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EXISTENCE OF SOLUTIONS FOR NON-LINEAR BOUNDARY VALUE PROBLEMS

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Existence of solutions for non-linear boundary value problems

Dna. Lorena Saavedra López

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Existence of solutions for non-linear boundary value problems

D. Alberto Cabada Fernández

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To Juan,
because he loves himself too

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Contents

Preface

It is now time to state what this project is, what it was and what it is meant to be. Through the last years I have developed, gathered and put together some results concerning the same specific aspect of the theory of differential equations, which can be roughly summarised as follows: ensuring the existence of constant sign solutions for different boundary value problems.

In 2014, when I was presented this project, I was supposed to study a family of differential equations coupled with the so called $(k, n-k)$ boundary conditions. The idea was to make a connection with the spectral theory to achieve our purposes. Once this connection was made, it revealed itself as a very useful tool to extrapolate the obtained results to many different boundary conditions.

Even though the final objective is the understanding of non-linear boundary value problems, the hardest and most important results deal with linear problems. This is due to the fact that, despite linear problems having significance and interest by themselves, we use them to make sure the existence of one or multiple solutions for several non-linear boundary value problems.

For instance, let us consider a general n^{th}-order differential equation as follows:

$$T_n\, u(t) = u^{(n)}(t) + p_1(t)\, u^{(n-1)}(t) + \cdots + p_{n-1}(t)\, u'(t) + p_n(t)\, u(t) = f\big(t, u(t)\big),$$

where $t \in I \equiv [a, b]$ and $p_j \in C^{n-j}(I)$ for $j = 1, \ldots, n$, coupled with some given boundary conditions.

It is well-known that the solutions of this problem coincide with the fixed points of the operator:

$$\mathcal{L}\, u(t) = \int_a^b g(t, s)\, f\big(s, u(s)\big)\, \mathrm{d}\, s,$$

in a suitable Banach space, where $g(t, s)$ is an integral kernel, known as Green's function, related to the linear problem:

$$T_n\, u(t) = 0, \quad t \in I,$$

with the considered boundary conditions.

Hence, the study of the linear problem gives a lot of information regarding the non-linear case. Indeed, the constant sign of the related Green's function ensures the existence of a constant sign solution under some additional hypotheses. This is the reason why the main aim of the first chapters is to characterise the constant sign of the Green's function related to several linear boundary value problems.

However, this is not always a suitable technique to study some kinds of problems. So, in Chapter 7, we develop a different method to prove the existence of solutions of non-linear boundary value problems involving the p-Laplacian: the variational approach.

To sum up, this Thesis contains a qualitative study of a wide range of linear and non-linear boundary value problems. We have achieved even more than the initial objectives which seemed so distant and difficult at first.

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Summary

As a motivation to start with the study of boundary value problems, we add a summary of all the chapters which this book contains with a final part of conclusions and future problems to consider.

This Thesis, compiled under the title "Existence of solutions for non-linear boundary value problems", contains a detailed collection of the different results proved by the author in her predoctoral stage.

The interest of the non-linear differential equations is well-known. This is due to their applications in different fields, such as physics, economy, medicine, biology or chemistry.

It is very important to make a precise study of the existence of solutions for this kind of problems, as well as their uniqueness or multiplicity. In this work, we focus on the qualitative analysis of diverse boundary value problems, both linear and non-linear ones. Indeed, in most of the cases our fundamental interest is to ensure the existence of constant sign solutions in their definition interval. This interest comes from the constant sign of many of the magnitudes which are modelled by this kind of problems.

Even though in the title we only refer to non-linear problems; in many cases, studying them is not possible without doing a previous study of an associated linear problem. So, the first chapters are devoted to linear boundary value problems. Using the previously obtained results, Chapters 6 and 7 contain the study of different non-linear boundary value problems.

Despite the fact that the linear problems have been presented as a tool in the study of non-linear problems, this kind of problems have interest by themselves. Indeed, along the first five chapters, we will see a wide number of examples where the usefulness of the results is shown.

Chapter 1: Preliminary results

In order to construct a self-contained work, Chapter 1 includes a collection of concepts and results which will be used along the different chapters.

First, our interest is focused on the concept of disconjugacy for a linear differential equation or order n:

$$T_n[M]\,u(t) = u^{(n)}(t) + p_1(t)\,u^{(n-1)}(t) + \cdots + p_{n-1}(t)\,u'(t) + \big(p_n(t) + M\big)\,u(t) = 0, \quad (1)$$

where $t \in I \equiv [a,b]$ and $p_j \in C^{n-j}(I)$. Disconjugacy gives an upper bound on the number of zeros which a differential equation can achieve.

Following the monograph of Coppel, see [42, Chapter 3], some properties of a differential equation in relation with this concept are established.

The aim of Section 1.2 is to introduce the concept of Green's function related to a two-point linear boundary value problem as follows:

$$\begin{aligned} T_n[M]\,u(t) &= \sigma(t), \quad \text{if } t \in I, \\ U_i(u) &= 0, \quad \text{if } i = 1, \ldots, n, \end{aligned} \tag{2}$$

where

$$U_i(u) = \sum_{j=0}^{n-1} \left(\omega_j^i \, u^{(j)}(a) + \nu_j^i \, u^{(j)}(b) \right), \quad i = 1, \ldots, n,$$

being ω_j^i and ν_j^i real constants for $i = 1, \ldots, n$, $j = 0, \ldots, n-1$ and $\sigma \in C(I)$.

It is well-known, see Theorem 1.2.4, that problem (2) has a unique solution if, and only if, there exists a unique associated Green's function, $g_M(t,s)$. In such a case, the unique solution is given by:

$$u(t) = \int_a^b g_M(t,s)\,\sigma(s)\,\mathrm{d}\,s.$$

From this expression, we can easily deduce that, if the related Green's function is of constant sign, then the solution of the related problem (2) has the same constant sign for every $\sigma \geqslant 0$. This property is known as inverse positive or negative character, depending on the sign.

In this section, we present a series of results related with the Green's function which appear in [16]. Furthermore, we show relation between disconjugacy and Green's function constant sign for the problem:

$$\begin{aligned} T_n[M]\,u(t) &= \sigma(t), \quad t \in I, \\ u(a) = u'(a) = \cdots &= u^{(k-1)}(a) = 0, \\ u(b) = u'(b) = \cdots &= u^{(n-k-1)}(b) = 0, \end{aligned} \tag{3}$$

see Lemma 1.2.14.

The boundary conditions considered in problem (3) are the so-called $(k, n-k)$ boundary conditions and we denote by X_k the set of functions of $C^n(I)$ which satisfy the such boundary conditions.

To finish the preliminary chapter, the expression of the adjoint operator to a given operator is introduced, together with a relation between the Green's function of both problems:

$$g_M^*(t,s) = g_M(s,t),$$

being $g_M^*(t,s)$ the Green's function related to the adjoint operator of (2).

Chapter 2: Disconjugacy

Now the relation between the constant sign Green's function for the $(k, n-k)$ problems and the disconjugacy of the associated linear differential equation, given by Lemma 1.2.14, is known. Thus, this chapter is devoted to characterising when the linear differential equation (1) is or not disconjugate. The characterisations are proved by means of spectral theory.

At first, we determine the parameter set for which the linear differential equation (1) is disconjugate.

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Theorem 1 (Theorem 2.1.1)**.** *Let $\bar{M} \in \mathbb{R}$ and $n \geqslant 2$ be such that* (1) *is a disconjugate equation on I for $M = \bar{M}$. Then,* (1) *is a disconjugate equation on I if, and only if, M belongs to $(\bar{M} - \lambda_1, \bar{M} - \lambda_2)$, where*

- *$\lambda_1 = +\infty$ if $n = 2$ and, for $n > 2$, $\lambda_1 > 0$ is the minimum of the least positive eigenvalues of $T_n[\bar{M}]$ in X_k, with $n - k$ even.*
- *$\lambda_2 < 0$ is the maximum of the biggest negative eigenvalues of $T_n[\bar{M}]$ in X_k, with $n - k$ odd.*

Since the main hypothesis used in previous result is that there exists $\bar{M} \in \mathbb{R}$ such that the linear differential equation (1) is disconjugate for $M = \bar{M}$, it is important to know when the disconjugacy parameter set is, or not, empty. The second part of Chapter 2 contains several results in this direction, finishing with two equivalent results where we obtain a spectral characterisation for this property.

Theorem 2 (Theorem 2.2.9)**.** *The linear differential equation* (1) *is not disconjugate on I for every $M \in \mathbb{R}$ if, and only if, there exist $c_1, c_2 \in (a, b]$ and $\bar{M} \in \mathbb{R}$ such that:*

- *There exist $k^* \in \{1, \ldots, n-1\}$, such that $n - k^*$ is even, and $\lambda_1 \leqslant 0$, an eigenvalue of $T[\bar{M}]$ in $X_{k^*[a,c_1]}$.*
- *There exist $k^{**} \in \{1, \ldots, n-1\}$, such that $n - k^{**}$ is odd, and $\lambda_2 \geqslant 0$, an eigenvalue of $T[\bar{M}]$ in $X_{k^{**}[a,c_2]}$.*

Remark 3. *Realise that in previous result $X_{k[c,d]}$ denotes the set X_k in $[c, d] \subset [a, b]$.*

Theorem 2.2.10 is an equivalent reformulation of Theorem 2 where a characterisation to ensure that there exist a minimum disconjugacy parameter set is proved.

Results from this chapter are published in [28, 31].

Chapter 3: Green's functions

This is the largest chapter of this Thesis, where we collect some of the most important results of this project. At the same time, many of these results are technical and with a difficult notation. Thus, along the chapter, we illustrate them with a particular recurrent example. The final objective is to characterise the constant sign of the Green's function related to the problem:

$$\begin{aligned} T_n[M] = \sigma(t), \quad t \in I \equiv [a, b], \\ u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0, \end{aligned} \tag{4}$$

where $\sigma_i, \varepsilon_j \in \mathbb{Z}$ for $i = 1, \ldots, k$, $j = 1, \ldots, n - k$ and

$$0 \leqslant \sigma_1 < \sigma_2 < \cdots < \sigma_k \leqslant n - 1, \quad 0 \leqslant \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leqslant n - 1.$$

Remark 4. *Denote as $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, the set of function of class $C^n(I)$ satisfying the considered boundary conditions. Moreover, let $\alpha,\ \beta \in \mathbb{Z}$ be such that:*

$$\alpha \notin \{\sigma_1,\dots,\sigma_k\},\ \text{and if}\ \alpha \neq 0,\ \text{then}\ \{0,\dots,\alpha-1\} \subset \{\sigma_1,\dots,\sigma_k\},$$
$$\beta \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\},\ \text{and if}\ \beta \neq 0,\ \text{then}\ \{0,\dots,\beta-1\} \subset \{\varepsilon_1,\dots,\varepsilon_{n-k}\}.$$

In fact, we will characterise when the related Green's function satisfies the following properties, stronger than having constant sign.

(P_{g_1}) There are three continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a,b)$ and $0 < k_1(t) < k_2(t)$ for all $t \in (a,b)$, satisfying:

$$\phi(s)\, k_1(t) \leqslant g_M(t,s) \leqslant \phi(s)\, k_2(t), \quad \text{for all } (t,s) \in I \times I.$$

(N_{g_1}) There are three continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a,b)$ and $k_2(t) < k_1(t) < 0$ for all $t \in (a,b)$, satisfying:

$$\phi(s)\, k_2(t) \leqslant g_M(t,s) \leqslant \phi(s)\, k_1(t), \quad \text{for all } (t,s) \in I \times I.$$

In Section 3.3, we deduce the imposed hypotheses to the operator and the boundary conditions. We describe them below.

(T_d) Let us say that the operator $T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, there exists the following decomposition:

$$T_0\, u(t) = u(t), \quad T_k\, u(t) = \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{T_{k-1}\, u(t)}{v_k(t)}\right),\ k = 1,\dots,n,$$

where $v_k > 0$, $v_k \in C^n(I)$ such that:

$$T_n[\bar{M}]\, u(t) = v_1(t)\ \dots\ v_n(t)\, T_n\, u(t),\ t \in I,$$

and, moreover, this decomposition satisfies, for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$,

$$T_{\sigma_1}\, u(a) = \dots = T_{\sigma_k}\, u(a) = 0,$$
$$T_{\varepsilon_1}\, u(b) = \dots = T_{\varepsilon_{n-k}}\, u(b) = 0.$$

(N_a) Let us say that $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy property (N_a) if, and only if,

$$\operatorname{card}\Big\{\ell \in \{\sigma_1,\dots,\sigma_k\} \mid \ell < h\Big\} + \operatorname{card}\Big\{\ell \in \{\varepsilon_1,\dots,\varepsilon_{n-k}\} \mid \ell < h\Big\} \geqslant h,$$

for all $h \in \{1,\dots,n-1\}$.

- The decomposition given by property (T_d) is not necessarily unique.
- If $T_n[\bar{M}]$ satisfies (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a) if, and only if, $\lambda = 0$ is not an eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

- In Section 3.8, we prove that hypothesis (T_d) cannot be weakened in general.

After a series of previous results, in Section 3.7, a characterisation of the parameter set for which Green's function fulfils either $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$ is obtained, being this property satisfied in a neighbourhood of $\bar{M} \in \mathbb{R}$ for which property (T_d) is fulfilled.

Theorem 5 (Theorem 3.7.1)**.** *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a).*

Let us denote by $g_M(t,s)$ the related Green's function of problem (3.0.1)–(3.0.3). *The following properties are fulfilled.*

- *If $n-k$ is even and $2 \leqslant k \leqslant n-1$, then $g_M(t,s)$ satisfies property $\left(P_{g_1}\right)$ if, and only if, $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, where:*
  - * *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * *$\lambda_2 < 0$ is the maximum between:*
    - · *$\lambda_2' < 0$, the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
    - · *$\lambda_2'' < 0$, the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *If $k=1$ and n is odd, then $g_M(t,s)$ satisfies property $\left(P_{g_1}\right)$ if, and only if, M belongs to $(\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, where:*
  - * *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
  - * *$\lambda_2 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.*
- *If $n-k$ is odd and $2 \leqslant k \leqslant n-2$, then $g_M(t,s)$ satisfies property $\left(N_{g_1}\right)$ if, and only if, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, where:*
  - * *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * *$\lambda_2 > 0$ is the minimum between:*
    - · *$\lambda_2' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
    - · *$\lambda_2'' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *If $k=1$ and $n>2$ is even, then $g_M(t,s)$ satisfies property $\left(N_{g_1}\right)$ if, and only if, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, where:*
  - * *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
  - * *$\lambda_2 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.*
- *If $k=n-1$ and $n>2$, then $g_M(t,s)$ satisfies property $\left(N_{g_1}\right)$ if, and only if, M belongs to $[\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, where:*
  - * *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.*

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* $\lambda_2 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-2}\}}^{\{\varepsilon_1|\beta\}}$.

- *If* $n = 2$*, then* $g_M(t,s)$ *satisfies* $\left(N_{g_1}\right)$ *if, and only if,* $M \in (-\infty, \bar{M} - \lambda_1)$*, where:*

    * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.

In the second part of Section 3.7, we study the related Green's function for a set of parameters which does not contain $\bar{M}$. In such a case, we cannot obtain a full characterisation of the parameter set for which the Green's functions fulfils one of the previously mentioned properties. However, we give an upper bound for this interval, together with some properties which are satisfied by the Green's function in such an interval.

The results of this Chapter appear in [29, 30].

Chapter 4: Strongly inverse positive (negative) operators

Operator $T_n[M]$ is said to be strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if for every function $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\,u \gneqq 0$ in I, then $u > 0$ on (a,b) and, in addition,

$$u^{(\alpha)}(a) > 0 \text{ and } \begin{cases} u^{(\beta)}(b) > 0, & \text{if } \beta \text{ is even,} \\ u^{(\beta)}(b) < 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

Analogously, we can define the concept of strongly inverse negative character.

In the same way, we post a relation between the constant sign Green's function and the strongly inverse positive or negative character of the associated problem. Using this relationship, in Chapter 4 we obtain a characterisation for these properties. Indeed, Theorem 5 can be rewritten, substituting properties $\left(P_{g_1}\right)$ and $\left(N_{g_1}\right)$ by strongly inverse positive or negative character, respectively (see Theorem 4.1.1).

As a direct consequence of this result, using the lower and upper solutions method, we obtain a result for the operator:

$$T_n[M,c]\,u(t) = T_n[M]\,u(t) + c(t)\,u(t), \quad t \in I, \tag{5}$$

with $c \in C(I)$.

Corollary 6 (Corollary 4.1.3)**.** *Consider the operator* $T_n[M,c]$ *defined in* (5).

Let $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and the set of indices* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfies* (N_a).

Let λ_1 *and* λ_2 *be as in Theorem 5. Then the following properties are fulfilled:*

- *If* $n-k$ *is even and* $-\lambda_1 < c(t) \leqslant -\lambda_2$ *for all* $t \in I$*, then* $T_n[\bar{M},c]$ *is strongly inverse positive in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

- *If* $n-k$ *is odd,* $n > 2$ *and* $-\lambda_2 \leqslant c(t) < -\lambda_1$ *for all* $t \in I$*, then* $T_n[\bar{M},c]$ *is strongly inverse negative in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

- *If* $n = 2$*, and* $c(t) < -\lambda_1$ *for all* $t \in I$*, then* $T_n[\bar{M},c]$ *is strongly inverse negative in* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.

In Section 4.3 we show several particular cases, where we prove the applicability of the achieved results. We point out that we avoid the calculus, and posterior study, of the expression of the Green's function to obtain the parameter set for which the operator is either strongly inverse positive or negative. Thence, the obtained results are shown as a very useful tool to ensure that a family of linear boundary value problems has a constant sign solution.

In the next section, we proceed to the study of problems with associated non-homogeneous boundary conditions. The main result of this section consists in a spectral characterisation of the strongly inverse positive or negative character for a wide range of non-homogeneous boundary value problems.

The Chapter finishes with a enumeration of examples, where the usefulness of the results is shown again, even for problems with non-constant coefficients.

As in Chapter 3, most of the results and examples here shown are published in [29, 30].

Chapter 5: Simply supported beam

This chapter is devoted to a fourth order problem coupled with the simply supported beam boundary conditions. This is a very interesting model in physics, since it represents the behaviour of a suspension bridge. In this case, the constant sign of the obtained solutions is equivalent to the case where the vertical displacement of the roadway occurs in only one direction (in the same direction of the applied forces when the sign is positive, downwards because of the gravity force). This is a fundamental fact in order to ensure the stability of the structure.

In [53, Section 2.6.6], it is considered a hinged beam (which models the roadway of the bridge) grabbed by non-linear forces along a two-sided string (the hangers of the suspension bridge). The unidimensional model for the vertical displacement of the roadway is:

$$\begin{cases} E\,I\,u^{(4)}(t) - T\,u''(t) + g(u(t)) = q(t), \quad t \in I, \\ u(a) = u(b) = u''(a) = u''(b) = 0, \end{cases} \tag{6}$$

where a and b are the extremes of the bridge, E and I are two positive constant related to the material of the beam (the Young's module and the moment of inertia), $T \geqslant 0$ is the strength tension constant, $q(t)$ is a downwards load distributed along the beam and g is the restoring force. If we consider g as a non-autonomous force in the sense $g(t, u(t)) = f(t)\,u(t)$, where $f \in C(I)$, dividing in (6) by $E\,I$ we have:

$$T_4[p,c]\,u(t) = u^{(4)}(t) - p\,u''(t) + c(t)\,u(t) = h(t), \quad t \in I,$$

where $p = \dfrac{T}{E\,I}$, $c(t) = \dfrac{f(t)}{E\,I}$ and $h(t) = \dfrac{q(t)}{E\,I}$.

The second part of the chapter is dedicated to proving the existence of constant sign solutions of the problem:

$$\begin{cases} T_4[p,c]\,u(t) = h(t)(\geqslant 0), \quad t \in I \\ u(a) = u''(a) = u(b) = u''(b) = 0, \end{cases} \tag{7}$$

for more general considerations of c than the previous ones.

In the first part of the chapter, we make a similar study to the one done in Chapters 3 and 4, for the particular problem:

$$\begin{cases} T_4[M]\,u(t) = u^{(4)}(t) + p_1(t)\,u'''(t) + p_2(t)\,u''(t) + M\,u(t) = h(t), \quad t \in I \\ u(a) = u''(a) = u(b) = u''(b) = 0, \end{cases}$$

where $p_1 \in C^3(I)$ and $p_2 \in C^2(I)$.

The obtained results for this problem and the strongly inverse positive character are a particular case of the ones proved in Chapter 4. However, in this case, for the strongly inverse negative character, we can prove that the maximum interval where this property can be fulfilled is indeed an optimal one. Thus, for this case we characterise the parameter set for both the strongly inverse positive character and the strongly inverse negative character. The results in this section can be seen in [32].

As we have said, the second part of Chapter 5 is devoted to the study of problem (7). In this chapter, we introduce a different technique which will be developed in Chapters 7: the variational approach.

All the previously proved results, in the same vein as Corollary 6, to make sure the existence of constant sign solution, impose that the function $c(t)$ remains between two eigenvalues of the operator $T_4[p,0]$ with different fixed boundary conditions. In this case, by means of variational approach, these values can be crossed in some sense, by imposing any additional hypotheses on its behaviour, for instance that the integral of the negative part of c is bounded (see Theorem 5.2.8). This fact increases significantly the number of functions which can be used to construct these physical models.

To finish the Chapter, we show a particular case of a suspension bridge where the previous results are applied. The results of this part are collected in [33].

With this chapter we finish our qualitative study of a wide number of linear boundary value problems, where we have obtained many results of which the usefulness is proved by means of numerous examples along the different chapters.

Chapter 6: Existence results for non-linear problems via constant sign Green's function

The aim of this chapter is to use the results from previous chapters to see when the following non-linear problem has one or multiple constant sign solutions for the different choices of the two-point boundary conditions:

$$\begin{cases} T_n[M]\,u(t) = f\big(t, u(t)\big),\ t \in I \equiv [a,b], \\ u^{(\sigma_1)}(a) = u^{(\sigma_2)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = u^{(\varepsilon_2)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0. \end{cases} \tag{8}$$

It is well-known that the solutions of the previous problem are given by the fixed point of the integral operator:

$$\mathcal{L}[M]\,u(t) = \int_a^b g_M(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s, \tag{9}$$

where $g_M(t,s)$ is the Green's function related to problem (8).

Under the hypothesis that $g_M(t,s)$ satisfies either $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$, by using different well-known results, such as the Krasnosel'skiĭ's Fixed Point Theorem, we prove diverse results which ensure the existence of one or multiple fixed points of the operator defined in (9) in some Banach spaces.

Once the results are shown and proved for the general problem (8), in Section 6.4, we study particular problems. We consider examples of which the linear part has been studied before. Indeed, we characterise the parameter set for which the related Green's function satisfies either $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$.

For instance, we consider the problem:

$$\begin{cases} u^{(4)}(t) - \pi^4 u(t) = f\big(t, u(t)\big), \ t \in [0,1], \\ u(0) = u''(0) = 0, \\ u'(1) = u''(1) = 0, \end{cases} \tag{10}$$

where f is a non-negative function on $[0,1] \times (-\infty, 0]$.

Using the following result we prove the existence of three non-positive solutions for the previous problem.

