof:oso/9780198508885.001.0001.

16. A. Alonso and A. Valli, "An Optimal Domain Decomposition Preconditioner for Low-Frequency Time-Harmonic Maxwell Equations," *Mathematics of Computation* 68, no. 226 (1999): 607–631, https://doi.org/10.1090/S0025-5718-99-01013-3.

17. A. Vourdas and K. Binns, "Magnetostatics With Scalar Potentials in Multiply Connected Regions," *IEE Proceedings: Science, Measurement & Technology* 136 (1989): 49–54.

18. V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms* (Springer Publishing Company, Incorporated, 2011).

19. H. Brezis, "Analyse fonctionnelle," *Collection Mathématiques Appliquées pour la Maîtrise* (Masson, 1983).

20. Altair Engineering, "Altair Flux 12.3," 2017, https://www.altair.com.es/flux/.

21. C. Pechstein, "Multigrid-Newton-Methods for Nonlinear Magnetostatic Problems," 2004.Ph.D. thesis.

Appendix A

Relationship Between the Properties of Functions $\mathcal{B}$ and b

Proposition 1. *Let $\mathcal{B}$ (resp. b) be of the form (17) (resp. (18)), where μ is a function, $\mu : \Omega_{mc} \times [0,\infty) \mapsto [0,\infty)$. Then:*

i. *$\mathcal{B}$ is a Caratheodory function if b is a Caratheodory function.*
ii. *Function $\mathcal{B}$ satisfies inequality (12) if function b satisfies*

$$|b(\boldsymbol{x},s)| \le c_1 s + k_1(\boldsymbol{x}) \quad \forall s \in [0,\infty) \quad a.e.\ \boldsymbol{x} \in \Omega_{mc}$$

iii. *Function $\mathcal{B}$ satisfies assumption B.3 if function $b(\boldsymbol{x},\cdot)$ is strictly increasing for a.e. $\boldsymbol{x} \in \Omega_{mc}$.*
iv. *Function $\mathcal{B}$ satisfies inequality (14) if function b satisfies*

$$s b(\boldsymbol{x},s) \ge c_2 s^2 - k_2(x) \quad \forall s \in [0,\infty) \quad a.e.\ \boldsymbol{x} \in \Omega_{mc}$$

v. *If function $\mathcal{B}$ satisfies inequality (15), then function b satisfies*

$$|b(\boldsymbol{x},s_1) - b(\boldsymbol{x},s_2)| \le \lambda |s_1 - s_2| \quad \forall s_2, s_2 \in [0,\infty) \quad a.e.\ \boldsymbol{x} \in \Omega_{mc}$$

with $\lambda = L$. Conversely, if function b satisfies this inequality, then function $\mathcal{B}$ satisfies (15) with $L = 3\lambda$.

vi. *Function $\mathcal{B}$ satisfies inequality (16) if the function b satisfies*

$$(b(\boldsymbol{x},s_1) - b(\boldsymbol{x},s_2)) \cdot (s_1 - s_2) \ge \omega |s_1 - s_2|^2 \quad \forall s_1, s_2 \in [0,\infty) \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{A1}$$

Proof. Let $\hat{\xi} \in \mathbb{R}^3$ such that $|\hat{\xi}| = 1$. By taking $\xi = s\hat{\xi}$ with $s \ge 0$, we easily obtain

$$b(\boldsymbol{x},s) = \mathcal{B}(\boldsymbol{x}, s\hat{\xi}) \cdot \hat{\xi} \tag{A2}$$

From this equation, the proof of the "only if" part of statements (i)–(iv) and (vi), and of the first part of (v) is immediate. Conversely, (17) and (18) obviously imply

$$\mathcal{B}(\boldsymbol{x},\xi) = \frac{b(\boldsymbol{x},|\xi|)}{|\xi|}\xi \quad \forall \xi \in \mathbb{R}^3, \xi \ne \boldsymbol{0} \quad a.e.\ \boldsymbol{x} \in \Omega_{mc} \tag{A3}$$

Using this relation, the proof of the "if" part of statements (i), (ii) and (iv) is easy. The proof of the "if" part of statement (vi) is contained in the proof of [21], Lemma 2.8. For the proof of the "if" part of statement (iii), we first note that the computation done in the proof of that lemma (replacing the monotonicity constant by zero) gives the inequality

$$(\mathcal{B}(\boldsymbol{x},\boldsymbol{\eta}) - \mathcal{B}(\boldsymbol{x},\xi)) \cdot (\boldsymbol{\eta} - \xi) \ge (b(\boldsymbol{x},|\boldsymbol{\eta}|) - b(\boldsymbol{x},|\xi|))(|\boldsymbol{\eta}| - |\xi|) \quad \forall \xi, \boldsymbol{\eta} \in \mathbb{R}^3$$

Since $b(\boldsymbol{x},\cdot)$ is strictly increasing, this implies (13) if $|\boldsymbol{\eta}| \ne |\xi|$. If $|\boldsymbol{\eta}| = |\xi|$ and $\boldsymbol{\eta} \ne \xi$, we have $|\boldsymbol{\eta}| > 0$ and hence $\mu(\boldsymbol{x},|\boldsymbol{\eta}|) > 0$. This fact and (17) imply (13).

The second part of statements (v) is proved in [21], Lemma 2.9. □