Theorem 7 (Theorem 6.4.4). *Let p, q and r be positive numbers satisfying the relation:*

$$0 < p < \frac{500}{147} p < q < \frac{500}{147} q \leqslant r.$$

Assume, moreover, that the function f satisfies the following conditions:

(a) $f(t,u) \leqslant \dfrac{3\,\pi^3}{2}\, r$ *for all* $t \in [0,1]$ *and* $u \in [-r, 0]$,

(b) $f(t,u) < \dfrac{3\,\pi^3}{2}\, p$ *for all* $t \in [0,1]$ *and* $u \in \left[-\dfrac{500}{147}p, 0\right]$,

(c) $f(t,u) \geqslant -214\, u$ *for all* $t \in \left[\dfrac{1}{3}, 1\right]$ *and* $u \in \left[-\dfrac{500}{147}q, -q\right]$.

Then, the problem (10) *has at least three solutions, u_1, u_2, u_3 such that $\|u_i\|_{C(I)} \leqslant r$ for $i = 1, 2, 3$ and:*

$$\min_{t \in \left[\frac{1}{3},1\right]} u_1(t) > -p > \min_{t \in \left[\frac{1}{3},1\right]} u_2(t), \qquad \max_{t \in \left[\frac{1}{3},1\right]} u_2(t) > -q > \max_{t \in \left[\frac{1}{3},1\right]} u_3(t).$$

Analogously, for the problem:

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) = u''(1) = 0, \end{cases}$$

where $f(t,u) \geqslant 0$ for all $(t,u) \in [0,1] \times [0, +\infty)$, we prove the existence of at least two positive solutions by means of the following result.

Theorem 8 (Theorem 6.4.16)**.** *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r.$$

Assume, moreover, that function f satisfies the following conditions:

(i) $f(t,u) \geqslant \dfrac{15625\,u}{66}$ *for all* $t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ *and* $u \in \left[r, \dfrac{125}{48}\, r\right]$*, being the inequality strict at* $u = r$*,*

(ii) $f(t,u) \leqslant 72\,q$ *for all* $t \in [0,1]$ *and* $u \in \left[0, \dfrac{125}{48}\, q\right]$*, being the inequality strict at* $u = q$*,*

(iii) $f(t,u) > \dfrac{1600000\,u}{8073}$ *for all* $t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ *and* $u \in \left[\dfrac{48}{125}\, p, p\right]$.

Then, the problem (6.4.37) *has at least two positive solution, u_1 and u_2, such that:*

$$p < \|u_1\|_{C(I)}, \qquad \max_{t \in \left[\frac{1}{5}, \frac{4}{5}\right]} u_1(t) < q < \max_{t \in \left[\frac{1}{5}, \frac{4}{5}\right]} u_2(t), \qquad \min_{t \in \left[\frac{1}{5}, \frac{4}{5}\right]} u_2(t) < r.$$

Despite that the calculus can be hard, due to the expression of the related Green's function, these results can be proved for a wide family of boundary value problems. Nevertheless, the existence of one constant sign solution can be proved if f satisfies one of the two following conditions:

$$f_0^+(t) := \limsup_{u \to 0^+} \frac{f(t,u)}{u} = 0, \quad \text{and} \quad f_\infty^-(t) := \liminf_{u \to \infty} \frac{f(t,u)}{u} = +\infty,$$
$$f_0^-(t) := \liminf_{u \to 0^+} \frac{f(t,u)}{u} = +\infty, \quad \text{and} \quad f_\infty^+(t) := \limsup_{u \to \infty} \frac{f(t,u)}{u} = 0,$$

without knowing the expression of the Green's function, see Corollaries 6.2.4 and 6.2.5.

The techniques here developed are suitable for problems with different boundary conditions, provided that the related Green's function satisfies either $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$.

Chapter 7: Existence results for non-linear problems via variational approach

This chapter is devoted to develop a different technique to the used before: the variational approach.

The non-linear problems studied in this chapter are different from all the studied before, since they involve non-linearities in all the non-null coefficients. Thus, we cannot construct a linear associated problem for which obtain the related Green's function as before. In this case, the solutions of the considered problems are given by critical points of associated operators in suitable Banach spaces.

This chapter is devoted to study, by means of variational approach, three different problems: one continuous one in Section? 7.1 and two discrete ones in Section 7.2. The results of the first section can be seen in [83] and the results related to the discrete problems are collected in [84].

Periodic $2\,n^{\text{th}}$-order non-linear p-Laplacian differential equations

In this section we study the problem:

$$\begin{cases} \left[\varphi_p\left(u^{(n)}(t)\right)\right]^{(n)} + \displaystyle\sum_{i=1}^{n-1}(-1)^i a_i \left[\varphi_p\left(u^{(n-i)}(t)\right)\right]^{(n-i)} \\ \qquad\qquad +(-1)^n\Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big) = 0,\ t\in[0,T], \\ u(T)-u(0) = \cdots = u^{(2\,n-1)}(T) - u^{(2\,n-1)}(0) = 0, \end{cases} \tag{11}$$

where $T \geqslant 0$, $a_i \geqslant 0$ for $i = 1, \ldots, n-1$ and φ_p denotes the p-Laplacian functional, defined as follows:

$$\varphi_p(t) = \begin{cases} t|t|^{p-2}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

We need to distinguish between a classical solution and a weak solution of problem (11).

Definition 9. *A function $u \in C^n([0,T])$ is said to be a classical solution of problem* (11) *if $\varphi_p\left(u^{(n)}(\cdot)\right) \in C^n([0,T])$ and it satisfies* (11).

Before saying what a weak solution is, we must introduce the space where they are defined in.

$$W_p := \left\{u \in W^{n,p}(0,T) \mid u(T)-u(0) = \cdots = u^{(n-1)}(T) - u^{(n-1)}(0) = 0\right\},$$

where $W^{n,p}(0,T)$ is the Sobolev space:

$$W^{n,p}(0,T) = \left\{u \in L^p(0,T) \mid u^{(i)} \in L^p(0,T),\ i = 1,\ldots,n\right\}.$$

Definition 10. *The function $u \in W_p$ is said to be a weak solution of* (11) *if, for every $v \in W_p$, it is satisfied the following equality:*

$$\begin{aligned} \int_0^T \varphi_p\left(u^{(n)}(t)\right) v^{(n)}(t)\,\mathrm{d}\,t + \sum_{i=1}^{n-1} a_i \int_0^T \varphi\left(u^{(n-i)}(t)\right)\, v^{(n-i)}(t)\,\mathrm{d}\,t \\ + \int_0^T \Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big)\, v(t)\,\mathrm{d}\,t = 0. \end{aligned} \tag{12}$$

The tools given by the variational approach allow us to prove the existence of weak solutions, provided that f and h satisfy a suitable property. However, we cannot always ensure that the obtained weak solutions are also classical solutions of problem (11). In this section, we obtain different existence results, both for weak and classical solutions.

At the end of the section, we see how the previously obtained results can be generalised for an impulsive problem.

Non-linear $2\,n^{\text{th}}$-order p-Laplacian difference equations

This last section is devoted to a different problem, in the sense that we are referring to a discrete problem. Here, one can see how the variational approach is also suitable, with some evident modifications, for a discrete problem.

We will study two different problems which involve the discrete equation:

$$\begin{aligned}\Delta^n\Big[\varphi_{p_n}\Big(\Delta^n u(k-n)\Big)\Big] + \sum_{i=1}^{n-1}(-1)^i a_i\,\Delta^{n-i}\Big[\varphi_{p_{n-i}}\Big(\Delta^{n-i}u(k-i)\Big)\Big]& \\ +(-1)^n\Big(V(k)\,\varphi_q\big(u(k)\big) - \lambda f\big(k,u(k)\big)\Big) &= 0,\end{aligned} \tag{13}$$

where $k \in \mathbb{Z}$ and:

- φ_p has been previously defined for $p > 1$,
- $V\colon \mathbb{Z} \to \mathbb{R}$ is a positive T-periodic function for T, a fixed integer,
- $f\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a given function with additional conditions in each case,
- $a_i \geqslant 0$ are real fixed numbers for each $i = 1, \ldots, n-1$,
- $p_i \geqslant q \geqslant 1$ for $i = 1, \ldots, n$,
- the forward difference operators are given by:

$$\begin{aligned}\Delta u(k) &= u(k+1) - u(k),\\ \Delta^i u(k) &= \Delta^{i-1}u(k+1) - \Delta^{i-1}u(k), \text{ if } i \geqslant 2.\end{aligned}$$

First, by imposing different hypotheses on f, the existence of non-zero homoclinic solutions for the equation (13) is ensured, or which is the same, we find non-trivial solutions of (13) such that

$$\lim_{|k|\to+\infty} \big|u(k)\big| = 0.$$

In the second part of the section, we consider a discrete boundary value problem:

$$\begin{aligned}\Delta^n\Big[\varphi_q\Big(\Delta^n u(k-n)\Big)\Big] + \sum_{i=1}^{n-1}(-1)^i a_i\Delta^{n-i}\Big[\varphi_q\Big(\Delta^{n-i}u\big(k-(n-i)\big)\Big)\Big]& \\ +(-1)^n\Big(V(k)\,\varphi_q\big(u(k)\big) - \lambda f\big(k,u(k)\big)\Big) &= 0,\\ u(0) = \Delta u(-1) = \Delta^2 u(-2) = \cdots = \Delta^{n-1}u(1-n) &= 0,\\ u(T+1) = \Delta u(T+1) = \Delta^2 u(T+1) = \cdots = \Delta^{n-1}u(T+1) &= 0,\end{aligned}$$

where $k \in [0,T] = \{0,1,2,\ldots,T\}$.

Here, under suitable hypotheses, we prove the existence of at least three solutions for the previous problem.

Conclusions and future problems

Along the seven chapters of this Thesis, we have proved a broad number or results for which the applicability is proved by means of the different examples.

One of the most important advances is the possibility of characterising the parameter set for constant sign solution without knowing its expression. Such an expression is usually hard to tackle, indeed, for the case of problems with non-constant coefficients we do not even have ensured being able of calculating its expression. Nevertheless, the calculus of the different eigenvalues is relatively easy. For the constant coefficient case it reduces to solving a linear system of equations and, for problems with non-constant coefficients, we can solve it by using software, such as *Mathematica*, with numerical methods. Thus, the different characterisations obtained along the first five chapters are useful and practical. We point out the one given in Theorem 5 for properties $\left(P_{g_1}\right)$ and $\left(N_{g_1}\right)$.

A possible problem to consider in the future is to generalise the results here proved for different boundary conditions, such as the periodic ones or two-point boundary conditions which do not fulfil (N_a). Along our study, we have seen that the techniques here used cannot be applied if the boundary conditions do not satisfy (N_a). One of the reasons is that for the $\bar{M} \in \mathbb{R}$ for which property (T_d) is fulfilled, $\lambda = 0$ is always an eigenvalue of the related operator with the considered boundary conditions. Thus, we do not have a unique Green's function defined for these value. On the other hand, for the periodic case, we can see several examples where the sign change of the Green's function does not appear at one of the vertices of $I \times I$. Then, our arguments are nor applicable to this case. Hence, the idea is to find a new technique or modify ours methods to obtain similar conclusions for these cases.

Even though, there are still cases to study, the amount of problems which can be analysed by using the results here collected is considerable, there are problems of all the orders with a high variety of boundary conditions, for instance in fourth order there are 40 possible boundary conditions to consider, including the well-known simply supported beam or the clamped beam boundary conditions.

In Chapter 6, using the results of previous chapters, we prove the existence of one or multiple constant sign solutions for a wide range of non-linear boundary value problems, we consider all the boundary conditions studied before. Indeed, if the result from Chapters 3 and 4 were generalised for different boundary conditions, then the results from Chapter 6 would be directly generalised.

Due to the big amount of different fixed point theorems, it is possible that the consideration of other results would prove the existence of solution under different hypotheses for f.

In Chapter 7, we introduce a different technique to achieve existence results. This is due to the different structure of the studied problems. Along this chapter, using the variational approach coupled with different results which ensure the existence of critical points for suitable operators, we arrive to some existence results. Here, we can see the difficulty of generalising a known result for higher order.

A problem we will study in the future is, following the lines initiated in [27], to combine techniques of variational approach and the Green's function constant sign to achieve results which make sure the existence of solutions for different non-linear boundary value problems.

Besides the future lines of investigation mentioned here, along the different chapters, several open problems are shown, coupled with the difficulties which kept us from solving them yet.

With the end of this project we arrive to a stop in our symbolic trip on the study of linear and non-linear boundary value problems, with a lot of route over, but still a long way to go. And, even though the obtained results are interesting, useful and powerful, as Robert Louis Stevenson wrote: "*For my part, I travel not to go anywhere, but to go. I travel for travel's sake. The great affair is to move*". Therefore, the study of linear and non-linear boundary value problems will continue to achieve new results and new open problems in a gratifying trip without a determined end.

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Chapter 1
Preliminary results

Contents

This chapter is devoted to introduce some definitions and previous results which we will use along the different chapters.

First, let us introduce a linear differential equation of order n, defined as follows:

$$T_n\, u(t) \equiv u^{(n)}(t) + p_1(t)\, u^{(n-1)}(t) + \;\cdots + p_{n-1}(t)\, u'(t) + p_n(t)\, u(t) = 0, \qquad (1.0.1)$$

where $t \in I \equiv [a,b]$ and $p_j \in C^{n-j}(I)$.

Instead of studying the operator T_n, let us consider the following family of n^{th}-order differential operators:

$$T_n[M]\, u(t) \equiv u^{(n)}(t) + p_1(t)\, u^{(n-1)}(t) + \cdots + p_{n-1}(t)\, u'(t) + (p_n(t) + M)\, u(t), \quad (1.0.2)$$

where $t \in I$ and $M \in \mathbb{R}$.

It is well-known that $p_n(t)$ in (1.0.2) can be uniquely decomposed as:

$$p_n(t) = \tilde{p}_n(t) + \frac{1}{b-a}\int_a^b p_n(s)\, \mathrm{d}\, s, \quad t \in I.$$

Thus, it is obvious that if we study a property on the general differential equation (1.0.1) is equivalent to determine the set of parameters M for which the linear differential equation:

$$T_n[M]\, u(t) = 0, \qquad (1.0.3)$$

satisfies such a property on I.

This chapter is divided in three sections: the first of them is devoted to the concept of disconjugacy of a linear differential equation of order n. After that, in Section 1.2 we define the Green's function related to a linear boundary value problem. Finally, Section 1.3 is devoted to introduce the adjoint operator of a given operator and some relations with the previous concepts.

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1.1 Disconjugacy

In this section, we introduce the concept of disconjugacy and some results collected on the monograph of Coppel [42, Chapter 3] about this concept.

Definition 1.1.1. *Let $p_k \in C^{n-k}(I)$ for $k = 1, \ldots, n$. The linear differential equation (1.0.3) of order n is said to be disconjugate on an interval I if every non-trivial solution has less than n zeros on I, multiple zeros being counted according to their multiplicity.*

Definition 1.1.2. *The functions $u_1, \ldots, u_n \in C^n(I)$ are said to form a Markov system on the interval I if the n Wronskians:*

$$W(u_1, \ldots, u_k) = \begin{vmatrix} u_1 & \cdots & u_k \\ \vdots & \cdots & \vdots \\ u_1^{(k-1)} & \cdots & u_k^{(k-1)} \end{vmatrix}, \quad k = 1, \ldots, n, \tag{1.1.1}$$

are positive throughout I.

Proposition 1.1.3. *If I is a compact interval, then there exists $\delta > 0$ such that the linear differential equation* (1.0.3) *is disconjugate on every subinterval of length less than δ.*

Theorem 1.1.4. *The linear differential equation (1.0.3) has a Markov system of solutions if, and only if, the operator $T_n[M]$ has a representation*

$$T_n[M]\, u \equiv v_1\, v_2\, \cdots\, v_n \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{v_n}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\cdots \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{v_2}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{v_1}\, u\right)\right)\right)\right), \tag{1.1.2}$$

where $v_k > 0$ on I and $v_k \in C^{n-k+1}(I)$ for $k = 1, \ldots, n$.

In order to see the construction of the different v_k, we give the proof of Theorem 1.1.4 which is collected on [42, Pages 91-93]. Before that, we show a previous Lemma which is used on the proof.

Lemma 1.1.5. *Let $u_1, \ldots, u_n, u \in C^k(I)$.*
If $W(u_1, \ldots, u_{k-1}) \neq 0$ and $W(u_1, \ldots, u_k) \neq 0$ for all $t \in I$, then

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{W(u_1, \ldots, u_{k-1}, u)}{W(u_1, \ldots, u_k)}\right) = \frac{W(u_1, \ldots, u_{k-1})\, W(u_1, \ldots, u_k, u)}{\Big(W(u_1, \ldots, u_k)\Big)^2}.$$

Proof of Theorem 1.1.4. Let $u_1, \ldots, u_n \in C^n(I)$ be a Markov fundamental system of solutions of (1.0.3). Let us denote $W_k = W(u_1, \ldots, u_k)$ for $k = 1, \ldots, n$ and $T_n \equiv T_n[M]$.

The linear differential equation with leading coefficient 1 which has $u_1, \ldots, u_k$ as a fundamental system of solutions is:

$$T_k\, u \equiv \frac{W(u_1, \ldots, u_k, u)}{W_k} = 0.$$

Define;

$$1 = W_0, \quad v_1 = W_1, \quad v_k = \frac{W_k\, W_{k-2}}{W_{k-1}^2}, \quad k = 2, \dots, n. \tag{1.1.3}$$

Using induction in k, the following equality is proved:

$$\frac{W_k}{W_{k-1}} = v_1\, \dots\, v_k. \tag{1.1.4}$$

We want to see that

$$T_k\, u \equiv v_1\, \cdots\, v_k \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{1}{v_k}\frac{\mathrm{d}}{\mathrm{d}\, t}\left(\cdots \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{1}{v_1}\, u\right)\right)\right) \text{ for } k = 1, \dots, n.$$

This is trivially true for $k = 1$.

Assume that this is satisfied for $k = h-1$. Then, using expression (1.1.4) and Lemma 1.1.5, for $k = h$, we have:

$$\begin{aligned} v_1\, \dots\, v_h \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{1}{v_h}\frac{\mathrm{d}}{\mathrm{d}\, t}\left(\cdots \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{1}{v_1}\, u\right)\right)\right) &= v_1\, \dots\, v_h \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{1}{v_1 \dots v_h}\, T_{h-1}\, u\right), \\ &= \frac{W_h}{W_{h-1}}\frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{W_{h-1}}{W_h}\, T_{h-1}\, u\right), \\ &= \frac{W_h}{W_{h-1}}\frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{W(u_1, \dots, u_{h-1}, u)}{W_h}\right), \\ &= \frac{W(u_1, \dots, u_h, u)}{W_h} = T_h\, u. \end{aligned}$$

Hence, we have obtained the representation (1.1.2).

Now, reciprocally, let us assume that the linear differential equation (1.0.3) has a representation of the form (1.1.2).

Define $u_1, \dots, u_n$ inductively, by setting $u_1 = v_1$ and taking u_k as a solution of the following equation:

$$\frac{W(u_1, \dots, u_{k-1}, u)}{W_{k-1}} = v_1\, \dots\, v_k, \quad k = 2, \dots, n.$$

Then, $u_1, \dots, u_n$ form a Markov system and taking the reciprocal of the above argument we obtain:

$$T_n\, u \equiv \frac{W(u_1, \dots, u_k, u)}{W_k},$$

so, $u_1, \dots, u_n$ is a fundamental system of solutions of (1.0.3). □

Theorem 1.1.6. *The linear differential equation (1.0.3) has a Markov fundamental system of solutions on the compact interval I if, and only if, it is disconjugate on I.*

Theorem 1.1.7. *Let $n,\, m \in \mathbb{N}$. If the equations $T_n^1[M]\, u(t) = 0$ and $T_m^2[M]\, u(t) = 0$ are disconjugate on the interval I, then the composite equation $T_n^1[M]\left(T_m^2[M]\, u(t)\right) = 0$ is also disconjugate on I.*

Defining the distance between two equations $(1.0.3)_1$ and $(1.0.3)_2$ by:

$$\sup_{t\in I}\sum_{k=1}^{n}\left|p_{k,1}(t)-p_{k,2}(t)\right|, \tag{1.1.5}$$

we have the following result in the corresponding metric space.

Proposition 1.1.8. *The set of all disconjugate equations* (1.0.3) *on a compact interval I is connected and open.*

Now, let us introduce a concept which is a useful tool to characterise the disconjugacy set of the linear differential equation (1.0.3).

Definition 1.1.9. *Let $a \in \mathbb{R}$, define the first right point conjugate of the point a, if it exists, for the linear differential equation (1.0.3) as:*

$$\eta_M(a) = \sup\left\{b > a \;\middle|\; \textit{equation (1.0.3) is disconjugate on } [a,b]\right\} \in (a,\infty).$$

Moreover, if the linear differential equation (1.0.3) is disconjugate on every interval $[a,b]$, with $b>a$, then $\eta_M(a)=\infty$.

Consider a fundamental system of solutions $y_1[M](t),\dots,y_n[M](t)$ of equation (1.0.3), where every $y_k[M](t)$ is uniquely determined by the following initial conditions:

$$y_k^{(n-k)}[M](a)=1,\quad y_k^{(n-j)}[M](a)=0,\; j=1,\dots,n,\; j\neq k. \tag{1.1.6}$$

Then, we denote the related $n-1$ Wronskians as:

$$W_k^n[M](t) := \begin{vmatrix} y_1[M](t) & \dots & y_k[M](t) \\ \vdots & \dots & \vdots \\ y_1[M]^{(k-1)}(t) & \dots & y_k[M]^{(k-1)}(t) \end{vmatrix}, \tag{1.1.7}$$

where $k=1,\dots,n-1$.

Proposition 1.1.10. *There exists a non-trivial solution of equation (1.0.3) which satisfies the boundary conditions $(k,n-k)$ on $[a,b]$ if, and only if, $W_{n-k}^n[M](b)=0$.*

Definition 1.1.11. *Denote $\omega_M(a)$ as the least $b>a$, if one exists, at which one of the Wronskians $W_1^n[M](b),\dots,W_{n-1}^n[M](b)$ vanishes.*

Next result gives a relation between this concept and the one given in Definition 1.1.9.

Proposition 1.1.12. $\eta_M(a)=\omega_M(a)$.

Proposition 1.1.13. *Let $b=\eta_M(a)<\infty$ and let $n-k\in\{1,\dots,n-1\}$ be such that $W_{n-k}^n[M](b)=0$ and $W_\ell^n[M](b)\neq 0$ for every $\ell<n-k$. The corresponding solution of equation* (1.0.3) *with $(k,n-k)$ boundary conditions is uniquely determined up to a constant factor, and it does not vanish on the open interval (a,b).*

Now, we introduce the set formed by a particular kind of functions, which satisfy the so-called $(k, n-k)$ boundary conditions:

$$X_k = \left\{ u \in C^n(I) \mid u(a) = \cdots = u^{(k-1)}(a) = u(b) = \cdots = u^{(n-k-1)}(b) = 0 \right\}, \quad (1.1.8)$$

where $1 \leqslant k \leqslant n-1$.

Disconjugacy gives a property for the eigenvalues of the n^{th}-order operators $T_n[M]$ in the sets X_k, see [76, Theorem 3.2].

Theorem 1.1.14. *Let $\bar{M} \in \mathbb{R}$ be such that equation* (1.0.3) *is disconjugate on I. Then for any $1 \leqslant k \leqslant n-1$ the following properties hold:*

- *If $n-k$ is even, then there is not any eigenvalue of $T_n[\bar{M}]$ on X_k such that $\lambda < 0$.*
- *If $n-k$ is odd, then there is not any eigenvalue of $T_n[\bar{M}]$ on X_k such that $\lambda > 0$.*

1.2 Green's function

The aim of this section is to introduce the concept of Green's function related to the following n^{th}-order two-point value problem.

$$T_n[M]\, u(t) = \sigma(t), \quad t \in I, \quad (1.2.1)$$

$$U_i(u) = 0, \quad i = 1, \dots, n, \quad (1.2.2)$$

where:

$$U_i(u) = \sum_{j=0}^{n-1} \left(\omega_j^i\, u^{(j)}(a) + \nu_j^i\, u^{(j)}(b) \right), \quad i = 1, \dots, n, \quad (1.2.3)$$

being ω_j^i and ν_j^i real constants for all $i = 1, \dots, n$, $j = 0, \dots, n-1$ and $\sigma \in C^n(I)$.

Now, following [16], in order to introduce the concept of Green's function related to the scalar problem (1.2.1)-(1.2.2) of order n, we consider the following equivalent first order vectorial problem for $\sigma \equiv 0$:

$$x'(t) = A(t)\, x(t),\ t \in I, \quad B\, x(a) + C\, x(b) = 0, \quad (1.2.4)$$

with $x(t) \in \mathbb{R}^n$, $A(t)$, B, $C \in \mathcal{M}_{n \times n}$, defined by:

$$x(t) = \begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix}, \quad A(t) = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline -\big(p_n(t) + M\big) & -p_{n-1}(t) \cdots - p_1(t) \end{array} \right),$$

$$B = \begin{pmatrix} \omega_0^1 & \cdots & \omega_{n-1}^1 \\ \vdots & \ddots & \vdots \\ \omega_0^n & \cdots & \omega_{n-1}^n \end{pmatrix}, \quad C = \begin{pmatrix} \nu_0^1 & \cdots & \nu_{n-1}^1 \\ \vdots & \ddots & \vdots \\ \nu_0^n & \cdots & \nu_{n-1}^n \end{pmatrix}. \quad (1.2.5)$$

Here I_j, $j = 1, \dots, n-1$, denotes the $j \times j$ identity matrix.

Realise that the correspondent boundary conditions (1.2.2) related to the definition set X_k introduced in (1.1.8) are defined as follows:

$$u(a) = \cdots = u^{(k-1)}(a) = u(b) = \cdots = u^{(n-k-1)}(b) = 0. \tag{1.2.6}$$

For this particular case, B and C read as follows:

$$B = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array} \right), \quad C = \left(\begin{array}{c|c} 0 & 0 \\ \hline I_{n-k} & 0 \end{array} \right).$$

Definition 1.2.1. *We say that G is a Green's function for problem* (1.2.4) *if it satisfies the following properties:*

(G1) $G \equiv (G_{i,j})_{i,j \in \{1,\dots,n\}} \colon (I \times I) \backslash \{(t,t),\ t \in I\} \to \mathcal{M}_{n \times n}$.

(G2) *G is a function of class C^1 on the triangles:*

$$\left\{(t,s) \in \mathbb{R}^2,\ a \leqslant s < t \leqslant b\right\} \text{ and } \left\{(t,s) \in \mathbb{R}^2,\ a \leqslant t < s \leqslant b\right\}.$$

(G3) *For all $i \neq j$ the scalar functions $G_{i,j}$ have a continuous extension to $I \times I$.*

(G4) *For all $s \in (a,b)$, the following equality holds:*

$$\frac{\partial}{\partial t} G(t,s) = A(t)\, G(t,s), \quad \text{for all } t \in I \backslash \{s\}.$$

(G5) *For all $s \in (a,b)$ and $i \in \{1,\dots,n\}$, the following equalities are fulfilled:*

$$\lim_{t \to s^+} G_{i,i}(t,s) = \lim_{t \to s^-} G_{i,i}(s,t) = 1 + \lim_{t \to s^+} G_{i,i}(s,t) = 1 + \lim_{t \to s^-} G_{i,i}(t,s).$$

(G6) *For all $s \in (a,b)$, the function $t \to G(t,s)$ satisfies the boundary conditions:*

$$B\, G(a,s) + C\, G(b,s) = 0.$$

Remark 1.2.2. *On the previous definition item* (G5) *can be modified to obtain the characterisation of the lateral limits for $s = a$ and $s = b$ as follows:*

$$\lim_{t \to a^+} G_{i,i}(t,a) = 1 + \lim_{t \to a^+} G_{i,i}(a,t), \quad \text{and} \quad \lim_{t \to b^-} G_{i,i}(b,t) = 1 + \lim_{t \to b^-} G_{i,i}(t,b).$$

It is very well-known that Green's function related to this problem follows the next expression [16, Section 1.4]:

$$G(t,s) = \begin{pmatrix} g_1(t,s) & \cdots & g_{n-1}(t,s) & g_M(t,s) \\ \frac{\partial}{\partial t} g_1(t,s) & \cdots & \frac{\partial}{\partial t} g_{n-1}(t,s) & \frac{\partial}{\partial t} g_M(t,s) \\ \vdots & \cdots & \vdots & \vdots \\ \frac{\partial^{n-1}}{\partial t^{n-1}} g_1(t,s) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}} g_{n-1}(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t,s) \end{pmatrix}, \tag{1.2.7}$$

where $g_M(t,s)$ is the scalar Green's function related to problem (1.2.1)-(1.2.2).

Using Definition 1.2.1, we can deduce the properties fulfilled by $g_M(t,s)$ and introduce the concept of Green's function related to the problem (1.2.1)-(1.2.2) (see [16, Section 1.4] for details).

Definition 1.2.3. *We say that g_M is a Green's function for problem* (1.2.1)-(1.2.2) *if it satisfies the following properties:*

(g_1) *g_M is defined on the square $I \times I$ (except $t = s$ if $n = 1$).*

(g_2) *For $\ell = 0, 1, \ldots, n-2$, the partial derivatives $\dfrac{\partial^\ell g_M}{\partial t^\ell}$ exist and they are continuous on $I \times I$.*

(g_3) *$\dfrac{\partial^{n-1} g_M}{\partial t^{n-1}}$ and $\dfrac{\partial^n g_M}{\partial t^n}$ exist and they are continuous on the triangles $a \leqslant s < t \leqslant b$ and $a \leqslant t < s \leqslant b$.*

(g_4) *For each $s \in (a,b)$, the function $t \to g_M(t,s)$ is a solution of the differential equation (1.2.1), with $\sigma \equiv 0$, on $[a,s)$ and on $(s,b]$.*

(g_5) *For each $t \in (a,b)$ there exist the lateral limits:*

$$\frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t^-,t) = \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t,t^+) \quad \text{and} \quad \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t,t^-) = \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t^+,t),$$

and, moreover,

$$\frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t^+,t) - \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t^-,t) = \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t,t^-) - \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t,t^+) = 1.$$

(g_6) *For each $s \in (a,b)$, the function $t \to g_M(t,s)$ satisfies the boundary conditions* (1.2.2).

In the sequel, there are shown some results about this concept which are collected on [16].

Theorem 1.2.4. *Problem* (1.2.1)-(1.2.2) *has a unique solution if, and only if, there exists a unique Green's function related to this problem.*

In such a case the unique solution is given by the following expression:

$$u(t) = \int_a^b g_M(t,s)\,\sigma(s)\,\mathrm{d}\,s.$$

First, let us consider the set where the operator $T_n[M]$ is defined as:

$$X = \Big\{ u \in C^n(I) \;\Big|\; U_i(u) = 0,\ i = 1, \ldots, n \Big\},$$

where $U_i(u)$ have been introduced in (1.2.3) for $i = 1, \ldots, n$.

Now, we introduce the concept of inverse positive (inverse negative) character of an operator.

Definition 1.2.5. *Operator $T_n[M]$ is said to be inverse positive (inverse negative) on X if every function $u \in X$ such that $T_n[M]\, u \geqslant 0$ on I, satisfies $u \geqslant 0$ ($u \leqslant 0$) on I.*

The next results are proved on [16, Section 1.6, Section 1.8] for several two-point n^{th}-order operators.

Theorem 1.2.6. *If $T_n[M]$ is inverse positive (inverse negative) on X, then the related Green's function is unique.*

Theorem 1.2.7. *Operator $T_n[M]$ is inverse positive (inverse negative) in X if, and only if, Green's function related to problem* (1.2.1)-(1.2.2) *is non-negative (non-positive) on its square of definition.*

Theorem 1.2.8. *Let M_1, $M_2 \in \mathbb{R}$ and suppose that operators $T_n[M_j]$, $j = 1, 2$, are invertible in X. Let g_j, $j = 1, 2$, be Green's functions related to operators $T_n[M_j]$ and suppose that both functions have the same constant sign on $I \times I$. Then, if $M_1 < M_2$, it is satisfied that $g_2 \leqslant g_1$ on $I \times I$.*

Theorem 1.2.9. *Let $M_1 < \bar{M} < M_2$ be three real constants. Suppose that operator $T_n[M]$ is invertible in X for $M = M_j$, $j = 1, 2$ and that the corresponding Green's function satisfies $g_2 \leqslant g_1 \leqslant 0$ (resp. $0 \leqslant g_2 \leqslant g_1$) on $I \times I$. Then the operator $T_n[\bar{M}]$ is invertible in X and the related Green's function $\bar{g}$ satisfies $g_2 \leqslant \bar{g} \leqslant g_1 \leqslant 0$ ($0 \leqslant g_2 \leqslant \bar{g} \leqslant g_1$) on $I \times I$.*

Now, we introduce two conditions on $g_M(t, s)$ that will be used along the work.

(P_g) There is a continuous function $\phi(t) > 0$ for all $t \in (a, b)$ and $k_1,\ k_2 \in L^1(I)$, such that $0 < k_1(s) < k_2(s)$ for a.e. $s \in I$, satisfying:

$$\phi(t)\, k_1(s) \leqslant g_M(t, s) \leqslant \phi(t)\, k_2(s), \quad \text{for a.e. } (t, s) \in I \times I.$$

(N_g) There is a continuous function $\phi(t) > 0$ for all $t \in (a, b)$ and $k_1,\ k_2 \in \mathcal{L}^1(I)$, such that $k_2(s) < k_2(s) < 0$ for a.e. $s \in I$, satisfying:

$$\phi(t)\, k_2(s) \leqslant g_M(t, s) \leqslant \phi(t)\, k_1(s), \quad \text{for a.e. } (t, s) \in I \times I.$$

Finally, we introduce the following sets, which describe the sets where the Green's function is of constant sign,

$$P_T = \Big\{ M \in \mathbb{R} \;\Big|\; \quad g_M(t, s) \geqslant 0, \quad \forall (t, s) \in I \times I \Big\}, \tag{1.2.8}$$

$$N_T = \Big\{ M \in \mathbb{R} \;\Big|\; \quad g_M(t, s) \leqslant 0, \quad \forall (t, s) \in I \times I \Big\}. \tag{1.2.9}$$

Realise that using Theorem 1.2.8 we can affirm that the two previous sets are real intervals (which can be empty or unbounded in some situations).

Next results describe one of the extremes of the two previous intervals (see [16, Theorems 1.8.31 and 1.8.23]).

Theorem 1.2.10. *Let $\bar{M} \in \mathbb{R}$ be fixed. If operator $T_n[\bar{M}]$ is invertible in X and its related Green's function satisfies condition (P_g), then the following statements hold.*

- *There exists $\lambda_1 > 0$, the least eigenvalue in absolute value of operator $T_n[\bar{M}]$ in X. Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue λ_1.*

- *The Green's function related to operator $T_n[M]$ is non-negative on $I \times I$ for all M in $(\bar{M} - \lambda_1, \bar{M}]$.*

- *The Green's function related to operator $T_n[M]$ cannot be non-negative on $I \times I$ for all $M < \bar{M} - \lambda_1$.*

- *If there is $M \in \mathbb{R}$ for which Green's function related to operator $T_n[M]$ is non-positive on $I \times I$, then $M < \bar{M} - \lambda_1$.*

Theorem 1.2.11. *Let $\bar{M} \in \mathbb{R}$ be fixed. If operator $T_n[\bar{M}]$ is invertible in X and its related Green's function satisfies condition (N_g), then the following statements hold:*

- *There exists $\lambda_1 < 0$, the least eigenvalue in absolute value of operator $T_n[\bar{M}]$ in X. Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue λ_1.*

- *The Green's function related to operator $T_n[M]$ is non-positive on $I \times I$ for all M in $[\bar{M}, \bar{M} - \lambda_1)$.*

- *The Green's function related to operator $T_n[M]$ cannot be non-positive on $I \times I$ for all $M > \bar{M} - \lambda_1$.*

- *If there is $M \in \mathbb{R}$ for which Green's function related to operator $T_n[M]$ is non-negative on $I \times I$, then $M > \bar{M} - \lambda_1$.*

Next results gives a property of the intervals N_T and P_T.

Theorem 1.2.12. *Let $\bar{M} \in \mathbb{R}$ be fixed. If operator $T_n[\bar{M}]$ is invertible in X and its related Green's function satisfies condition (P_g), then $\sup(N_T) = \inf(P_T)$, whenever the interval $N_T \neq \emptyset$.*

Theorem 1.2.13. *Let $\bar{M} \in \mathbb{R}$ be fixed. If operator $T_n[\bar{M}]$ is invertible in X and its related Green's function satisfies condition (N_g), then $\sup(N_T) = \inf(P_T)$, whenever the interval $P_T \neq \emptyset$.*

We also mention a result which appears on [42, Chapter 3, Section 6] and that connects the disconjugacy and the sign of the Green's function related to the problem (1.2.1)-(1.2.2).

Lemma 1.2.14. *If the linear differential equation* (1.0.3) *is disconjugate and $g_M(t,s)$ is the Green's function related to the problem* (1.2.1)-(1.2.2)*, then:*

$$g_M(t,s)\,p(t) \geqslant 0, \quad (t,s) \in I \times I,$$

$$\frac{g_M(t,s)}{p(t)} > 0, \quad (t,s) \in [a,b] \times (a,b).$$

where $p(t) = (t-a)^k\,(t-b)^{n-k}$.

Remark 1.2.15. *On previous Lemma, by means of expression:*

$$\frac{g_M(t,s)}{p(t)} > 0, \quad (t,s) \in [a,b] \times (a,b).$$

we mean that:

$$\frac{g_M(t,s)}{p(t)} > 0, \quad \textit{for all } (t,s) \in (a,b) \times (a,b).$$

and:

$$\infty > \lim_{t \to a^+} \frac{g_M(t,s)}{p(t)} > 0 \quad \textit{and} \quad \infty > \lim_{t \to b^-} \frac{g_M(t,s)}{p(t)} > 0, \quad \textit{for all } s \in (a,b).$$

Moreover, due to the regularity of function g_M, we conclude that there is a positive constant K such that the following properties hold for all $s \in (a,b)$:

$$0 < \ell_1(s) = \lim_{t \to a^+} \frac{g_M(t,s)}{p(t)} = \frac{(-1)^{n-k} \dfrac{\partial^k}{\partial t^k} g_M(t,s)_{|t=a}}{k!\,(b-a)^{n-k}} \leqslant K,$$

and,

$$0 < \ell_2(s) = \lim_{t \to b^-} \frac{g_M(t,s)}{p(t)} = \frac{\dfrac{\partial^{n-k}}{\partial t^k} g_M(t,s)_{|t=b}}{(b-a)^k (n-k)!} \leqslant K.$$

We note that such properties imply the following inequalities:

$$\begin{aligned} (-1)^{n-k}\, g_M(t,s) > 0, &\quad (t,s) \in (a,b) \times (a,b), \\ (-1)^{n-k} \frac{\partial^k}{\partial t^k} g_M(t,s)_{|t=a} > 0, &\quad s \in (a,b), \\ \frac{\partial^{n-k}}{\partial t^{n-k}} g_M(t,s)_{|t=b} > 0, &\quad s \in (a,b). \end{aligned}$$

1.3 Adjoint operator

The adjoint operator of $T_n[M]$ is given by the next expression, see for details [16, Section 1.4] or [42, Chapter 3, Section 5],

$$T_n^*[M]v(t) \equiv (-1)^n\, v^{(n)}(t) + \sum_{j=1}^{n-1} (-1)^j \, \left(p_{n-j}\, v\right)^{(j)}(t) + \big(p_n(t) + M\big)\, v(t), \tag{1.3.1}$$

and its domain of definition is:

$$\begin{aligned} D\big(T_n^*[M]\big) = \Bigg\{ v \in C^n(I) \ \mid \sum_{j=1}^{n} \sum_{i=0}^{j-1} (-1)^{j-1-i} (p_{n-j}\, v)^{(j-1-i)}(b)\, u^{(i)}(b) \\ = \sum_{j=1}^{n} \sum_{i=0}^{j-1} (-1)^{j-1-i} (p_{n-j}\, v)^{(j-1-i)}(a)\, u^{(i)}(a) \ \forall u \in D(T_n[M]) \Bigg\}, \end{aligned} \tag{1.3.2}$$

with $p_0 = 1$.

Next result appears in [42, Chapter 3, Theorem 9].

Theorem 1.3.1. *Equation (1.0.3) is disconjugate on an interval I if, and only if, the adjoint equation, $T_n^*[M]\,u(t) = 0$, is disconjugate on I.*

Indeed, in [42, Chapter 3, Theorem 10], we have the following property.

Theorem 1.3.2. *If $T_n[M]\,u$ has a representation* (1.1.2)*, then*

$$T_n^*[M]\,u \equiv \frac{1}{v_1}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{v_2}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\cdots\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{v_n}\frac{\mathrm{d}}{\mathrm{d}\,t}\,(v_1\,v_2\,\ldots\,v_n\,u)\right)\right)\right),$$

provided that $v_k > 0$ on I and $v_k \in C^n(I)$ for $k = 1, \ldots, n$.

We denote $g_M^*(t,s)$ as the Green's function related to the adjoint operator, $T_n^*[M]$.

In [16, Section 1.4], it is proved the following relationship:

$$g_M^*(t,s) = g_M(s,t). \tag{1.3.3}$$

Now, let us define the following operator:

$$\widehat{T}_n\big[(-1)^n\,M\big] := (-1)^n T_n^*[M], \tag{1.3.4}$$

we deduce, from the previous expressions, that:

$$\widehat{g}_{(-1)^n\,M}(t,s) = (-1)^n\,g_M^*(t,s) = (-1)^n\,g_M(s,t), \tag{1.3.5}$$

where $\widehat{g}_{(-1)^n M}(t,s)$ is the scalar Green's function related to operator $\widehat{T}_n\big[(-1)^n M\big]$ in the set $D\big(T_n^*[M]\big)$.

Obviously, we have the analogous of Theorem 1.3.1 for operator $\widehat{T}_n\big[(-1)^n\,M\big]$.

Theorem 1.3.3. *Equation (1.0.3) is disconjugate on an interval I if, and only if, the equation $\widehat{T}_n\big[(-1)^n M\big]\,u(t) = 0$, is disconjugate on I.*

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Chapter 2
Disconjugacy

Contents

There is a huge number of works related to disconjugacy and its properties, see for instance [42, 65, 88]. From the last half of the past century until now this subject of investigation has attracted important researchers who have established very interesting criteria to ensure such a property for particular equations.

This is the case of [75], where the disconjugacy of the second order equation:

$$y''(t) + f(t)\,y(t) = 0, \quad t \in I \equiv [a,b] \ \text{ with } f \geqslant 0,$$

is characterized in terms of the least eigenvalue of the corresponding eigenvalue problem

$$y''(t) + \lambda\, f(t)\,y(t) = 0, \quad t \in I, \quad y(a) = y'(b) = 0.$$

In [11], some characterisation of the disconjugacy is given for the general second order equation:

$$\big(p(t)y'(t)\big)' + f(t)\,y(t) = 0, \quad t \in I,$$

by means of variational approach.

More recently, in [41], there are shown sufficient conditions to ensure the disconjugacy of some second and fourth order equations.

In [77], there are obtained sufficient conditions (and different necessary ones) for disconjugacy on $[a,+\infty)$ for even order linear differential equations of the form:

$$y^{(2\,n)}(t) - (-1)^n\, p(t)\,y(t) = 0, \quad t \in I \ \text{ with } p \geqslant 0.$$

Also, in [51], sufficient conditions for disconjugacy of the linear differential equation:

$$y^{(n)}(t) + p(t)\,y(t) = 0, \quad t \in [a,+\infty),$$

with $p(t)$ a constant sign function are given.

In [86], it is studied the disconjugacy of the second order linear differential equation:

$$\big(r(t)\,x'(t)\big)' + p(t)\,x'(t) + q(t)\,x(t) = 0, \quad t \in I.$$

In addition, in [43], the authors achieve sufficient conditions to ensure the disconjugacy of the non-linear p-Laplacian equation

$$\big(|u'(t)|^{p-1}\,u'(t)\big)' + q(t)\,|u(t)|^{p-2}\,u(t) = 0, \quad t \in [a, +\infty).$$

The aim of this chapter consists in the characterisation of the disconjugacy of the general n^{th}-order linear differential equation (1.0.1) on any arbitrary interval I. It is obvious that such a problem is equivalent to study the set of parameters M for which the linear differential equation (1.0.3) is disconjugate on I. To this end, we assume that this set is not empty, i.e., there exists at least a $\bar{M}$ such that $T_n[\bar{M}]\,u(t) = 0$ is disconjugate on I. We have obtained such a characterisation in [28].

Moreover, since the main hypothesis of obtaining this characterisation is that there exists $\bar{M} \in \mathbb{R}$ such that the linear differential equation (1.0.3) is disconjugate for $M = \bar{M}$, it is important to make sure that the parameter set of disconjugacy is or not empty. Such a property is characterised in [31].

Next result, which appears in [29], gives a property of the operator under the disconjugacy hypothesis.

Lemma 2.0.1. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]\,u(t) = 0$ is disconjugate on I. Then the following properties are fulfilled:*

- *If $n-k$ is even, then $T_n[\bar{M}]$ is an inverse positive operator on X_k, defined in expression* (1.1.8)*, and its related Green's function, $g_{\bar{M}}(t,s)$, satisfies (P_g).*
- *If $n-k$ is odd, then $T_n[\bar{M}]$ is an inverse negative operator on X_k, defined in expression* (1.1.8)*, and its related Green's function satisfies (N_g).*

Proof. By Lemma 1.2.14 and Remark 1.2.15 we have that for all $s \in (a,b)$ function $\dfrac{g_{\bar{M}}(t,s)}{p(t)}$ can be extended to a strictly positive and continuous function in I, thus:

$$0 < k_1(s) = \min_{t\in I} \frac{g_{\bar{M}}(t,s)}{p(t)} < \max_{t \in I} \frac{g_{\bar{M}}(t,s)}{p(t)} = k_2(s), \quad s \in (a,b). \tag{2.0.1}$$

Since $g_{\bar{M}}$ is a continuous function in $I \times I$, we have that k_1 and k_2 are continuous functions too.

If $n-k$ is even, we take $\phi(t) = p(t)$ and condition (P_g) is trivially fulfilled.

If $n-k$ is odd, we take $\phi(t) = -p(t)$ and multiplying equation (2.0.1) by -1, condition (N_g) holds immediately. □

From this result coupled with Theorems 1.2.4 and 1.2.6, we have that if equation (1.0.3) is disconjugate on I and u is a solution of problem (1.2.1) for $t \in I$, with boundary conditions $(k, n-k)$, it is uniquely determined by the expression:

$$u(t) = \int_a^b g_{\bar{M}}(t,s)\,\sigma(s)\,\mathrm{d}\,s.$$

2.1 Characterisation of disconjugacy

This section is devoted to characterise the parameter set for which the n^{th}-order linear differential equation (1.0.3) is disconjugate on $I \equiv [a, b]$. This result has been published in [28].

Theorem 2.1.1. *Let $\bar{M} \in \mathbb{R}$ and $n \geqslant 2$ be such that* (1.0.3) *is a disconjugate equation on I for $M = \bar{M}$. Then,* (1.0.3) *is a disconjugate equation on I if, and only if, M belongs to $(\bar{M} - \lambda_1, \bar{M} - \lambda_2)$, where:*

- *$\lambda_1 = +\infty$ if $n = 2$ and, for $n > 2$, $\lambda_1 > 0$ is the minimum of the least positive eigenvalues on $T_n[\bar{M}]$ in X_k, with $n - k$ even.*
- *$\lambda_2 < 0$ is the maximum of the biggest negative eigenvalues on $T_n[\bar{M}]$ in X_k, with $n-k$ odd.*

2.1.1 Derivative of Wronskian

Before proving our main result let us see by induction that:

$$\frac{\partial}{\partial t} W_\ell^n[M](t) = \begin{vmatrix} y_1[M](t) & \dots & y_\ell[M](t) \\ \vdots & \ddots & \vdots \\ y_1^{(\ell-2)}[M](t) & \dots & y_\ell^{(\ell-2)}[M](t) \\ y_1^{(\ell)}[M](t) & \dots & y_\ell^{(\ell)}[M](t) \end{vmatrix}. \tag{2.1.1}$$

For $\ell = 2$, we have:

$$W_2^n[M](t) = \begin{vmatrix} y_1[M](t) & y_2[M](t) \\ y_1'[M](t) & y_2'[M](t) \end{vmatrix} = y_1[M](t)\, y_2'[M](t) - y_2[M](t)\, y_1'[M](t),$$

and, trivially:

$$W_2^{n\prime}[M](t) = \begin{vmatrix} y_1[M](t) & y_2[M](t) \\ y_1''[M](t) & y_2''[M](t) \end{vmatrix}.$$

Suppose that the equality (2.1.1) is fulfilled for $k = 1, \dots, \ell$, then $W_{\ell+1}^n[M_1](t)$ is given by the following expression:

$$y_1[M_1](t) \begin{vmatrix} y_2'[M_1](t) & \dots & y_{\ell+1}'[M_1](t) \\ & \ddots & \\ y_2^{(\ell)}[M_1](t) & \dots & y_{\ell+1}^{(\ell)}[M_1](t) \end{vmatrix} + \dots + (-1)^\ell y_{\ell+1}[M_1](t) \begin{vmatrix} y_1'[M_1](t) & \dots & y_\ell'[M_1](t) \\ & \ddots & \\ y_1^{(\ell)}[M_1](t) & \dots & y_\ell^{(\ell)}[M_1](t) \end{vmatrix},$$

using the induction hypothesis we arrive at:

$$\frac{\partial}{\partial t} W_{\ell+1}^n[M_1](t) = \begin{vmatrix} y_1[M_1](t) & \dots & y_{\ell+1}[M_1](t) \\ & \ddots & \\ y_1^{(\ell-1)}[M_1](t) & \dots & y_{\ell+1}^{(\ell-1)}[M_1](t) \\ y_1^{(\ell+1)}[M_1](t) & \dots & y_{\ell+1}^{(\ell+1)}[M_1](t) \end{vmatrix} + \begin{vmatrix} y_1'[M_1](t) & \dots & y_{\ell+1}'[M_1](t) \\ y_1'[M_1](t) & \dots & y_{\ell+1}'[M_1](t) \\ & \ddots & \\ y_1^{(\ell)}[M_1](t) & \dots & y_{\ell+1}^{(\ell)}[M_1](t) \end{vmatrix},$$

so, since the second addend in the previous equality is null, our assertion is proved.

Proof of Theorem 2.1.1. Let $n > 2$. First, let us see that the optimal interval of disconjugacy, which we denote by $D_{\bar{M}}$, must necessarily be a subset of $(\bar{M} - \lambda_1, \bar{M} - \lambda_2)$.

Using, Lemma 1.2.14, we know that if $M \in D_{\bar{M}}$, then the Green's function related to operator $T_n[M]$ in X_k is of constant sign, positive if $n-k$ is even and negative if $n-k$ is odd.

Let $\widehat{k} \in \{1, \dots, n-1\}$ be such that $n - \widehat{k}$ is even and λ_1 is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\widehat{k}}$. Using Lemma 2.0.1 and Theorem 1.2.10, we can affirm that $g_{\widehat{M},\widehat{k}}$ changes sign on $I \times I$ for $\widehat{M} \leqslant \bar{M} - \lambda_1$, then $\widehat{M} \notin D_{\bar{M}}$ for every $\widehat{M} \leqslant \bar{M} - \lambda_1$.

In an analogous way, let $\widetilde{k} \in \{1, \dots, n-1\}$ be such that $n - \widetilde{k}$ is odd and λ_2 is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\widetilde{k}}$. Using the same arguments, with Lemma 2.0.1 and Theorem 1.2.11, we can affirm that $g_{\widetilde{M},\widetilde{k}}$ has not constant sign on $I \times I$ for $\widetilde{M} \geqslant \bar{M} - \lambda_2$, then $\widetilde{M} \notin D_{\bar{M}}$ for every $\widetilde{M} \geqslant \bar{M} - \lambda_2$.

Hence, we have proved that $D_{\bar{M}} \subset \big(\bar{M} - \lambda_1, \bar{M} - \lambda_2\big)$.

Let us see that $D_{\bar{M}} = \big(\bar{M} - \lambda_1, \bar{M} - \lambda_2\big)$. Denote $M_1 = \inf D_{\bar{M}}$ and $M_2 = \sup D_{\bar{M}}$. Because of Proposition 1.1.8, $D_{\bar{M}}$ should be an open interval, in particular $M_j \neq \bar{M}$ for $j = 1, 2$.

If $D_{\bar{M}} \neq (\bar{M} - \lambda_1, \bar{M} - \lambda_2)$ then, at least one (or both) of the two following inequalities holds: either $M_1 > \bar{M} - \lambda_1$ or $M_2 < \bar{M} - \lambda_2$.

Suppose that first inequality is fulfilled.

Since $T_n[M_1]\, u(t) = 0$ is not a disconjugate equation on I, we have that $c = \eta_{M_1}(a) \leqslant b$.

Using Propositions 1.1.10 and 1.1.12, we can ensure the existence of $\ell \in \{1, \dots, n-1\}$ such that there exists a non-trivial solution of $T_n[M_1]\, u(t) = 0$, satisfying boundary conditions $(n - \ell, \ell)$ on $[a, c]$. Choosing the least ℓ which satisfies the previous assertion, due to Proposition 1.1.13, we can affirm that the solution which satisfies boundary conditions $(n - \ell, \ell)$ on $[a, c]$ is uniquely determined up to a constant factor and it does not vanish on (a, c).

If $c = b$, then $\bar{M} - M_1 \in (\lambda_2, \lambda_1)$ is an eigenvalue of $T_n[\bar{M}]$ in $X_{n-\ell}$, and it contradicts the definition of λ_1 when ℓ is even and λ_2, if ℓ is odd.

So, we have that $c < b$.

Using Proposition 1.1.10, we know that $W_\ell^n[M_1](c) = 0$.

And, since $T_n[M]\, u(t) = 0$ is a disconjugate equation on I for $M \in (M_1, M_2)$, we can affirm that $W_\ell^n[M_1 + \delta](t) \neq 0,\ t \in (a, b]$ for every $0 < \delta < M_2 - M_1$.

Since $W_\ell^n[M](t)$ is a continuous function of M, we have that $W_\ell^n[M_1](t)$ is of constant sign on a neighbourhood of c, so it has a double zero at c as a function of t.

Using the expression of the derivative of the Wronskian given in (2.1.1), we know that:

$$0 = \frac{\partial}{\partial t} W_\ell^n[M_1](t)_{|t=c} = \begin{vmatrix} y_1[M_1](c) & \dots & y_\ell[M_1](c) \\ & \ddots & \\ y_1^{(\ell-2)}[M_1](c) & \dots & y_\ell^{(\ell-2)}[M_1](c) \\ y_1^{(\ell)}[M_1](c) & \dots & y_\ell^{(\ell)}[M_1](c) \end{vmatrix}. \tag{2.1.2}$$

We take the following solution of (1.0.3):

$$y(t) = \begin{vmatrix} y_1[M_1](c) & \dots & y_\ell[M_1](c) \\ & \ddots & \\ y_1^{(\ell-2)}[M_1](c) & \dots & y_\ell^{(\ell-2)}[M_1](c) \\ y_1[M_1](t) & \dots & y_\ell[M_1](t) \end{vmatrix}.$$

Since it is a linear combination of $y_1[M_1], \dots y_\ell[M_1]$, it is obvious that it has $n-\ell$ zeros at a.

Taking into account that $W_\ell^n[M_1](c) = 0$, function y trivially satisfies the boundary conditions $(n-\ell,\ell)$ at $[a,c]$. And, using Proposition 1.1.13, since $c = \eta_{M_1}(a)$ we know that it does not vanish on the open interval (a,c).

Because of equality (2.1.2) it is not difficult to verify that such a function also satisfies the boundary conditions $(n-\ell-1,\ell+1)$ on $[a,c]$. If $\ell = n-1$, this is not possible, because it cannot exist any non-trivial solution with n zeros at c.

Also, if $\ell < n-1$, denoting as $g_{\bar{M},n-\ell}$ and $g_{\bar{M},n-\ell-1}$, the related Green's functions to problem (1.2.1), coupled with the boundary conditions (1.2.6) for $M = \bar{M}$, $b = c$ and $k = \ell$ or $k = \ell+1$, respectively, we deduce the following equalities for all $t \in [a,c]$

$$y(t) = \int_a^c g_{\bar{M},n-\ell}(t,s)\,(\bar{M}-M_1)\,y(s)\,\mathrm{d}\,s,$$

and

$$y(t) = \int_a^c g_{\bar{M},n-\ell-1}(t,s)\,(\bar{M}-M_1)\,y(s)\,\mathrm{d}\,s.$$

Using Lemma 1.2.14 we know that $g_{\bar{M},n-\ell}(t,s)$ and $g_{\bar{M},n-\ell-1}(t,s)$ have different constant sign on $[a,c]\times(a,c)$, so last equalities cannot be satisfied at the same time. Then we can affirm that $M_1 = \bar{M}-\lambda_1$.

With analogous arguments we conclude that $M_2 = \bar{M}-\lambda_2$.

If $n=2$, the argument related to λ_2 is the same.

Suppose that there exist $M^* < \bar{M}$ such that the equation (1.0.3) is not disconjugate on $[a,b]$, then, necessarily, $M_1 \in (M^*,\bar{M})$ and $c = \eta_{M_1}(a) \in (a,b]$. If $c = b$ it implies the existence of a positive eigenvalue of $T_n[\bar{M}]$ in X_1, which contradicts Theorem 1.1.14.

Then, we can proceed analogously to the case where $n > 2$ with $c < b$ and arrive to a contradiction. So, we have completely proved our result. □

2.1.2 Particular cases

This section is devoted to show several particular cases, where the previous results are applied. In order to obtain the eigenvalues of particular problems, we calculate a fundamental system of solutions $y_1[M](t),\dots,y_n[M](t)$ of equation (1.0.3), where every $y_k[M](t)$ satisfies the initial conditions (1.1.6) and $W_k^n[M]$ denotes the correspondent Wronskians previously introduced in (1.1.7).

From Proposition 1.1.10, we know that the eigenvalues of operator $T_n[\bar{M}]$ in X_k are given as the $\lambda \in \mathbb{R}$ for which $W_{n-k}^n[\bar{M}-\lambda](b) = 0$. So, in the sequel, we will use this method to find the eigenvalues of the different considered problems. We have developed a program in *Mathematica* to obtain the eigenvalues of these problems, see Appendix A.

Operator $T_n^0[M] \equiv \dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n} + M$

First of all, we consider problems where $T_n^0[M]\,u(t) \equiv u^{(n)}(t) + M\,u(t)$, with $[a,b] = [0,1]$. In this kind of problems, for $M = 0$, $u^{(n)}(t) = 0$ is always disconjugate on every real interval I, see [42, Chapter 3]. So, the hypotheses of Theorem 2.1.1 are satisfied and, in order to construct the optimal parameter set of disconjugacy, we only need to calculate the closest to zero eigenvalues in each set.

Remark 2.1.2. *Note that adjoint equation to problem $T_n^0[M]\,u(t) = 0, u \in X_k$ is given by:*

$$T_n^{0^*}[M]\,u(t) = (-1)^n\,u^{(n)}(t) + M\,u(t) = 0, \quad u \in X_{n-k}.$$

So, if we have that λ_i is an eigenvalue of $T_n^0[0] \equiv \dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n}$ in X_k, it is also an eigenvalue of $(-1)^n\,\dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n}$ in X_{n-k}. Thus, $(-1)^n\,\lambda_i$ is an eigenvalue of $\dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n}$ in X_{n-k}.

As a straight consequence, we only need to obtain the first $\left\lfloor \dfrac{n}{2} \right\rfloor$ Wronskians, where $\lfloor\cdot\rfloor$ means the floor function.

- 2^{nd}-order

The eigenvalues of operator $T_2^0[0] \equiv \dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in X_1 must satisfy $W_1^2[-\lambda](1) = 0$, which can be replaced by the following equation:

$$\sin\left(\sqrt{-\lambda}\right) = 0, \tag{2.1.3}$$

so its closest to zero negative eigenvalue is $\lambda_2^1 = -\pi^2$.

Thus, we conclude that $u''(t) + M\,u(t) = 0$ is a disconjugate equation on $[0,1]$ if, and only if,

$$M \in \big(-\infty, \pi^2\big).$$

- 3^{rd}-order

$\lambda_3^1 \approxeq 4.23321$ is the least positive solution of $W_1^3[\lambda^3](1) = 0$, which is given by the equation:

$$\cos\left(\frac{1}{2}\sqrt{3}\lambda\right) - \sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}\lambda\right) = e^{\frac{-3\,\lambda}{2}}. \tag{2.1.4}$$

Then, the least positive eigenvalue of operator $T_3^0[0] \equiv \dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in X_1 is $\left(\lambda_3^1\right)^3$ and the biggest negative eigenvalue of operator $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in X_2 is $-\left(\lambda_3^1\right)^3$.

So, we conclude that $u^{(3)}(t) + M\,u(t) = 0$ is disconjugate on $[0,1]$ if, and only if,

$$M \in \left(-\left(\lambda_3^1\right)^3, \left(\lambda_3^1\right)^3\right) \approxeq \big(-4.233^3, 4.233^3\big).$$

- 4^{th}-order

$\lambda_4^1 \approxeq 5.553$ is the least positive solution of $W_1^4[\lambda^4](1) = 0$, which follows the expression:

$$\tan\left(\frac{\lambda}{\sqrt{2}}\right) = \tanh\left(\frac{\lambda}{\sqrt{2}}\right). \tag{2.1.5}$$

$\lambda_4^2 \approxeq 4.73004$ is the least positive solution of $W_2^4[-\lambda^4](1) = 0$, which we can express as:

$$\cos(\lambda)\cosh(\lambda) = 1. \tag{2.1.6}$$

The biggest negative eigenvalue of operator $T_4^0[0] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in X_1 and X_3 is $-\left(\lambda_4^1\right)^4$.

The least positive eigenvalue of operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in X_2 is $\left(\lambda_4^2\right)^4$.

Therefore, we can affirm that the linear differential equation $u^{(4)}(t) + M\,u(t) = 0$ is disconjugate on $[0,1]$ if, and only if,

$$M \in \left(-\left(\lambda_4^2\right)^4, \left(\lambda_4^1\right)^4\right) \approxeq \left(-4.73^4, 5.55^4\right).$$

- 5^{th}-order

In this case, we also obtain $\lambda_5^1 \approxeq 6.94867$ and $\lambda_5^2 \approxeq 5.64117$ as the least positive solutions of $W_1^5[\lambda^5](1) = 0$ and $W_2^5[-\lambda^5](1) = 0$, respectively. But the equations obtained are too complicate to show here and they have not so much interest.

The least positive eigenvalue of operator $T_5^0[0] \equiv \dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}$ in X_1 is $\left(\lambda_5^1\right)^5$.

The biggest negative eigenvalue of operator $\dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}$ in X_2 is $-\left(\lambda_5^2\right)^5$.

The least positive eigenvalue of operator $\dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}$ in X_3 is $\left(\lambda_5^2\right)^5$.

The biggest negative eigenvalue of operator $\dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}$ in X_4 is $-\left(\lambda_5^1\right)^5$.

Thus, we conclude that the linear differential equation $u^{(5)}(t) + M\,u(t) = 0$ is disconjugate on $[0,1]$ if, and only if,

$$M \in \left(-\left(\lambda_5^2\right)^5, \left(\lambda_5^2\right)^5\right) \approxeq \left(-5.64^5, 5.64^5\right).$$

- 6^{th}-order

$\lambda_6^1 \approxeq 8.3788$ is the least positive solution of $W_1^6[\lambda^6](1) = 0$, which is equivalent to:

$$\sin(\lambda) - \sqrt{3}\cos\left(\frac{\lambda}{2}\right)\sinh\left(\frac{\sqrt{3}\lambda}{2}\right) + \sin\left(\frac{\lambda}{2}\right)\cosh\left(\frac{\sqrt{3}\lambda}{2}\right) = 0. \tag{2.1.7}$$

$\lambda_6^2 \approxeq 6.70763$ is the least positive solution of $W_2^6[-\lambda^6](1) = 0$, which we can express as:

$$-3e^{\lambda/2}\left(e^{2\lambda}+1\right)+\sqrt{3}\left(e^{\lambda}-1\right)^3 \sin\left(\frac{\sqrt{3}\lambda}{2}\right)+\left(e^{\lambda}+1\right)^3 \cos\left(\frac{\sqrt{3}\lambda}{2}\right)-2e^{3\lambda/2}\cos\left(\sqrt{3}\lambda\right)=0. \quad (2.1.8)$$

$\lambda_6^3 \approxeq 6.28319^6$ is the least positive solution of $W_3^6[\lambda^6](1) = 0$, which can be represented as the first positive root of the following equation:

$$\sin(\lambda)\left(-\cos(\lambda)+\cosh\left(\sqrt{3}\lambda\right)+4\right)-8\sin\left(\frac{\lambda}{2}\right)\cosh\left(\frac{\sqrt{3}\lambda}{2}\right)=0. \quad (2.1.9)$$

The biggest negative eigenvalue of operator $T_6^0[0] \equiv \dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in X_1 and X_5 is $-\left(\lambda_6^1\right)^6$.

The least positive eigenvalue of operator $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in X_2 and X_4 is $\left(\lambda_6^2\right)^6$.

The biggest negative eigenvalue of operator $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in X_3 is $-\left(\lambda_6^3\right)^6$.

Hence, we can affirm that $u^{(6)}(t) + M\,u(t) = 0$ is a disconjugate equation on $[0,1]$ if, and only if,

$$M \in \left(-\left(\lambda_6^2\right)^6, \left(\lambda_6^3\right)^6\right) \approxeq \left(-6.71^6, 6.28^6\right).$$

- 7^{th}-order

We are not able to obtain analytically the eigenvalues of operator $T_7^0[0] \equiv \dfrac{\mathrm{d}^7}{\mathrm{d}\,t^7}$, but we can obtain them numerically.

The least positive eigenvalue of this operator in X_1 is $\left(\lambda_7^1\right)^7$, where $\lambda_7^1 \approxeq 9.82677$.

The biggest negative eigenvalue in X_2 is $-\left(\lambda_7^2\right)^7$, where $\lambda_7^2 \approxeq 7.85833$.

The least positive eigenvalue in X_3 is $\left(\lambda_7^3\right)^7$, where $\lambda_7^3 \approxeq 7.1347$.

The biggest negative eigenvalue in X_4 is $-\left(\lambda_7^3\right)^7$.

The least positive eigenvalue in X_5 is $\left(\lambda_7^2\right)^7$.

The biggest negative eigenvalue in X_6 is $-\left(\lambda_7^1\right)^7$.

So, we conclude that the linear differential equation $u^{(7)}(t) + M\,u(t) = 0$ is disconjugate on $[0,1]$ if, and only if,

$$M \in \left(-\left(\lambda_7^3\right)^7, \left(\lambda_7^3\right)^7\right) \approxeq \left(-7.13^7, 7.13^7\right).$$

- 8^{th}-order

$\lambda_8^1 \approxeq 11.2846$, $\lambda_8^2 \approxeq 9.06306$, $\lambda_8^3 \approxeq 8.09971$ and $\lambda_8^4 \approxeq 7.81871$ can be obtained analytically as the least positive solution of $W_1^8[\lambda^8](1) = 0$, $W_2^8[-\lambda^8](1) = 0$, $W_3^8[\lambda^8](1) = 0$ and $W_4^8[-\lambda^8](1) = 0$, respectively, but their expressions are too big to show it here and they do not bring any important information.

The biggest negative eigenvalue of operator $T_8^0[0] \equiv \dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}$ in X_1 and X_7 is $-(\lambda_8^1)^8$.

The least positive eigenvalue of operator $\dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}$ in X_2 and X_6 is $(\lambda_8^2)^8$.

The biggest negative eigenvalue of operator $\dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}$ in X_3 and X_5 is $-(\lambda_8^3)^8$.

The least positive eigenvalue of operator $\dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}$ in X_4 is $(\lambda_8^4)^8$.

So, we can affirm that $u^{(8)}(t) + M\,u(t) = 0$ is a disconjugate equation on $[0,1]$ if, and only if,

$$M \in \left(-\left(\lambda_8^4\right)^8, \left(\lambda_8^3\right)^8\right) = \left(-7.82^8, 8.1^8\right).$$

Remark 2.1.3. *In this kind of problems, if λ is an eigenvalue on $[0,1]$, then $\dfrac{\lambda}{(b-a)^n}$ is an eigenvalue on $[a,b]$.*

So, we can extend our conclusions about disconjugacy on any arbitrary interval $[a,b]$.

Operators with constant coefficients

This characterisation of the disconjugacy is also useful for those problems which have more non-nulls coefficients. Let us see some particular cases.

- $T_4^1[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 10\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 10\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + 10\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

For example, we can consider the operator of order 4:

$$T_4^1[M]\,u(t) \equiv u^{(4)}(t) + 10\,u^{(3)}(t) + 10\,u''(t) + 10\,u'(t) + M\,u(t), \quad t \in [0,1]. \tag{2.1.10}$$

We can show, using the characterisation of the first right point conjugate of 0, given in Proposition 1.1.12, that:

$$u^{(4)}(t) + 10\,u^{(3)}(t) + 10\,u''(t) + 10\,u'(t) = 0,$$

is a disconjugate equation on $[0,1]$ and, so, we can apply Theorem 2.1.1.

First, we calculate numerically, by means of the *Mathematica* program developed in Appendix A, the closest to zero eigenvalues in each X_k, for $k = 1,2,3$.

The biggest negative eigenvalue in X_1 is -7.02782^4.

The least positive eigenvalue in X_2 is 5.27208^4.

The biggest negative eigenvalue in X_3 is -5.97041^4.

Realise that, in this case, we need to obtain the three correspondents Wronskians because it is not possible to connect the eigenvalues in X_1 with those in X_3 by means of its corresponding adjoint equation.

So, we conclude that $u^{(4)}(t) + 10\,u^{(3)}(t) + 10\,u''(t) + 10\,u'(t) + M\,u(t) = 0$ is a disconjugate equation on $[0,1]$ if, and only if,

$$M \in \left(-5.27208^4, 5.97041^4\right).$$

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- $T_4^2[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 10\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 550\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

This is an example where operator $T_n[M]$ does not satisfy disconjugacy hypothesis for $\bar{M} = 0$.

If we choose the fourth order equation:

$$T_4^2[M]\,u(t) \equiv u^{(4)}(t) + 10\,u^{(3)}(t) + 550\,u'(t) + M\,u(t) = 0, \quad t \in [0,1]. \tag{2.1.11}$$

We obtain that such an equation is not disconjugate on $[0,1]$ for $M = 0$, since we can find non-trivial solutions with fourth zeros or more. However, if we analyse the equation:

$$T_4^2[-600]\,u(t) = 0,$$

we can affirm, by means of Proposition 1.1.12, that it is disconjugate on $[0,1]$.

We can apply Theorem 2.1.1 to the equation (2.1.11) for $\bar{M} = -600$.

The biggest negative eigenvalue of $T_4^2[-600]\,u(t)$ in X_1 is -9556.55.

The least positive eigenvalue in X_2 is 11.5685.

The biggest negative eigenvalue in X_3 is -28.9753.

Hence, using Theorem 2.1.1, we can affirm that (2.1.11) is a disconjugate equation on $[0,1]$ if, and only if,

$$M \in (-600 - 11.5685, -600 + 28.9753) = (-611.5685, -571.0247).$$

- $T_6^1[M] \equiv \dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} - 8\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$

Consider, now, a sixth order example:

$$T_6^1[M]\,u(t) = u^{(6)}(t) - 8\,u^{(3)}(t) + M\,u(t), \quad t \in [0,1]. \tag{2.1.12}$$

It is not difficult to verify, by means of the characterisation of the first right point conjugate of a, given in Proposition 1.1.12, that:

$$u^{(6)}(t) - 8\,u^{(3)}(t) = 0,$$

is a disconjugate equation on $[0,1]$. So, we can apply Theorem 2.1.1.

Numerically, we obtain that the biggest negative eigenvalue related to the boundary conditions $(5,1)$ is -8.40247^6.

Moreover, the least positive eigenvalue related to the boundary conditions $(4,2)$ is 6.717^6.

The biggest negative eigenvalue related to the boundary conditions $(3,3)$ is -6.2835^6.

The least positive eigenvalue related to the boundary conditions $(2,4)$ is 6.698^6.

Finally, the biggest negative eigenvalue related to the boundary conditions $(1,5)$ is given by -8.355^6.

Then, we conclude that (2.1.12) is a disconjugate equation on $[0,1]$ if, and only if,

$$M \in \left(-6.698^6, 6.2835^6\right).$$

- $T_4^3[M] \equiv \dfrac{d^4}{d\,t^4} + 50\dfrac{d^2}{d\,t^2} + M$

Finally, let us consider the operator:

$$T_4^3[M]\,u(t) = u^{(4)}(t) + 50\,u''(t) + M\,u(t), \quad t \in [0,1]. \tag{2.1.13}$$

In this case, if we study the operator for $M = 0$, we obtain:

$$W_2^4[0](t) = \frac{5\sqrt{2}t\sin\left(5\sqrt{2}t\right) + 2\cos\left(5\sqrt{2}t\right) - 2}{2500},$$

which has at least a sign change on $[0,1]$. So, $T_4^3[0]\,u(t) = 0$ is not a disconjugate equation on $[0,1]$.

But, if we take $\bar{M} = 200$, we can verify, studying its different Wronskians, see Propositions 1.1.10 and 1.1.12, that $T_4^3[200]\,u(t) = 0$ is a disconjugate equation on $[0,1]$. Hence, we can apply Theorem 2.1.1 to this problem.

Due to the fact that it is a self adjoint problem, we only need to obtain the eigenvalues related to the boundary conditions $(3,1)$ and $(2,2)$ because the eigenvalues related to the boundary conditions $(3,1)$ and $(1,3)$ are the same.

The biggest negative eigenvalue related to the boundary conditions $(3,1)$ is given by $-\left(\lambda_1\right)^4$, where $\lambda_1 \approxeq 3.71137$ is the least positive solution of the following equation:

$$\sqrt{25-\sqrt{425-\lambda^4}}\sin\left(\sqrt{\sqrt{425-\lambda^4}+25}\right) = \sqrt{\sqrt{425-\lambda^4}+25}\sin\left(\sqrt{25-\sqrt{425-\lambda^4}}\right). \tag{2.1.14}$$

The least positive eigenvalue related to the boundary conditions $(2,2)$ is given by $\left(\lambda_2\right)^4$, where $\lambda_2 \approxeq 2.77939$ is the least positive solution of the following equation:

$$\begin{aligned} -2\sqrt{200-\lambda^4} + 50\sin\left(\sqrt{25-\sqrt{\lambda^4+425}}\right)\sin\left(\sqrt{\sqrt{\lambda^4+425}+25}\right) \\ +2\sqrt{200-\lambda^4}\cos\left(\sqrt{25-\sqrt{\lambda^4+425}}\right)\cos\left(\sqrt{\sqrt{\lambda^4+425}+25}\right) = 0. \end{aligned} \tag{2.1.15}$$

Hence we conclude that (2.1.13) is a disconjugate equation if, and only if,

$$M \in (200-\lambda_2^4, 200+\lambda_1^4) \approxeq (140.324, 389.73).$$

Operators with non-constant coefficients

We have already seen that applying Theorem 2.1.1 is a very useful tool to obtain the parameter set for which the linear differential equation (1.0.3) is disconjugate. This result is, even, more useful for problems with non-constant coefficients.

- $T_3^1[M] \equiv \dfrac{d^3}{d\,t^3} + \cos(10\,t)\dfrac{d^2}{d\,t^2} + M$

Consider, for instance, the third order linear differential equation:

$$T_3^1[M]\,u(t) = u^{(3)}(t) + \cos(10\,t)\,u''(t) + M\,u(t) = 0, \quad t \in [0,1]. \tag{2.1.16}$$

Let us see that it is disconjugate for $M = 0$.

Since every solution of the first order linear differential equation

$$L_1\, u(t) = u'(t) + \cos(10\, t)\, u(t) = 0,$$

follows the expression $u(t) = c_1\, e^{\frac{-\sin(10\,t)}{10}}$, with $c_1 \neq 0$, we conclude that it is disconjugate on any real interval. Also, it is well-known that the equation $L_2\, u(t) = u''(t) = 0$ is also disconjugate on any real interval. So, as a direct application of Theorem 1.1.7, we can affirm that $L_1\, L_2\, u(t) = T_3^1[0]\, u(t) = 0$ is a disconjugate equation on any real interval.

Now, we obtain numerically the closest to zero eigenvalues related to the boundary conditions $(2,1)$ and $(1,2)$, which are -4.33149^3 and 4.29055^3, respectively. So, we can affirm that (2.1.16) is a disconjugate equation in $[0,1]$ if, and only if, $M \in \big(-4.29055^3, 4.33149^3\big)$.

- $T_3^2[M] \equiv \dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + t\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

Consider, now, the following third order linear differential equation

$$T_3^2[M]\, u(t) \equiv u^{(3)}(t) + t\, u'(t) + M\, u(t) = 0, \quad t \in [0,1], \tag{2.1.17}$$

for which, by means of Proposition 1.1.12, we can verify that it is disconjugate on $[0,1]$ for $M = 0$.

If we calculate numerically the closest to zero eigenvalues of operator defined in (2.1.17) for $M = 0$, we obtain that 4.19369^3 is the least positive eigenvalue of operator $T_3^2[0]\, u(t)$ in X_1.

Moreover, -4.21255^3 is the biggest negative eigenvalue of operator $T_3^2[0]\, u(t)$ in X_2.

So, we can affirm that (2.1.17) is a disconjugate equation on $[0,1]$ if, and only if, M belongs to $\big(-4.19369^3, 4.21255^3\big)$.

- $T_4^4[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + e^{2\,t}\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

We can also apply it to a fourth order operator whose eigenvalues were also obtained numerically.

$$T_4^4[M] \equiv u^{(4)}(t) + e^{2\,t} u'(t) + M\, u(t) = 0, \quad t \in [0,1]. \tag{2.1.18}$$

We can verify, by means of Proposition 1.1.12 again, that (2.1.18) is disconjugate on $[0,1]$.

The biggest negative eigenvalue in X_1 is -5.5325^4.

The least positive eigenvalue in X_2 is 4.7235^4.

The biggest negative eigenvalue in X_3 is -5.5815^4.

So, as straight consequence of Theorem 2.1.1, we conclude that equation (2.1.18) is disconjugate on $[0,1]$ if, and only if, $M \in \big(-4.7235^4, 5.5325^4\big)$.

2.2 Non-disconjugacy criteria

In this section, we give some results about disconjugacy and we obtain sufficient criteria to affirm that a linear differential equation is not disconjugate for every $M \in \mathbb{R}$. Finally, we prove that one of these sufficient criteria gives also a necessary condition to obtain the characterisation of non-disconjugacy.

In this section, we remark the intervals where the functions in X_k, previously introduced in (1.0.2), are defined, as follows:

$$X_{k\,[a,b]} = \left\{ u \in C^n(I) \;\middle|\; u(a) = \cdots = u^{(k-1)}(a) = u(b) = \cdots = u^{(n-k-1)}(b) = 0 \right\}. \tag{2.2.1}$$

Lemma 2.2.1. *Assume that the linear differential equation* (1.0.3) *is disconjugate on I for $M = \bar{M}$. Then, if $n-k$ is even, $W^n_{n-k}[M](t) \neq 0$ for every $t \in (a,b]$ and $M \geqslant \bar{M}$.*

Proof. Trivially, for $M = \bar{M}$ the assertion is true, see Proposition 1.1.12.

If there exist $c \in (a,b]$ and $M^* > \bar{M}$, such that $W^n_{n-k}[M^*](c) = 0$, with $n-k$ even, we have, using Proposition 1.1.10, that $\bar{M} - M^* < 0$ is and eigenvalue of $T_n[\bar{M}]$ in $X_{k\,[a,c]}$, with $n-k$ even.

Hence, using Theorem 1.1.14, we can affirm that the linear differential equation (1.0.3) is not disconjugate on $[a,c]$ for $M = \bar{M}$. But in such a case, since $[a,c] \subset I$, the linear differential equation (1.0.3) cannot be disconjugate for $M = \bar{M}$ on I, which contradicts our assumption. □

As a consequence, we deduce the following result.

Corollary 2.2.2. *If $W^n_{n-k}[M^*](c) = 0$ for any $c \in (a,b]$, with $n-k$ even, then the linear differential equation* (1.0.3) *is not disconjugate for every $M \leqslant M^*$.*

Now, if $n-k$ odd, in an analogous way we can prove the following results.

Lemma 2.2.3. *Assume that the linear differential equation* (1.0.3) *is disconjugate on I for $M = \bar{M}$. Then, if $n-k$ is odd, $W^n_{n-k}[M](t) \neq 0$ for every $t \in (a,b]$ and $M \leqslant \bar{M}$.*

Corollary 2.2.4. *If $W^n_{n-k}[M^*](c) = 0$ for any $c \in (a,b]$ and $n-k$ odd, then the linear differential equation* (1.0.3) *is not disconjugate for every $M \geqslant M^*$.*

Now, combining Corollaries 2.2.2 and 2.2.4, we obtain the following result which gives a sufficient criterion to ensure that the linear differential equation (1.0.3) is not disconjugate for every $M \in \mathbb{R}$.

Corollary 2.2.5. *If there exist $c_1, c_2 \in (a,b]$ and $M_1 \geqslant M_2$ such that:*

- *There exists $n-k^*$ even such that $W^n_{n-k^*}[M_1](c_1) = 0$.*
- *There exits $n-k^{**}$ odd such that $W^n_{n-k^{**}}[M_2](c_2) = 0$.*

Then, the linear differential equation (1.0.3) *is not disconjugate for every $M \in \mathbb{R}$.*

And, as a particular case, we have the next result.

Corollary 2.2.6. *Suppose that there exists $\bar{M} \in \mathbb{R}$ such that there are two different Wronskians satisfying $W^n_{k^*}[\bar{M}](c_1) = W^n_{\tilde{k}}[\bar{M}](c_2) = 0$ for some $c_1, c_2 \in (a, b]$, with $k^* - \tilde{k}$ an odd number.*

Then, there is not any $M \in \mathbb{R}$ such that $T_n[M]\, u(t) = 0$ is a disconjugate equation on I.

Now, using Corollary 2.2.5, we introduce another sufficient criterion to ensure that the equation (1.0.3) is not disconjugate for every $M \in \mathbb{R}$.

Corollary 2.2.7. *If there exists $\bar{M} \in \mathbb{R}$ such that $W^n_{n-k}[\bar{M}]$ has a double zero at $c \in (a, b]$, for any $n-k \in \{1, \dots, n-1\}$. Then, there is not any $M \in \mathbb{R}$ such that the linear differential equation* (1.0.3) *is disconjugate on I.*

Proof. We have that $W^n_{n-k}[\bar{M}](c) = 0$.

If $W^n_{n-k-1}[\bar{M}](c) = 0$, we can apply Corollary 2.2.5 with $M_1 = M_2 = \bar{M}$ together with $c_1 = c_2 = c$ and we conclude that the linear differential equation (1.0.3) is not disconjugate for every $M \in \mathbb{R}$.

If $W^n_{n-k-1}[\bar{M}](c) \neq 0$, we consider the following non-trivial solution of equation (1.0.3):

$$y[\bar{M}](t) := \begin{vmatrix} y_1[\bar{M}](c) & \dots & y_{n-k}[\bar{M}](c) \\ \vdots & \ddots & \vdots \\ y_1^{(n-k-2)}[\bar{M}](c) & \dots & y_{n-k}^{(n-k-2)}[\bar{M}](c) \\ y_1[\bar{M}](t) & \dots & y_{n-k}[\bar{M}](t) \end{vmatrix}.$$

By construction, this solution satisfies the boundary conditions $(k, n-k)$ on $[a, c]$ and, from the expression of a derivative of a Wronskian given in (2.1.1), it also satisfies the boundary conditions $(k-1, n-k+1)$. Hence, using Proposition 1.1.10, we can affirm that $W^n_{n-k+1}[\bar{M}](c) = 0$, and then we can apply again Corollary 2.2.5 to affirm that the linear differential equation (1.0.3) is not disconjugate for every $M \in \mathbb{R}$. □

Now, let us see that the reciprocal of Corollary 2.2.5 is true, i.e., we can state a sufficient and necessary criterion to ensure that the parameter set where the linear differential equation (1.0.3) is disconjugate on I, is, or not, empty.

Theorem 2.2.8. *The linear differential equation* (1.0.3) *is not disconjugate on I for every $M \in \mathbb{R}$ if, and only if, there exist $c_1, c_2 \in (a, b]$ and $M_1 \geqslant M_2$ such that:*

- *There exists $n - k^*$ even such that $W^n_{n-k^*}[M_1](c_1) = 0$.*
- *There exits $n - k^{**}$ odd such that $W^n_{n-k^{**}}[M_2](c_2) = 0$.*

Proof. We only need to prove the first implication, since the other one is given in Corollary 2.2.5.

Let us denote the following parameters:

$$\widehat{M}_1 = \sup\left\{M \in \mathbb{R} \mid \exists\, c \in (a, b], W^n_{n-k}[M](c) = 0, \text{ with } n-k \text{ even}\right\},$$
$$\widehat{M}_2 = \inf\left\{M \in \mathbb{R} \mid \exists\, c \in (a, b], W^n_{n-k}[M](c) = 0, \text{ with } n-k \text{ odd}\right\}.$$

We can affirm that both previous sets are not empty because, on the other hand, there is not any eigenvalue of $T_n[M]\,u(t)$ in $X_{k\,[a,c]}$ for some $k \in \{1, \dots, n-1\}$, every $c \in (a, b]$ and every $M \in \mathbb{R}$. But, we know that for every $M \in \mathbb{R}$ there exists an interval, which we denote by $\big[a, \alpha_M(a)\big] \subset [a, b]$, where the linear differential equation (1.0.3) is disconjugate, see Proposition 1.1.3. By Lemma 2.0.1 and Theorems 1.2.10 and 1.2.11, it must exist an eigenvalue of $T[M]$ in $X_{k\,[a,\alpha_M(a)]}$ Hence, by Proposition 1.1.10, there exist $\bar{M} \in \mathbb{R}$ such that $W^n_{n-k}[\bar{M}]\big(\alpha_M(a)\big) = 0$.

Suppose now that the assertion of the theorem is not true, then necessarily $\widehat{M_1} < \widehat{M_2}$, and we have that for $M \in (\widehat{M_1}, \widehat{M_2})$ all Wronskians do not vanish for $t \in (a, b]$. So we can affirm that for those M the linear differential equation (1.0.3) is disconjugate on $[a, b]$, by means of Proposition 1.1.12. Then we have that the disconjugacy set is not empty and the proof is complete. □

Now, by using Proposition 1.1.10, we can reformulate previous result in terms of spectral theory.

Theorem 2.2.9. *The linear differential equation* (1.0.3) *is not disconjugate on I for every $M \in \mathbb{R}$ if, and only if, there exist $c_1, c_2 \in (a, b]$ and $\bar{M} \in \mathbb{R}$ such that:*

- *There exist $k^* \in \{1, \dots, n-1\}$, such that $n - k^*$ is even, and $\lambda_1 \leqslant 0$, an eigenvalue of $T[\bar{M}]$ in $X_{k^*\,[a,c_1]}$.*

- *There exist $k^{**} \in \{1, \dots, n-1\}$, such that $n - k^{**}$ is odd, and $\lambda_2 \geqslant 0$, an eigenvalue of $T[\bar{M}]$ in $X_{k^{**}\,[a,c_2]}$.*

We can rewrite previous result to characterise when the disconjugacy set is non-empty as follows.

Theorem 2.2.10. *There exists an interval (M_1, M_2), such that the linear differential equation* (1.0.3) *is disconjugate on I for every $M \in (M_1, M_2)$ if, and only if, for all $M \in \mathbb{R}$ one of the two following assertions is satisfied*

- *If $n - k$ is even, there is not any eigenvalue of $T_n[M]$ on $X_{k\,[a,c]}$ such that $\lambda \leqslant 0$ for all $c \in (a, b]$.*

- *If $n - k$ is odd, there is not any eigenvalue of $T_n[M]$ on $X_{k\,[a,c]}$ such that $\lambda \geqslant 0$ for all $c \in (a, b]$.*

In fact, by Theorem 2.2.9 we can only ensure the existence of at least one $\bar{M}$ for which equation (1.0.3) is disconjugate on I. The open character of the disconjugacy set, which has been shown in Proposition 1.1.8, allows us to make sure the existence of such an interval.

Realise that, to ensure that the disconjugacy set is not empty, from this result, we obtain that for all $M \in \mathbb{R}$ one of the assertions of Theorem 1.1.14 must be satisfied. Moreover, using Theorem 1.1.14, we deduce that if for $M = \bar{M}$ the linear differential equation (1.0.3) is disconjugate on I, then, since the disconjugacy on I implies the disconjugacy on every interval $[a, c] \subset I$, both assertions of Theorem 2.2.10 are satisfied for such $\bar{M}$.

2.2.1 Particular cases

In this section we present three examples, where the different criteria given in the previous section are applied.

Example 1 is a fourth order linear differential equation where we apply Corollary 2.2.6 to see that two consecutive Wronskians oscillate for an interval of $M \in \mathbb{R}$.

Example 2 is an eight order linear differential equation where we find two non consecutive Wronskians which oscillate simultaneously for an interval of $M \in \mathbb{R}$.

Example 3 is a fourth order linear differential equation where, in order to apply Corollary 2.2.7, we find a Wronskian with a double zero for a $\widehat{M}$. Furthermore, we prove that this $\widehat{M}$ is the unique $M \in \mathbb{R}$ for which the sufficient and necessary condition given in Theorem 2.2.8 is satisfied.

Before beginning with the explicit examples we show a result which follows from the proof of Corollary 2.2.7.

Lemma 2.2.11. *If there exists $M \in \mathbb{R}$, such that $W_k^n[M]$ has a double zero at $t \in (0,1]$ and $W_{k-1}[M](t) \neq 0$, then $W_{k+1}^n[M](t) = 0$.*

- $T_4^5[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 200\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$

Firstly, we show an example where the non-disconjugacy criteria holds as an straight consequence of Corollary 2.2.6.

We consider the operator:

$$T_4^5[M]\,u(t) = u^{(4)}(t) + 200\,u''(t) + M\,u(t),\ t \in [0,1].$$

Let us see that for $\bar{M} = 8^4$, both $W_1^4[\bar{M}](t)$ and $W_2^4[\bar{M}](t)$ oscillate on $[0,1]$.

We obtain that they are given by the following expressions

$$W_1^4[\bar{M}](t) = \frac{\left(3+\sqrt{41}\right)\sin\left(\sqrt{2}\left(\sqrt{41}-3\right)t\right) - \left(\sqrt{41}-3\right)\sin\left(\sqrt{2}\left(3+\sqrt{41}\right)t\right)}{768\sqrt{82}},$$

$$W_2^4[\bar{M}](t) = \frac{41\cos\left(6\sqrt{2}t\right) - 9\cos\left(2\sqrt{82}t\right) - 32}{377856},$$

and, in particular, they satisfy:

$$\begin{aligned} W_1^4[\bar{M}](1/3) &= \frac{1}{384}\left(\frac{3\sin\left(\dfrac{\sqrt{82}}{3}\right)\cos\left(\sqrt{2}\right)}{\sqrt{82}} - \frac{\sin\left(\sqrt{2}\right)\cos\left(\dfrac{\sqrt{82}}{3}\right)}{\sqrt{2}}\right) \\ &\approxeq 0.00182165 > 0, \\ W_1^4[\bar{M}](1) &= \frac{\left(3+\sqrt{41}\right)\sin\left(\sqrt{2}(\sqrt{41}-3)\right) - \left(\sqrt{41}-3\right)\sin\left(\sqrt{2}(3+\sqrt{41})\right)}{768\sqrt{82}} \\ &\approxeq -0.00167221 < 0, \end{aligned}$$

and:

$$W_2^4[\bar{M}](4/5) = \frac{-32 + 41\cos\left(\dfrac{24\sqrt{2}}{5}\right) - 9\cos\left(\dfrac{8\sqrt{82}}{5}\right)}{377856} \approxeq 0.0000184718 > 0,$$
$$W_2^4[\bar{M}](1) = \frac{-32 + 41\cos\left(6\sqrt{2}\right) - 9\cos\left(2\sqrt{82}\right)}{377856} \approxeq -0.000166337 < 0.$$

So, both of them oscillate on $[0,1]$ and Corollary 2.2.6 allows us to affirm that it does not exist $M \in \mathbb{R}$ such that $u^{(4)}(t) + 200\,u''(t) + M\,u(t) = 0$ is a disconjugate equation on $[0,1]$.

Realise that in this case, by means of the continuity of the Wronskians as functions of M, we can affirm that there exists $\varepsilon > 0$, such that the non-disconjugacy criterion is satisfied for $M \in (8^4 - \varepsilon, 8^4 + \varepsilon)$.

In order to apply the spectral characterisation given in Theorem 2.2.9, we obtain numerically, for this case, that $7.03^4 > 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{3[0,1]}$ and $-5.075^4 < 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{2[0,0.9]}$.

- $T_8^1[M] \equiv \dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8} + 11^4\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$

In this example we consider the eighth-order linear differential equation:

$$T_8^1[M]\,u(t) = u^{(8)}(t) + 11^4\,u^{(4)}(t) + M\,u(t) = 0, \quad t \in [0,1]. \tag{2.2.2}$$

Let us see that for $\bar{M} = -40917842$, $W_1^8[\bar{M}](t)$ and $W_4^8[\bar{M}](t)$ oscillate simultaneously.

In order to obtain the expression of $W_1^8[\bar{M}]$ and $W_4^8[\bar{M}]$, we calculate the solutions $y_k[\bar{M}]$, $k = 1,\dots,4$, defined in (1.1.6):

$$\begin{aligned}
y_1[\bar{M}](t) =& \frac{\sinh(7t)}{13337898} + \frac{\cos\left(\sqrt[4]{\frac{8521}{2}}t\right)\sinh\left(\sqrt[4]{\frac{8521}{2}}t\right) - \sin\left(\sqrt[4]{\frac{8521}{2}}t\right)\cosh\left(\sqrt[4]{\frac{8521}{2}}t\right)}{38886\sqrt[4]{2}\cdot 8521^{3/4}} \\
& - \frac{\sin(7t)}{13337898}, \\
y_2[\bar{M}](t) =& \frac{\cosh(7t) - \cos(7t)}{1905414} - \frac{\sin\left(\sqrt[4]{\frac{8521}{2}}t\right)\sinh\left(\sqrt[4]{\frac{8521}{2}}t\right)}{19443\sqrt{17042}}, \\
y_3[\bar{M}](t) =& \frac{\sin(7t) + \sinh(7t)}{272202} - \frac{\cos\left(\sqrt[4]{\frac{8521}{2}}t\right)\sinh\left(\sqrt[4]{\frac{8521}{2}}t\right) - \sin\left(\sqrt[4]{\frac{8521}{2}}t\right)\cosh\left(\sqrt[4]{\frac{8521}{2}}t\right)}{19443\cdot 2^{3/4}\sqrt[4]{8521}}, \\
y_4[\bar{M}](t) =& \frac{\cos(7t) + \cosh(7t) - 2\cos\left(\sqrt[4]{\frac{8521}{2}}t\right)\cosh\left(\sqrt[4]{\frac{8521}{2}}t\right)}{38886}.
\end{aligned}$$

Since $W_1^8[\bar{M}](t) = y_1[\bar{M}](t)$, we have that:

$$y_1[\bar{M}]\left(\frac{4}{5}\right) = \frac{\sinh\left(\frac{28}{5}\right) - \sin\left(\frac{28}{5}\right)}{13337898} + \frac{\cos\left(\frac{2}{5}\cdot 2^{3/4}\sqrt[4]{8521}\right)\sinh\left(\frac{2}{5}\cdot 2^{3/4}\sqrt[4]{8521}\right)}{38886\sqrt[4]{2}\cdot 8521^{3/4}}$$
$$- \frac{\sin\left(\frac{2}{5}\cdot 2^{3/4}\sqrt[4]{8521}\right)\cosh\left(\frac{2}{5}\cdot 2^{3/4}\sqrt[4]{8521}\right)}{38886\sqrt[4]{2}\cdot 8521^{3/4}} \approxeq 0.0000164747 > 0,$$

$$y_1[\bar{M}](1) = \frac{\sinh(7) - \sin(7)}{13337898} + \frac{\cos\left(\sqrt[4]{\frac{8521}{2}}\right)\sinh\left(\sqrt[4]{\frac{8521}{2}}\right)}{38886\sqrt[4]{2}\cdot 8521^{3/4}}$$
$$- \frac{\sin\left(\sqrt[4]{\frac{8521}{2}}\right)\cosh\left(\sqrt[4]{\frac{8521}{2}}\right)}{38886\sqrt[4]{2}\cdot 8521^{3/4}} \approxeq -0.000006063524 < 0,$$

hence $W_1^8[\bar{M}]$ oscillates on $(0,1]$.

$W_4^8[\bar{M}]$ is given by:

$$W_4^8[\bar{M}](t) = \begin{vmatrix} y_1[\bar{M}](t) & y_2[\bar{M}](t) & y_3[\bar{M}](t) & y_4[\bar{M}](t) \\ y_1'[\bar{M}](t) & y_2'[\bar{M}](t) & y_3'[\bar{M}](t) & y_4'[\bar{M}](t) \\ y_1''[\bar{M}](t) & y_2''[\bar{M}](t) & y_3''[\bar{M}](t) & y_4''[\bar{M}](t) \\ y_1'''[\bar{M}](t) & y_2'''[\bar{M}](t) & y_3'''[\bar{M}](t) & y_4'''[\bar{M}](t) \end{vmatrix}.$$

One can verify that $W_4^8[\bar{M}](t) = \dfrac{w_4(t)}{4677952554370173403681473 6}$, where $w_4(t)$ is a function whose explicit expression is obtained by means of *Mathematica* program, but its expression is to complicated to show here. However, it is not difficult to verify that:

$$w_4\left(\frac{4}{5}\right) = 2.05505 \times 10^{15} > 0, \qquad \text{and} \qquad w_4(1) = -9.23372 \times 10^{16} < 0,$$

hence it also oscillates on $[0,1]$, see Figure 2.2.1.

So, by means of Corollary 2.2.6 again, we can affirm that the linear differential equation (2.2.2) is not disconjugate on $[0,1]$ for all $M \in \mathbb{R}$. As in previous example we obtain an interval of $M \in \mathbb{R}$ for which the non-disconjugacy hypothesis is satisfied.

Remark 2.2.12. *In this example we have applied the non-disconjugacy criterion to two non consecutive Wronskians. However, we did not say anything about the others Wronskians.*

First, realise that, since the problem is self-adjoint, we only need to study the Wronskians $W_1^8[M]$, $W_2^8[M]$, $W_3^8[M]$ *and* $W_4^8[M]$.

By numerical studies, we observe that:

- $W_1^8[M]$ *does not have any zero on* $(0,1]$ *for every* $M \leqslant M_1 = -\dfrac{235165923}{4}$,
- $W_2^8[M]$ *does not have any zero on* $(0,1]$ *for every* $M \geqslant M_1$,

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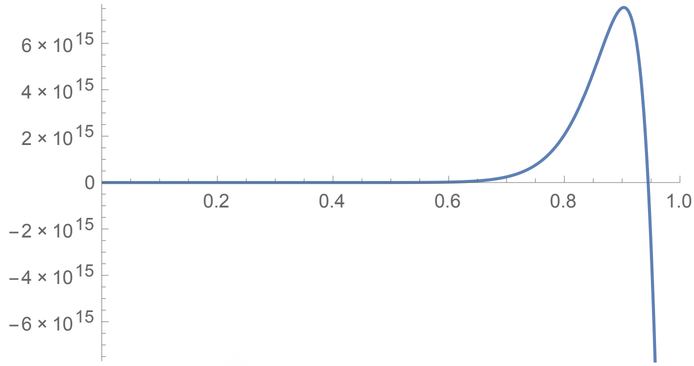

Figure 2.2.1: $w_4(t)$ with $t \in [0,1]$.

- *$W_3^8[M]$ does not have any zero on $(0,1]$ for every $M \leqslant 0$,*
- *$W_4^8[M]$ does not have any zero on $(0,1]$ for every $M \leqslant 0$.*

With analytical calculus, see Appendix B, we are able to prove first and fourth items. Moreover, we also can prove:

- *$W_2^8[M]$ does not have any zero on $(0,1]$ for $M = M_1$,*
- *$W_3^8[M]$ does not have any zero on $(0,1]$ for $M = 0$,*

However, because of the hardness of the calculus, we have not been able to completely prove analytically that $W_2^8[M]$ does not have any zero on $(0,1]$ for every $M \geqslant M_1$ and $W_3^8[M]$ does not have any zero on $(0,1]$ for every $M \leqslant 0$.

Open Problem 1. *If we could prove, by analytical calculus, the numerical studies of previous remark, by Lemma 2.2.11, we have that, in this case, two consecutive Wronskian do not oscillate simultaneously for any $M \in \mathbb{R}$ and allow us to prove that it does not exist any $M \in \mathbb{R}$ such that the disconjugacy criterion of Corollary 2.2.7 is satisfied.*

So, we would have proved that Corollaries 2.2.5 and 2.2.7 are not equivalent. In particular, Corollary 2.2.7 would be a sufficient but not necessary condition. In any case, an analytical proof remains as an open problem.

In this case, in order to apply Theorem 2.2.9, we obtain numerically that $7.6^8 > 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{7[0,1]}$ and $-7.7^8 < 0$ is an eigenvalue of $T[\bar{M}]$ in $X_{4[0,1]}$.

- $T_4^6[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 1000\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

Next example follows from Corollary 2.2.7.

We consider the fourth order equation:

$$T_4^6[M]\,u(t) = u^{(4)}(t) + 1000\,u'(t) + M\,u(t) = 0, \quad t \in [0,1]. \tag{2.2.3}$$

To prove the non-disconjugacy property for this equation is really complicated. This is due, in part, from the fact that the parameter $\bar{M}$, for which the non-disconjugacy criteria given in Theorem 2.2.8 follows, is unique. The calculus made on this case are very hard to deal with, so they will be avoided here, for details see Appendix C.

Next, we summarise the steps that we have followed in order to obtain the unique $\bar{M} \in \mathbb{R}$ for which the hypotheses of Theorem 2.2.8 are fulfilled:

- First, we prove that the first Wronskian, $W_1^4[M]$ has at least a zero on $(0,1]$ for $M = 0$.
- Then, we obtain the expression of such a Wronskian for negative values of M and we prove that there exists $M^* < 0$ such that $W_1^4[M^*](t) \neq 0$ for all $t \in (0,1]$.
- After that, we verify that $W_1^4[M](1) \neq 0$ for all $M \in [M^*, 0]$. Since $W_1^4[M]$ is a continuous function on M, we conclude that there exist $\widehat{M} \in (M^*, 0)$ and $\widehat{t} \in (0,1)$ such that $W_1^4[\widehat{M}]$ has a double zero at $\widehat{t}$ and that $W_1^4[M]$ oscillates on $[0,1]$ for all $0 \geqslant M > \widehat{M}$.

Hence, as a consequence of the previous assertions, using Corollary 2.2.7, we conclude that the linear differential equation (2.2.3) is not disconjugate for every $M \in \mathbb{R}$.

To see that there is not any $M \in \mathbb{R}$, $M \neq \widehat{M}$ such that the non-disconjugacy criteria given in Theorem 2.2.8 could be applied, we argue as follows:

- First, we see that if $M < \widehat{M}$, then $W_1^4[M] > 0$ on $(0,1]$.
- After that, we verify that $W_3^4[M]$ is of constant sign for every $M \leqslant 0$.
- Finally, we prove the constant sign of $W_2^4[M] > 0$ on $(0,1]$ for $M \geqslant \widehat{M}$.

This three last items allow us to affirm that the hypotheses of Corollaries 2.2.5, 2.2.6 and 2.2.7 and Theorem 2.2.8 are only satisfied for $\widehat{M}$.

Because of the complexity and extension of the calculus, they are shown on Appendix C.

Chapter 3
Green's functions

Contents

In this chapter, the largest of the project, it appears a collection of important results which are very useful in the study of non-linear problems too. It is very well-known that the applicability of the method of lower and upper solutions, coupled with the monotone iterative techniques [44, 62], is equivalent to the constant sign of the Green's function related to the linear part of the studied problem [16, 15].

Moreover, by using the constant sign of the related Green's function, non-existence, existence and multiplicity results are derived, by means of the well-known Krasnosel'skiĭ contraction/expansion fixed point theorem [61], from the construction of suitable cones on Banach spaces. Such a construction follows by using adequate properties of Green's function, one of them is its constant sign [4, 18, 54, 55, 92].

The combination of these two methods has also been proved as a useful tool to ensure the existence of solution [17, 19, 39, 52, 80].

Having in mind the power of this constant sign property, we will describe the interval of parameters for which the Green's function related to the general linear operator $T_n[M]$ of order n, introduced in (1.0.2), with many different two-point boundary conditions.

In the sequel, we introduce two sets of indices which characterise de boundary conditions in each case.

Let $k \in \{1,\dots,n-1\}$ and consider two set of indices, $\{\sigma_1,\dots,\sigma_k\} \subset \{0,\dots,n-1\}$ and $\{\varepsilon_1,\dots,\varepsilon_{n-k}\} \subset \{0,\dots,n-1\}$, such that:

$$0 \leqslant \sigma_1 < \sigma_2 < \cdots < \sigma_k \leqslant n-1, \quad 0 \leqslant \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leqslant n-1.$$

We study the problem:

$$T_n[M] = \sigma(t), \quad t \in I \equiv [a,b], \tag{3.0.1}$$

coupled with the following boundary conditions:

$$u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \tag{3.0.2}$$

$$u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0. \tag{3.0.3}$$

This boundary conditions cover many different problems. As an example, we can consider $n = 4$, and:

$$\{\sigma_1, \sigma_2\} = \{0, 2\} \text{ and } \{\varepsilon_1, \varepsilon_2\} = \{0, 2\},$$

which correspond to the simply supported beam boundary conditions. We have studied such kind of problems in [32]. Moreover, in Chapter 5 we will show a deeper study for the simply supported beam boundary conditions.

In addition, for the choice:

$$\{\sigma_1, \ldots, \sigma_k\} = \{0, \ldots, k-1\} \text{ and } \{\varepsilon_1, \ldots, \varepsilon_{n-k}\} = \{0, \ldots, n-k-1\},$$

we define the so-called $(k, n-k)$ boundary conditions which correspond to have a zero of multiplicity k at the left extreme of the interval, a, and a zero of multiplicity $n-k$ at the right extreme, b. We have considered these problems in [29]. We collect the generalisation of the result for the different choices of $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ in [30].

The study of the constant sign of the related Green's function has been widely developed along the literature. From Theorem 1.2.9, we know that the parameter set for which the Green's function is of constant sign, H_T, is an interval. Moreover, by using Theorems 1.2.10 and 1.2.11, under the hypothesis that the related Green's function satisfies either condition (P_g) or (N_g) one of the extremes of H_T is characterised as the opposed of the closest to zero eigenvalue of $T_n[M]$, in the correspondent space of definition, given by:

$$\begin{aligned} X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}} = \Big\{ u \in C^n(I) \mid u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = u^{(\varepsilon_1)}(b) \\ = \cdots = u^{(\varepsilon_{n-k})}(b) = 0 \Big\}. \end{aligned} \tag{3.0.4}$$

Remark 3.0.1. *Realise that from* (1.1.8)*,* $X_k \equiv X_{\{0,1,\ldots,k-1\}}^{\{0,1,\ldots,n-k-1\}}$.

Thus, the difficulty remains in the characterisation of the other extreme of the interval H_T. In this case, as it is shown in [16, Section 1.8], such an extreme is not an eigenvalue of the considered problem, so to attain its exact value is not immediate. In practical situations it is necessary to obtain the expression of the Green's function, which is, in general, a difficult matter to deal with, see for instance [22, 66]. In [21], provided operator $T_n[M]$ has constant coefficients, it is developed a computer algorithm that calculates the exact expression of a Green's function coupled with two-point boundary value conditions. However, such an expression is often too complicated to manage, and to describe the interval H_T is really very difficult in practical situations. In fact, there is not a direct method of construction for non-constant coefficients.

We mention that in [72], it is used the disconjugacy theory to obtain the values for which the third order operators $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M \dfrac{\mathrm{d}^i}{\mathrm{d}\,t^i}$, for $i = 0, 1, 2$, coupled with conditions $(1, 2)$ and $(2, 1)$ have constant sign Green's function. In [23], a similar procedure is used for the fourth order operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$, coupled with conditions $(2, 2)$ and, more recently, in [24] with conditions $(1, 3)$ and $(3, 1)$. In all the situations, the interval of disconjugacy is obtained and then, by means of the expression of the Green's function, it is proved that such an interval is optimal. This coincidence holds only in particular cases as the ones treated in these papers, in general the intervals of disconjugacy and constant sign Green's functions do not coincide for the n^{th} - order operator $T_n[M]$, even for the $(k, n - k)$ boundary conditions.

It is for this that we make in this work a general characterisation of the regular extreme of the interval of constant sign H_T by means of the spectral theory. We will show that it is an eigenvalue of the same operator $T_n[M]$ but related to different suitable two-point boundary value conditions.

Moreover, our results show that the Green's function satisfies a stronger condition than (P_g) or (N_g) in the whole intervals for which it is of constant sign. Such a property will be very useful in Chapter 6 where several results for non-linear two point boundary value problems will be obtained.

3.1 Green's function of the vectorial problem

This subsection is devoted to express, as functions of $g_M(t, s)$, the functions $g_1(t, s), \dots, g_{n-1}(t, s)$, defined on (1.2.7), as the first row components of the matrix Green's function related to the vectorial system (1.2.4)-(1.2.5). In such a relation the considered boundary conditions are not relevant.

By studying the adjoint operator as in [16, Section 1.3], we know that the related Green's function of the adjoint operator G^* satisfies that $G^*(t, s) = G^T(s, t)$. Moreover, the following equality holds:

$$\frac{\partial}{\partial t}\Big(- G^*(t, s)\Big) = -A^T(t)\Big(- G^*(t, s)\Big), \quad t \in I \setminus \{s\}.$$

So, we can transform previous equality in:

$$\begin{aligned}\left(-\frac{\partial}{\partial t}\, G(s, t)\right)^T &= -\frac{\partial}{\partial t}\, G^T(s, t) = -A^T(t)\left(-G^T(s, t)\right) \\ &= A^T(t)\, G^T(s, t) = \big(G(s, t)\, A(t)\big)^T.\end{aligned}$$

Hence:

$$\frac{\partial}{\partial t}\, G(s, t) = -G(s, t)\, A(t),$$

or, which is the same,

$$\frac{\partial}{\partial s}\, G(t, s) = -G(t, s)\, A(s). \tag{3.1.1}$$

Using this equality, we will prove by induction the following ones:

$$g_{n-j}(t,s) = (-1)^j \,\frac{\partial^j}{\partial s^j}\, g_M(t,s) + \sum_{k=0}^{j-1} \alpha_k^j(s)\, \frac{\partial^k}{\partial s^k}\, g_M(t,s), \quad j = 1,\dots,n-1. \tag{3.1.2}$$

Here $\alpha_i^j(s)$ are functions of $p_1(s),\dots,\, p_j(s)$ and of its derivatives up to order $(j-1)$ and follow the recurrence formula:

$$\alpha_0^0(s) = 0, \tag{3.1.3}$$

$$\alpha_k^{j+1}(s) = 0, \quad k \geqslant j+1 \geqslant 1\,, \tag{3.1.4}$$

$$\alpha_0^{j+1}(s) = p_{j+1}(s) - \left(\alpha_0^j\right)'(s), \quad j \geqslant 0, \tag{3.1.5}$$

$$\alpha_k^{j+1}(s) = -\left(\alpha_{k-1}^j(s) + \left(\alpha_k^j\right)'(s)\right), \quad 1 \leqslant k \leqslant j. \tag{3.1.6}$$

Indeed, using equality (3.1.1), we deduce that the Green's matrix terms which are on position $(1,i)$, $i = 1,\dots,n$, satisfy the following equality:

$$g_{i-1}(t,s) = -\frac{\partial}{\partial s}\, g_i(t,s) + p_{n-i+1}(s)\, g_M(t,s), \quad i = 2,\dots,n, \tag{3.1.7}$$

where $g_M(t,s) \equiv g_n(t,s)$.

If we take $i = n$ in equation (3.1.7) we deduce:

$$g_{n-1}(t,s) = -\frac{\partial}{\partial s}\, g_M(t,s) + p_1(s)\, g_M(t,s),$$

which gives equation (3.1.2) for $j = 1$.

Assume now that equalities (3.1.2) – (3.1.6) are fulfilled for some $j \in \{1,\dots,n-2\}$. Let us see that they hold again for $j+1$.

$$\begin{aligned}
g_{n-j-1}(t,s) =& p_{j+1}(s)\, g_M(t,s) - \frac{\partial}{\partial s}\left((-1)^j \frac{\partial^j}{\partial s^j}\, g_M(t,s) + \sum_{k=0}^{j-1} \alpha_k^j(s) \frac{\partial^k}{\partial s^k}\, g_M(t,s)\right) \\
=& p_{j+1}(s)\, g_M(t,s) + (-1)^{j+1} \frac{\partial^{j+1}}{\partial s^{j+1}}\, g_M(t,s) - \sum_{k=0}^{j-1} \left(\alpha_k^j\right)'(s) \frac{\partial^k}{\partial s^k}\, g_M(t,s) \\
&- \sum_{k=0}^{j-1} \alpha_k^j(s) \frac{\partial^{k+1}}{\partial s^{k+1}}\, g_M(t,s) \\
=& (-1)^{j+1} \frac{\partial^{j+1}}{\partial s^{j+1}}\, g_M(t,s) + p_{j+1}(s)\, g_M(t,s) \\
&- \sum_{k=0}^{j-1} \left(\alpha_k^j\right)'(s) \frac{\partial^k}{\partial s^k}\, g_M(t,s) - \sum_{k=1}^{j} \alpha_{k-1}^j(s) \frac{\partial^k}{\partial s^k}\, g_M(t,s) \\
=& (-1)^{j+1} \frac{\partial^{j+1}}{\partial s^{j+1}}\, g_M(t,s) + \sum_{k=0}^{j} \alpha_k^{j+1}(s)\, \frac{\partial^k}{\partial s^k}\, g_M(t,s).
\end{aligned}$$

Now, we can express the Green's matrix related to problem (1.2.4)-(1.2.5), $G(t,s)$, as:

$$\left(\begin{array}{ccc} (-1)^{n-1}\,\dfrac{\partial^{n-1}}{\partial s^{n-1}}\,g_M(t,s)+\displaystyle\sum_{k=0}^{n-2}\alpha_k^{n-1}(s)\,\dfrac{\partial^k}{\partial s^k}\,g_M(t,s) & \cdots & g_M(t,s) \\ (-1)^{n-1}\,\dfrac{\partial^{n}}{\partial t\,\partial s^{n-1}}\,g_M(t,s)+\displaystyle\sum_{k=0}^{n-2}\alpha_k^{n-1}(s)\,\dfrac{\partial^{k+1}}{\partial t\,\partial s^k}\,g_M(t,s) & \cdots & \dfrac{\partial}{\partial t}\,g_M(t,s) \\ & \cdots & \\ \vdots & & \vdots \\ & \cdots & \\ (-1)^{n}\,\dfrac{\partial^{2n-2}}{\partial t^{n-1}\partial s^{n-1}}\,g_M(t,s)+\displaystyle\sum_{k=0}^{n-2}\alpha_k^{n-1}(s)\dfrac{\partial^{n-1+k}}{\partial t^{n-1}\partial s^k}\,g_M(t,s) & \cdots & \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,g_M(t,s) \end{array}\right). \tag{3.1.8}$$

If the coefficients $p_1(s),\ldots,p_{n-1}(s),\ p_n(s)$ are constants, $p_1,\ldots,p_{n-1},p_n$, we can solve explicitly the recurrence formula (3.1.3) – (3.1.6) and deduce that $\alpha_k^j(s)=(-1)^k\,p_{j-k}$. So, we have that:

$$g_{n-j}(t,s)=\sum_{k=0}^{j}(-1)^k\,p_{j-k}\,\frac{\partial^k}{\partial s^k}\,g_M(t,s),\quad \text{with } p_0=1,$$

and we can rewrite $G(t,s)$ as:

$$\left(\begin{array}{cccc} \displaystyle\sum_{k=0}^{n-1}(-1)^k\,p_{n-1-k}\,\frac{\partial^k}{\partial s^k}\,g_M(t,s) & \cdots & \displaystyle\sum_{k=0}^{1}(-1)^k\,p_{1-k}\,\frac{\partial^k}{\partial s^k}\,g_M(t,s) & g_M(t,s) \\ \displaystyle\sum_{k=0}^{n-1}(-1)^k\,p_{n-1-k}\,\frac{\partial^{k+1}}{\partial t\partial s^k}\,g_M(t,s) & \cdots & \displaystyle\sum_{k=0}^{1}(-1)^k\,p_{1-k}\,\frac{\partial^{k+1}}{\partial t\partial s^k}\,g_M(t,s) & \dfrac{\partial}{\partial t}\,g_M(t,s) \\ \vdots & & \vdots & \\ \displaystyle\sum_{i=0}^{n-1}(-1)^k\,p_{n-1-k}\,\frac{\partial^{n-1+k}}{\partial t^{n-1}\partial s^k}\,g_M(t,s) & \cdots & \displaystyle\sum_{k=0}^{1}(-1)^k\,p_{1-k}\,\frac{\partial^{n-1+k}}{\partial t^{n-1}\partial s^k}\,g_M(t,s) & \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,g_M(t,s) \end{array}\right).$$

In particular, if $T_n[M]\,u(t)\equiv u^{(n)}(t)+M\,u(t)$ we conclude that

$$g_{n-j}(t,s)=(-1)^j\,\frac{\partial^j}{\partial s^j}\,g_M(t,s),$$

so the Green's matrix, $G(t,s)$, is given by expression:

$$\left(\begin{array}{cccc} (-1)^{n-1}\,\dfrac{\partial^{n-1}}{\partial s^{n-1}}\,g_M(t,s) & \cdots & -\dfrac{\partial}{\partial s}\,g_M(t,s) & g_M(t,s) \\ (-1)^{n-1}\,\dfrac{\partial^{n}}{\partial t\partial s^{n-1}}\,g_M(t,s) & \cdots & -\dfrac{\partial^2}{\partial t\partial s}\,g_M(t,s) & \dfrac{\partial}{\partial t}\,g_M(t,s) \\ \vdots & & \vdots & \\ (-1)^{n-1}\,\dfrac{\partial^{2\,n-2}}{\partial t^{n-1}\partial s^{n-1}}\,g_M(t,s) & \cdots & -\dfrac{\partial^{n}}{\partial t^{n-1}\partial s}\,g_M(t,s) & \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,g_M(t,s) \end{array}\right).$$

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Remark 3.1.1. *We note that in the general case it is possible to obtain some of the components of system* (3.1.3) – (3.1.6).

$$\alpha_0^j(s) = \sum_{i=0}^{j-1} (-1)^i \, p_{j-i}^{(i)}(s),$$
$$\alpha_1^j(s) = \sum_{i=1}^{j-1} (-1)^i \, i \, p_{j-i}^{(i-1)}(s),$$
$$\alpha_j^{j+1}(s) = (-1)^j \, p_1(s).$$

As we will see in Section 3.5, the usefulness of this decomposition remains in the study of the different derivatives of the Green's function related to $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ making a relationship with the associated vectorial problem.

3.2 Green's function sign and strongly inverse positive (negative) character

We have already mentioned many of the properties which constant sign Green's function implies on the related differential operators and their solutions. One of them is studied in Theorem 1.2.7 for either the inverse positive or negative character. In the sequel we introduce a stronger concept than inverse positive (negative) character for the operator $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. In order to do that, we introduce some notation.

Notation 3.2.1. *Let* α, $\beta \in \{0,\dots,n-1\}$ *be such that:*

$$\alpha \notin \{\sigma_1,\dots,\sigma_k\}, \text{ and if } \alpha \neq 0, \text{ then } \{0,\dots,\alpha-1\} \subset \{\sigma_1,\dots,\sigma_k\}, \tag{3.2.1}$$
$$\beta \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\}, \text{ and if } \beta \neq 0, \text{ then } \{0,\dots,\beta-1\} \subset \{\varepsilon_1,\dots,\varepsilon_{n-k}\}. \tag{3.2.2}$$

Realise that $\alpha \leqslant k$ and $\beta \leqslant n-k$. As an example we can say that for the choice $n=5$, $k=3$ and $\{\sigma_1,\sigma_2,\sigma_3\} = \{0,1,2\}$ and $\{\varepsilon_1,\varepsilon_2\} = \{1,2\}$, we have $\alpha = 3 = k$ and $\beta = 0 < n-k = 2$.

Remark 3.2.2. *Along this work we consider different choices of boundary conditions. Sometimes, we do not know the order of the given indices which define the spaces of definition. For instance, we consider the following boundary conditions:*

$$u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{k-1})}(a) = 0,$$
$$u^{(\alpha)}(a) = 0,$$
$$u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0.$$

In order to point out this setting of the indices we use the following notation:

$$X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}} = \Big\{ u \in C^n(I) \;\Big|\; u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{k-1})}(a) = 0,\ u^{(\alpha)}(a) = 0,$$
$$u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0 \Big\}.$$

Definition 3.2.3. *Operator $T_n[M]$ is said to be strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if every function $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\,u \gneqq 0$ on I, must satisfy $u > 0$ on (a,b) and moreover,*

$$u^{(\alpha)}(a) > 0 \text{ and } \begin{cases} u^{(\beta)}(b) > 0, & \text{if } \beta \text{ is even,} \\ u^{(\beta)}(b) < 0, & \text{if } \beta \text{ is odd,} \end{cases}$$

where α and β are defined in (3.2.1) *and* (3.2.2)*, respectively.*

Definition 3.2.4. *Operator $T_n[M]$ is said to be strongly inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if every function $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\,u \gneqq 0$ on I, must satisfy $u < 0$ on (a,b) and, moreover,*

$$u^{(\alpha)}(a) < 0 \text{ and } \begin{cases} u^{(\beta)}(b) < 0, & \text{if } \beta \text{ is even,} \\ u^{(\beta)}(b) > 0, & \text{if } \beta \text{ is odd,} \end{cases}$$

where α and β are defined in (3.2.1) *and* (3.2.2)*, respectively.*

We can prove analogous results to Theorem 1.2.7 for these concepts.

Theorem 3.2.5. *Operator $T_n[M]$ is strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, Green's function related to problem* (3.0.1)–(3.0.3)*, $g_M(t,s)$ satisfies the following properties:*

- *$\forall\, t \in (a,b)$, $g_M(t,s) > 0$ for a.e. $s \in (a,b)$.*
- *$\dfrac{\partial^\alpha}{\partial t^\alpha} g_M(t,s)_{|t=a} > 0$ for a.e. $s \in (a,b)$.*
- $$\begin{cases} \dfrac{\partial^\beta}{\partial t^\beta} g_M(t,s)_{|t=b} > 0, & \text{if } \beta \text{ is even,} \\ \dfrac{\partial^\beta}{\partial t^\beta} g_M(t,s)_{|t=b} < 0, & \text{if } \beta \text{ is odd,} \end{cases} \quad \text{for a.e. } s \in (a,b).$$

Proof. First, let us prove the first implication.

From Theorem 1.2.7, we know that $g_M(t,s) \geqslant 0$ on $I \times I$.

From equality (1.3.3) we have that $g_M(t,s) = g_M^*(s,t)$, where $g_M^*(t,s)$ is the Green's function related to the adjoint operator, defined in (1.3.1)-(1.3.2). Moreover, from item (g_4) in Definition 1.2.3, for each fixed $\bar t \in (a,b)$, $g_M^*(s,\bar t)$ satisfies:

$$T_n^* g_M^*(s,\bar t) = 0, \quad s \in [a,\bar t) \cup (\bar t, b]. \tag{3.2.3}$$

Let us see that for each $\bar t \in (a,b)$ $g_M(\bar t,s) = g_M^*(s,\bar t)$ has a finite number of zeros. Suppose, on the contrary, that there exists $\bar t \in (a,b)$ such that $g_M(\bar t,s)$ has infinite zeros. Without lost of generality, assume that $g_M(\bar t,s)$ has infinite zeros on $[a,\bar t)$. Then, since it is a solution of the differential equation (3.2.3), necessarily $g_M(\bar t,s) = 0$ for all $s \in [a,\bar t]$, see Proposition 1.1.3. Consider on the problem (3.0.1)–(3.0.3):

$$\sigma(t) = \begin{cases} t-\bar t, & a \leqslant t \leqslant \bar t, \\ \\ 0, & \bar t \leqslant t \leqslant b. \end{cases}$$

So, from Theorem 1.2.4, we have:

$$u(\bar{t}) = \int_a^b g_M(\bar{t}, s)\, \sigma(s)\, \mathrm{d}\, s = \int_a^{\bar{t}} g_M(\bar{t}, s)(s - \bar{t})\, \mathrm{d}\, s = 0.$$

Reciprocally, if the Green's function satisfies the hypotheses of the theorem, we clearly have the next conclusions for $\sigma \gneqq 0$ on I:

$$\begin{aligned} u(t) &= \int_a^b g_M(t,s)\, \sigma(s)\, \mathrm{d}\, s > 0,\ \forall t \in (a,b), \\ u^{(\alpha)}(a) &= \int_a^b \frac{\partial^\alpha}{\partial t^\alpha} g_M(t,s)_{|t=a}\, \sigma(s)\, \mathrm{d}\, s > 0, \\ u^{(\beta)}(b) &= \begin{cases} \displaystyle\int_a^b \frac{\partial^\beta}{\partial t^\beta} g_M(t,s)_{|t=b}\, \sigma(s)\, \mathrm{d}\, s > 0, & \text{if } \beta \text{ is even,} \\ \displaystyle\int_a^b \frac{\partial^\beta}{\partial t^\beta} g_M(t,s)_{|t=b}\, \sigma(s)\, \mathrm{d}\, s < 0, & \text{if } \beta \text{ is odd.} \end{cases} \end{aligned}$$

□

Analogously, we obtain a result for the strongly inverse negative case.

Theorem 3.2.6. *Operator $T_n[M]$ is strongly inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, Green's function related to problem* (3.0.1)–(3.0.3)*, $g_M(t,s)$ satisfies the following properties:*

- $\forall\, t \in (a,b)$, $g_M(t,s) < 0$ *for a.e.* $s \in (a,b)$.
- $\dfrac{\partial^\alpha}{\partial t^\alpha} g_M(t,s)_{|t=a} < 0$ *for a.e.* $s \in (a,b)$.
- $\begin{cases} \dfrac{\partial^\beta}{\partial t^\beta} g_M(t,s)_{|t=b} < 0, & \textit{if } \beta \textit{ is even,} \\ \dfrac{\partial^\beta}{\partial t^\beta} g_M(t,s)_{|t=b} > 0, & \textit{if } \beta \textit{ is odd,} \end{cases}$ *for a.e.* $s \in (a,b)$.

Remark 3.2.7. *Realise that in two previous results we have, indeed, that the inequalities are fulfilled for all $s \in (a,b)$, except for a finite number of points on (a,b).*

3.3 Hypotheses on operator $T_n[\bar{M}]$

This section is devoted to describe the hypotheses needed to obtain the main result of the chapter.

In [29], the studied problems are the related to the so-called $(k, n-k)$ boundary conditions which correspond to $\{\sigma_1,\dots,\sigma_k\} = \{0,\dots,k-1\}$ and $\{\varepsilon_1,\dots,\varepsilon_{n-k}\} = \{0,\dots,n-k-1\}$.

By using Theorems 1.1.4 and 1.1.6, under the hypothesis that equation (1.0.3) is disconjugate on $[a,b]$ for $M=\bar{M}$, we prove the existence of a decomposition as follows:

$$T_0\,u(t)=u(t),\quad T_k\,u(t)=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{T_{k-1}\,u(t)}{v_k(t)}\right),\ k=1,\dots,n,$$

where $v_k>0$, $v_k\in C^n(I)$ are such that:

$$T_n[\bar{M}]\,u(t)=v_1(t)\,\dots\,v_n(t)\,T_n\,u(t),\ t\in I,$$

and, moreover, this decomposition satisfies, for every $u\in X_{\{0,\dots,k-1\}}^{\{0,\dots,n-k-1\}}$:

$$\begin{aligned} T_0\,u(a)&=\dots=T_{k-1}\,u(a)=0,\\ T_0\,u(b)&=\dots=T_{n-k-1}\,u(b)=0.\end{aligned}$$

In [32], we study a fourth order problem coupled with the simply supported beam boundary conditions, that is, $\{\sigma_1,\sigma_2\}=\{\varepsilon_1,\varepsilon_2\}=\{0,2\}$. We also obtain a decomposition as follows:

$$T_0\,u(t)=u(t),\quad T_k\,u(t)=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{T_{k-1}\,u(t)}{v_k(t)}\right),\ k=1,\dots,4,$$

where $v_k>0$, $v_k\in C^n(I)$ are such that:

$$T_4[\bar{M}]\,u(t)=v_1(t)\,\dots\,v_4(t)\,T_4\,u(t),\ t\in I,$$

and, moreover, this decomposition satisfies, for every $u\in X_{\{0,2\}}^{\{0,2\}}$:

$$\begin{aligned} T_0\,u(a)&=T_2\,u(a)=0,\\ T_0\,u(b)&=T_2\,u(b)=0.\end{aligned}$$

Furthermore, the simplest n^{th}-order operator which we can study is $T_n^0[0]\,u(t)=u^{(n)}(t)$. It is obvious that such an operator satisfies:

$$T_0\,u(t)=u(t),\quad T_k\,u(t)=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{T_{k-1}\,u(t)}{v_k(t)}\right),\ k=1,\dots,n,$$

where $v_k\equiv 1$ on I and,

$$T_n^0[0]\,u(t)=v_1(t)\,\dots\,v_n(t)\,T_n\,u(t),\ t\in I,$$

and, moreover, this decomposition satisfies, for every $u\in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$\begin{aligned} T_{\sigma_1}\,u(a)&=u^{(\sigma_1)}(a)=0,\dots,T_{\sigma_k}\,u(a)=u^{(\sigma_k)}(a)=0,\\ T_{\varepsilon_1}\,u(b)&=u^{(\varepsilon_1)}(b)=0,\dots,T_{\varepsilon_{n-k}}\,u(b)=u^{(\varepsilon_{n-k})}(b)=0.\end{aligned}$$

Thus, it is natural to impose that the operator $T_n[\bar{M}]$ satisfies the following property.

Definition 3.3.1. *Let us say that the operator $T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, there exists the following decomposition:*

$$T_0\, u(t) = u(t), \quad T_k\, u(t) = \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{T_{k-1}\, u(t)}{v_k(t)}\right), \; k = 1, \dots, n, \tag{3.3.1}$$

where $v_k > 0$, $v_k \in C^n(I)$ are such that

$$T_n[\bar{M}]\, u(t) = v_1(t)\, \dots\, v_n(t)\, T_n\, u(t), \; t \in I, \tag{3.3.2}$$

and, moreover, this decomposition satisfies, for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$\begin{aligned} T_{\sigma_1}\, u(a) &= \cdots = T_{\sigma_k}\, u(a) = 0, \\ T_{\varepsilon_1}\, u(b) &= \cdots = T_{\varepsilon_{n-k}}\, u(b) = 0. \end{aligned}$$

As we have shown above, the operator $T_n^0[M]\, u(t) \equiv u^{(n)}(t) + M\, u(t)$ satisfies property (T_d) for $M = 0$. Indeed, the existence of such a decomposition for $M = \bar{M}$ allows us to express the operator $T_n[\bar{M}]$ as a composition of operators of first order satisfying the boundary conditions given on (3.0.2)-(3.0.3). That is, in order to study the oscillation, we can think on the operator as it was $T_n[\bar{M}] \equiv \dfrac{\mathrm{d}^n}{\mathrm{d}\, t^n}$.

Remark 3.3.2. *Realise that due to Theorems 1.1.4 and 1.1.6, the disconjugacy of the linear differential equation* (1.0.3) *for $M = \bar{M}$ on I is a necessary condition for the operator $T_n[\bar{M}]$ to satisfy property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Remark 3.3.3. *Realise that there may exist different decompositions* (3.3.1) *depending on the choice of v_k for $k = 1, \dots, n$.*

Moreover, even if we are not able to obtain such a decomposition, we cannot ensure that it does not exist, unless we prove that the linear differential equation (1.0.3) *is not disconjugate.*

Open Problem 2. *It will be very useful to have a property, such as the disconjugacy for the $(k, n-k)$ boundary conditions, which ensure that an operator fulfils or not property (T_d). However, by the moment, we have not found such a property. As we have mentioned, disconjugacy is a necessary but not sufficient condition to make sure the property (T_d) in more general boundary conditions.*

In the sequel, we describe the different operators given by the decomposition (3.3.1) for $\ell = 0, \dots, n$. Let us see that:

$$T_\ell u(t) = \frac{1}{v_1(t)\dots v_\ell(t)}\, u^{(\ell)}(t) + p_{\ell_1}(t)\, u^{(\ell-1)}(t) + \cdots + p_{\ell_\ell}(t)\, u(t), \tag{3.3.3}$$

where $p_{\ell_i} \in C^{n-\ell}(I)$, for every $i = 1, \dots, \ell$, and $\ell = 0, \dots, n$ are defined as follows:

$$p_{\ell_1}(t) = a_1^1\big(v_1(t), \dots, v_\ell(t)\big)\, v_1'(t) + \cdots + a_1^\ell\big(v_1(t), \dots, v_\ell(t)\big)\, v_\ell'(t), \tag{3.3.4}$$

$$\begin{aligned} p_{\ell_2}(t) =& a_2^1\big(v_1(t),\ldots,v_\ell(t)\big)\, v_1''(t) + \cdots + a_2^{\ell-1}\big(v_1(t),\ldots,v_\ell(t)\big)\, v_{\ell-1}''(t) \\ &+ f_2\big(v_1(t),\ldots,v_\ell(t),v_1'(t),\ldots,v_\ell'(t)\big), \end{aligned} \tag{3.3.5}$$

$$\begin{aligned} p_{\ell_3}(t) =& a_3^1\big(v_1(t),\ldots,v_\ell(t)\big)\, v_1'''(t) + \cdots + a_3^{\ell-2}\big(v_1(t),\ldots,v_\ell(t)\big)\, v_{\ell-2}'''(t) \\ &+ f_3\big(v_1(t),\ldots,v_\ell(t),v_1'(t),\ldots,v_\ell'(t),v_1''(t),\ldots,v_{\ell-1}''(t)\big), \end{aligned} \tag{3.3.6}$$

$$\vdots$$

$$\begin{aligned} p_{\ell_\ell}(t) =& a_\ell^1\big(v_1(t),\ldots,v_\ell(t)\big)\, v_1^{(\ell)}(t) + \\ &+ f_\ell\big(v_1(t),\ldots,v_\ell(t),v_1'(t),\ldots,v_\ell'(t),\ldots,v_1^{(\ell-1)}(t),v_2^{(\ell-1)}(t)\big), \end{aligned} \tag{3.3.7}$$

where $a_i^j \in C^\infty\left((0,+\infty)^\ell\right)$, $f_i \in C^\infty\left((0,\infty)^\ell \times \mathbb{R}^{(i-1)\frac{2\ell-i+2}{2}}\right)$ for every $\ell = 0,\ldots,n$, $i = 1,\ldots,\ell$ and $j = 1,\ldots,\ell-i+1$. Let us prove that equations (3.3.3)–(3.3.7) are fulfilled.

We can see that for $\ell = 1$ the result is true:

$$T_1\, u(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{u(t)}{v_1(t)}\right) = \frac{u'(t)}{v_1(t)} - \frac{v_1'(t)}{v_1^2(t)}\, u(t), \tag{3.3.8}$$

hence $a_1^1(x) = -\dfrac{1}{x^2}$.

Suppose, by induction hypothesis that the result is true for a given $\ell \geqslant 1$. Then, let us see what happens for $\ell + 1$.

$$T_{\ell+1}u(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{v_1(t)\ldots v_{\ell+1}(t)}\, u^{(\ell)}(t) + \frac{p_{\ell_1}(t)}{v_{\ell+1}(t)}\, u^{(\ell-1)}(t) + \cdots + \frac{p_{\ell_\ell}(t)}{v_{\ell+1}(t)}\, u(t)\right),$$

or, which is the same,

$$T_{\ell+1}u(t) = \frac{1}{v_1(t)\ldots v_{\ell+1}(t)}\, u^{(\ell+1)}(t) + p_{\ell+1_1}(t)\, u^{(\ell)}(t) + \cdots + p_{\ell+1_{\ell+1}}(t)\, u(t),$$

where

$$\begin{aligned} p_{\ell+1_1}(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{v_1(t)\ldots v_{\ell+1}(t)}\right) + \frac{p_{\ell_1}(t)}{v_{\ell+1}(t)}, \\ p_{\ell+1_j}(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{p_{\ell_{j-1}}(t)}{v_{\ell+1}(t)}\right) + \frac{p_{\ell_j}(t)}{v_{\ell+1}(t)}, \quad 2 \leqslant j \leqslant \ell, \\ p_{\ell+1_{\ell+1}}(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{p_{\ell_\ell}(t)}{v_{\ell+1}(t)}\right), \end{aligned}$$

which clearly satisfy (3.3.4)–(3.3.7) for $\ell + 1$. Thus, (3.3.3) is proved.

Example 3.3.4. *Now, let us show, as an example, the expression of* $T_2 u(t)$*:*

$$\begin{aligned} T_2\, u(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{T_1 u(t)}{v_2(t)}\right) =& \frac{u''(t)}{v_1(t)v_2(t)} - u'(t)\frac{2v_2(t)v_1'(t) + v_1(t)v_2'(t)}{v_1^2(t)v_2^2(t)} \\ &+ u(t)\frac{v_1(t)v_1'(t)v_2'(t) + v_2(t)\left(2{v_1'}^2(t) - v_1(t)v_1''(t)\right)}{v_1^3(t)v_2^2(t)}. \end{aligned} \tag{3.3.9}$$

Thus, in this case, $a_1^1(x,y) = -\dfrac{2}{x^2\,y}$, $a_1^2(x,y) = -\dfrac{1}{x\,y^2}$, $a_2^1(x,y) = -\dfrac{1}{x^2\,y}$ *and* $f(x,y,z,t) = \dfrac{x\,z\,t + 2\,y\,z^2}{x^3\,y^2}$.

Remark 3.3.5. *Realise that, from the arbitrariness of the choice of* $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$*, if the operator* $T_n[\bar{M}]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$*, then for each* $\ell \in \{\sigma_1,\dots,\sigma_k\}$ *we have:*

$$T_\ell u(a) = \frac{1}{v_1(a)\dots v_\ell(a)}\,u^{(\ell)}(a) + p_{\ell_1}(a)\,u^{(\ell-1)}(a) + \cdots p_{\ell_\ell}(a)\,u(a) = 0,$$

this equality implies that $p_{\ell_h}(a) = 0$ *for each* $h \in \{1,\dots,\ell\}$ *such that* $\ell - h \notin \{\sigma_1,\dots,\sigma_k\}$.

Analogously, for each $\ell \in \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$*, it is satisfied:*

$$T_\ell u(b) = \frac{1}{v_1(b)\dots v_\ell(b)}\,u^{(\ell)}(b) + p_{\ell_1}(b)\,u^{(\ell-1)}(b) + \cdots p_{\ell_\ell}(b)\,u(b) = 0,$$

which implies that $p_{\ell_h}(b) = 0$ *for each* $h \in \{1,\dots,\ell\}$ *such that* $\ell - h \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$.

Now, we deduce two results which are a straight consequence of property (T_d) and Remark 3.3.5.

Lemma 3.3.6. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$*. If* $u \in C^n\big([a,c)\big)$*, where* $c > a$*, is a function that satisfies* $u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{\ell-1})}(a) = 0$*, for* $\ell = 1,\dots,k$*, then:*

$$T_{\sigma_1}\,u(a) = \cdots = T_{\sigma_{\ell-1}}\,u(a) = 0,$$

and,

$$T_{\sigma_\ell}\,u(a) = f(a)\,u^{(\sigma_\ell)}(a),$$

where:

$$f(t) = \frac{1}{v_1(t)\;\dots\;v_{\sigma_\ell}(t)} > 0,\ t \in I.$$

In particular, $u^{(\sigma_\ell)}(a) = 0$ *if, and only if,* $T_{\sigma_\ell}u(a) = 0$.

Proof. The result is a direct consequence of expression (3.3.3) and Remark 3.3.5. □

We have an analogous result for $t = b$.

Lemma 3.3.7. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$*. If* $u \in C^n\big((c,b]\big)$*, where* $c < b$*, is a function that satisfies* $u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{\ell-1})}(b) = 0$*, for* $\ell \in \{1,\dots,n-k\}$*, then:*

$$T_{\varepsilon_1}\,u(b) = \cdots = T_{\varepsilon_{\ell-1}}\,u(b) = 0,$$

and,

$$T_{\varepsilon_\ell}\,u(b) = g(b)\,u^{(\varepsilon_\ell)}(b),$$

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where:

$$g(t) = \frac{1}{v_1(t)\dots v_{\varepsilon_\ell}(t)} > 0,\ t \in I.$$

In particular, if $u^{(\varepsilon_\ell)}(b) = 0$ *if, and only if,* $T_{\varepsilon_\ell} u(b) = 0$.

Proof. The proof follows, as in Lemma 3.3.6, from (3.3.3) and Remark 3.3.5. □

In order to ensure that Green's function is well-defined for $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, a new hypothesis is required.

Definition 3.3.8. *Let us say that* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy property* (N_a) *if, and only if,*

$$\operatorname{card}\Big\{\ell \in \{\sigma_1,\dots,\sigma_k\} \mid \ell < h\Big\} + \operatorname{card}\Big\{\ell \in \{\varepsilon_1,\dots,\varepsilon_{n-k}\} \mid \ell < h\Big\} \geqslant h, \tag{3.3.10}$$

for all $h \in \{1,\dots,n-1\}$.

Realise that, in the second order case, the Neumman boundary conditions do not satisfy property (N_a). However, Dirichlet and Mixed conditions are included.

Now, we show a result which ensures the existence of a unique Green's function for $T_n[\bar{M}]$, provided that property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ is fulfilled by $T_n[\bar{M}]$ and the set of indices $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies (N_a).

Lemma 3.3.9. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *Then* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a) *if, and only if,* $\lambda = 0$ *is not an eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. In order to prove the sufficient condition, let us consider $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, such that

$$T_n[\bar{M}]\,u(t) = 0,\ t \in I.$$

We will see that necessarily $u \equiv 0$ in I.

Since the operator $T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we can use the decomposition given in (3.3.1). So, we have:

$$0 = T_n[\bar{M}]\,u(t) = v_1(t)\ \dots\ v_n(t)\,T_n\,u(t),\ t \in I,$$

which, since $v_1,\dots,v_n > 0$ on I, implies that:

$$T_n\,u(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{T_{n-1}\,u(t)}{v_n(t)}\right) = 0,\ t \in I,$$

hence $\dfrac{T_{n-1}\,u}{v_n}$ is a constant function on I. So, since $v_n > 0$ on I, $T_{n-1}\,u$ is of constant sign on I.

Thus, $\dfrac{T_{n-2}\,u}{v_{n-2}}$ is a monotone function, with at most one zero on I. As before, since $v_{n-2} > 0$ on I, we can conclude that $T_{n-2}\,u$ can have at most one zero on I.

Proceeding analogously, we conclude that u can have at most $n-1$ zeros on I.

If $T_\ell u \not\equiv 0$ for all $\ell = 1,\dots,n-1$, then each time that either $T_\ell u(a) = 0$ or $T_\ell u(b) = 0$, a possible oscillation is lost. Indeed, if the maximum number of zeros for $T_\ell\, u$ on I is h and one of them is found at $t = a$ or $t = b$, then $T_\ell\, u$ can have at most $h-1$ sign changes on I ($h-2$ if both $T_\ell\, u(a) = T_\ell u(b) = 0$).

Since $T_n[\bar M]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, $T_\ell\, u(a) = 0$ or $T_\ell\, u(b) = 0$, at least n times.

If $T_\ell\, u(t) \not\equiv 0$ for all $\ell = 1,\dots,n-1$, from property (T_d), n possible sign changes are lost. However, we have already seen, without taking into account he boundary conditions, that the maximum number of zeros for u without being a trivial function is $n-1$. This implies that, necessarily, $u \equiv 0$.

If there exists some $\ell \in \{1,\dots,n-1\}$, such that $T_\ell\, u \equiv 0$ on I, let us choose the least ℓ that satisfies this property. With the same arguments as before, we can conclude that with maximal oscillation u can have at most $\ell - 1$ zeros.

Using the fact that $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a), we have $T_h\, u(a) = 0$ or $T_h\, u(b) = 0$ at least ℓ times from $h = 0$ to ℓ. Therefore, we lose ℓ possible sign changes. Thence, arguing as before we conclude that $u \equiv 0$ on I. And, as a consequence, $\lambda = 0$ is not an eigenvalue of $T_n[\bar M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Reciprocally, to prove the necessary condition, assume $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ do not satisfy (N_a). Then, there exists $h_0 \in \{1,\dots,n-1\}$ such that:

$$\operatorname{card}\Big\{\ell \in \{\sigma_1,\dots,\sigma_k\} \mid \ell < h_0\Big\} + \operatorname{card}\Big\{\ell \in \{\varepsilon_1,\dots,\varepsilon_{n-k}\} \mid \ell < h_0\Big\} \geqslant h_0.$$

As a consequence, there always exists a non-trivial function satisfying the boundary conditions (3.0.2)-(3.0.3) for $\sigma_\ell < h_0$ and $\varepsilon_\ell < h_0$ such that:

$$T_{h_0} u(t) = 0, \quad t \in I.$$

Trivially, $T_{\sigma_\ell}\, u(a) = 0$ and $T_{\varepsilon_\ell}\, u(b) = 0$ for all $\sigma_\ell > h_0$ and $\varepsilon_\ell > h_0$. Thus, by applying Lemmas 3.3.6 and 3.3.7 inductively, we conclude that $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

As a consequence, it is obvious that $\lambda = 0$ is an eigenvalue of $T_n[\bar M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. $\square$

Now, let us see that, under the considered hypotheses, the Green's function related to operator $T_n[\bar M]$ satisfies a suitable property.

Theorem 3.3.10. *Let $\bar M \in \mathbb{R}$ be such that $T_n[\bar M]$ satisfies condition (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy condition (N_a). Then, the following assertions are fulfilled.*

- *If $n-k$ is even, then $T_n[\bar M]$ is strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, the related Green's function, $g_{\bar M}(t,s)$, satisfies (P_g).*
- *If $n-k$ is odd, then $T_n[\bar M]$ is strongly inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, the related Green's function, $g_{\bar M}(t,s)$, satisfies (N_g).*

Proof. First, let us verify the strongly inverse positive (negative) character.

To this end, we use the decomposition of $T_n[\bar{M}]$ given on (3.3.1).

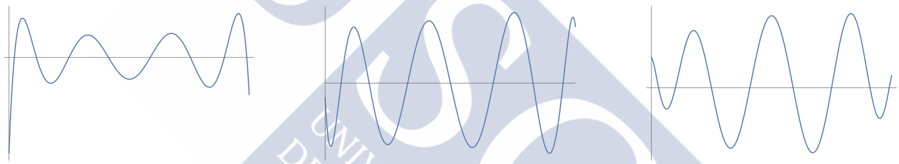
Since $v_1(t)\,\dots\,v_n(t) > 0$; if $T_n[\bar{M}]\,u \gneqq 0$ on I, from (3.3.2), we conclude that $T_n u \gneqq 0$ on I.

Hence, from (3.3.1) we know that $\dfrac{T_{n-1}\,u}{v_n}$ is a non-trivial non-decreasing function, with at most a sign change on I. Therefore, since $v_n > 0$, we can affirm that $T_{n-1}u$ can have at most a sign change, being negative at $t = a$ and positive at $t = b$.

Repeating this process for $T_{n-\ell}u$, with $\ell = 1, \dots n$, we can affirm that $T_0 u = u$ can have at most n zeros on (a, b), whenever the following inequalities are satisfied for every $\ell = 1, \dots, n$:

$$\begin{cases} T_{n-\ell}\,u(a) > 0, & \text{if } \ell \text{ is even,} \\ T_{n-\ell}\,u(a) < 0, & \text{if } \ell \text{ is odd,} \end{cases} \qquad \text{and} \qquad T_{n-\ell}\,u(b) > 0. \tag{3.3.11}$$

This behaviour is illustrated in Figure 3.3.1.



(a) Oscillation of $T_{n-\ell}u(t)$ **(b)** Oscillation of $T_{n-(\ell+1)}u(t)$ **(c)** Oscillation of $T_{n-(\ell+2)}u(t)$

Figure 3.3.1: These three graphics illustrate the different behaviours of $T_{n-\ell}$ at $t = a$ and $t = b$. In **(a)** and **(b)**, we observe that $T_{n-\ell}u(a)\,T_{n-(\ell+1)}u(a) > 0$ and $T_{n-\ell}u(b)\,T_{n-(\ell+1)}u(b) < 0$. Taking into account that $T_{n-\ell}u(a) < 0$, then $T_{n-(\ell+1)}u$ is decreasing in a neighbourhood of $t = a$. This fact, coupled with $T_{n-(\ell+1)}u(a) < 0$ inhibits the existence of a possible sign change of $T_{n-(\ell+1)}u$ on a neighbourhood of $t = a$. Analogously, the fact that $T_{n-\ell}u(b) < 0$ implies that $T_{n-(\ell+1)}u$ is decreasing in a neighbourhood of $t = b$. But, taking into account that $T_{n-(\ell+1)}u(b) > 0$, another possible sign change is lost. Thus, since $T_{n-\ell}u$ has 8 zeros, with maximal oscillation $T_{n-(\ell+1)}u$ could have 9. However, from the boundary conditions, just 7 oscillations are allowed. On the other hand, from **(b)** to **(c)**, we can see that $T_{n-(\ell+1)}u(a)\,T_{n-(\ell+2)}u(a) < 0$ and $T_{n-(\ell+1)}u(b)\,T_{n-(\ell+2)}u(b) > 0$. Thus, in this case, the maximal oscillation is allowed and $T_{n-(\ell+1)}u$ achieves 8 sign changes.

Repeating the same argument as in Lemma 3.3.9, we can affirm that each time that $T_{n-\ell}\,u(a) = 0$ or $T_{n-\ell}\,u(b) = 0$, we lose a possible oscillation and, therefore, a possible zero of u in (a, b).

From property (T_d), we know that for all $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, it is satisfied that:

$$T_{\sigma_1}\,u(a) = \cdots = T_{\sigma_k}\,u(a) = T_{\varepsilon_1}\,u(b) = \cdots = T_{\varepsilon_{n-k}}\,u(b) = 0, \tag{3.3.12}$$

i.e, we lose the n possible zeros which u could ever have. Thus, we can conclude that u cannot have any zero on (a, b).

Let us see the sign of $u^{(\alpha)}(a)$ and $u^{(\beta)}(b)$ which gives the sign of u.

Realise that, since $u(a) = \cdots = u^{(\alpha-1)}(a) = 0$ and $u(b) = \cdots = u^{(\beta-1)}(b) = 0$, from (3.3.3) we have

$$T_\alpha u(a) = \frac{u^{(\alpha)}(a)}{v_1(a)\dots v_\alpha(a)}, \quad T_\beta u(b) = \frac{u^{(\beta)}(b)}{v_1(b)\dots v_\beta(b)}, \tag{3.3.13}$$

hence, $u^{(\alpha)}(a)$ and $T_\alpha u(a)$, and $u^{(\beta)}(b)$ and $T_\beta u(b)$ have the same sign, respectively.

If either, $T_\ell u(a) = 0$ for any $\ell \notin \{\sigma_1, \dots, \sigma_k\}$, or $T_\ell u(t) = 0$ for any $\ell \notin \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$, then we lose another possible zero and, necessarily, $u \equiv 0$ on I which is a contradiction with $T_n[\bar{M}]\, u \gneqq 0$.

Moreover, taking into account (3.3.12), the sign of $T_\ell u(a)$ must allow the maximum number of oscillations for $T_\ell u$. Otherwise $u \equiv 0$ on I which is again a contradiction with $T_n[\bar{M}]\, u \gneqq 0$.

Notation 3.3.11. *Along this work, we understand for conditions of maximal oscillation those which allow u to have the maximum number of zeros depending on the fixed boundary conditions without being a trivial solution.*

Hence $T_{n-\ell}$ must satisfy the conditions for maximal oscillation. That is, $T_{n-\ell} u(a)$ must change its sign each time that it is not null, i.e., if $T_{n-\ell} u(a) > 0$ for a given $\ell = 1, \dots, n$, then $T_{n-\ell-1} u(a) \leqslant 0$ and if $T_{n-\ell-1} u(a) = 0$, we consider $\tilde{\ell} \in \{\ell+1, \dots, n\}$ such that $T_{n-\tilde{\ell}} u(a) \neq 0$ and $T_{n-h}\, u(a) = 0$ for $h \in \{\ell+1, \dots, \tilde{\ell}-1\}$, then $T_{n-\tilde{\ell}} u(a) < 0$. On the contrary, $T_{n-\ell} u(b)\, T_{n-(\ell+1)} u(b) \geqslant 0$. And, if $T_{n-\ell} u(b) \neq 0$ and $T_{n-(\ell+1)} u(b) = 0$, then $(-1)^{h-1} T_{n-\ell} u(b)\, T_{n-(\ell+h)} u(b) > 0$ if $T_{n-(\ell+i)} u(b) = 0$ for all $i = 1, \dots, h-1$ and $T_{n-(\ell+h)} u(b) \neq 0$, see Figure 3.3.2.

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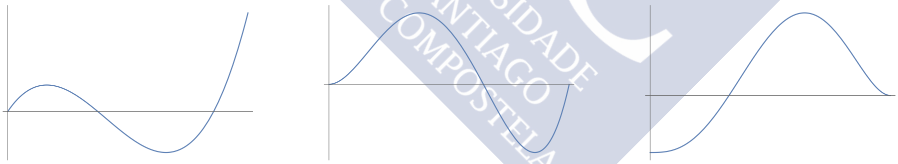
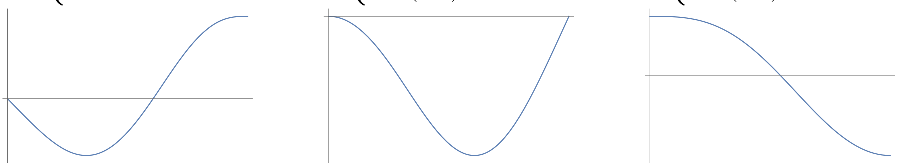


(a) $\begin{cases} T_{n-\ell} u(a) = 0, \\ T_{n-\ell} u(b) > 0. \end{cases}$ **(b)** $\begin{cases} T_{n-(\ell+1)} u(a) = 0, \\ T_{n-(\ell+1)} u(b) = 0. \end{cases}$ **(c)** $\begin{cases} T_{n-(\ell+2)} u(a) < 0, \\ T_{n-(\ell+2)} u(b) = 0. \end{cases}$



(d) $\begin{cases} T_{n-(\ell+3)} u(a) = 0, \\ T_{n-(\ell+3)} u(b) > 0. \end{cases}$ **(e)** $\begin{cases} T_{n-(\ell+4)} u(a) = 0, \\ T_{n-(\ell+4)} u(b) = 0. \end{cases}$ **(f)** $\begin{cases} T_{n-(\ell+5)} u(a) > 0, \\ T_{n-(\ell+5)} u(b) < 0. \end{cases}$

Figure 3.3.2: Possible behaviour of $T_h\, u$ for $h = n-\ell, \dots, n-(\ell-1)$ with maximal oscillation taking into account the boundary conditions

From property (T_d), we know that $T_{n-\ell} u(a)$ vanishes $k-\alpha$ times for $\ell \in \{1, \dots, n-\alpha\}$. Hence, taking into account the previous argument and the conditions given in (3.3.11), we have:

$$\begin{cases} T_\alpha\, u(a) > 0, & \text{if } n-\alpha-(k-\alpha) = n-k \text{ is even,} \\ \\ T_\alpha\, u(a) < 0, & \text{if } n-k \text{ is odd.} \end{cases}$$

Realise that, to obtain the previous inequalities, there are considered as many sign changes for $T_h u(a)$ as times that it is non-null from $h=\alpha$ to $h=n-1$. That is, the $n-\alpha$ steps minus the $k-\alpha$ zeros that are found. Thus, from (3.3.13):

$$\begin{cases} u^{(\alpha)}(a) > 0, & \text{if } n-k \text{ is even,} \\ \\ u^{(\alpha)}(a) < 0, & \text{if } n-k \text{ is odd.} \end{cases} \tag{3.3.14}$$

From this, since $u \neq 0$ on (a,b), we already conclude that:

$$\begin{cases} u(t) > 0, & \forall\, t \in (a,b), \quad \text{if } n-k \text{ is even,} \\ \\ u(t) < 0, & \forall\, t \in (a,b), \quad \text{if } n-k \text{ is odd.} \end{cases} \tag{3.3.15}$$

Taking into account that, since $\beta \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$, necessarily $T_\beta u(b) \neq 0$ from (3.3.13) and (3.3.15) we have:

- If $n-k$ is even:

$$\begin{cases} u^{(\beta)}(b) > 0, & \text{if } \beta \text{ is even,} \\ \\ u^{(\beta)}(b) < 0, & \text{if } \beta \text{ is odd.} \end{cases} \tag{3.3.16}$$

- If $n-k$ is odd:

$$\begin{cases} u^{(\beta)}(b) < 0, & \text{if } \beta \text{ is even,} \\ \\ u^{(\beta)}(b) > 0, & \text{if } \beta \text{ is odd.} \end{cases} \tag{3.3.17}$$

Hence, from (3.3.14)–(3.3.17), we conclude that if $n-k$ is even, then the operator $T_n[\bar{M}]$ is a strongly inverse positive operator in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and if $n-k$ is odd, then the operator $T_n[\bar{M}]$ is a strongly inverse negative operator in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Let us see that $g_{\bar{M}}(t,s)$ satisfies condition (P_g) or (N_g), respectively.

Using Theorems 3.2.5 and 3.2.6, we know that $(-1)^{n-k}\, g_{\bar{M}}(t,s) > 0$ for a.e. (t,s) in $(a,b)\times(a,b)$ $\big($indeed, we know that for all $t \in (a,b)$ $(-1)^{n-k}\, g_{\bar{M}}(t,s) > 0$ for a.e. $s \in (a,b)\big)$. Let us see that, in fact, this inequality holds for all $(t,s) \in (a,b)\times(a,b)$.

For each fixed $s \in (a,b)$, let us define $u_s(t) = (-1)^{n-k}\, g_{\bar{M}}(t,s)$. From de definition of the Green's function, we have that $u_s \in C^{n-2}(I)$ and $u_s \in C^n\big([a,s)\cup(s,b]\big)$.

We know that $u_s(t) \geqslant 0$ on I, and it satisfies the boundary conditions (3.0.2)-(3.0.3).

Moreover, since $g_{\bar{M}}(t,s)$ is the Green's function related to $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we have:

$$T_n[\bar{M}]\, u_s(t) = v_1(t)\dots v_n(t)\, T_n\, u_s(t) = 0, \quad t \neq s.$$

Since $v_1 \cdots v_n > 0$ on I, we have that $T_n\, u_s(t) = 0$ if $t \neq s$. Hence,

$$\begin{cases} \dfrac{1}{v_n(t)}\, T_{n-1} u_s(t) = c_1, & t < s, \\ \\ \dfrac{1}{v_n(t)}\, T_{n-1} u_s(t) = c_2, & t > s, \end{cases} \tag{3.3.18}$$

where, to allow the maximal oscillation, c_1, $c_2 \in \mathbb{R}$ are of different sign.

Since $v_n > 0$, $T_{n-1}\, u_s$ has the same sign as c_1 or c_2, if $t < s$ or $t > s$, respectively, i.e., in order to have maximal number of oscillations, it has two components of constant different sign.

Then, since $\dfrac{1}{v_{n-1}}\, T_{n-2}\, u_s$ is a continuous function, it can have at most two sign changes and the same happens with $T_{n-2}\, u_s$.

Proceeding in a similar way, we conclude that, with maximal oscillation, $T_{n-\ell}\, u_s$ can have at most ℓ zeros, for $\ell = 2, \dots, n$. In particular, u_s has at most n sign changes on I.

Arguing as before, each time that $T_{n-\ell}\, u_s(a) = 0$ or $T_{n-\ell}\, u_s(b) = 0$ a possible oscillation is lost.

Taking into account that $T_n[\bar{M}]$ satisfies (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we use Lemmas 3.3.6 and 3.3.7 to affirm that u_s satisfies (3.3.12). Thus, $T_{n-\ell}\, u_s(a)$ or $T_{n-\ell}\, u_s(b)$ vanish n times for $\ell = 1, \dots, n$. So, we have lost the n possibles zeros and we can affirm that $u_s > 0$ on (a,b). Or, which is the same, $(-1)^{n-k}\, g_{\bar{M}}(t,s) > 0$ for all $(t,s) \in (a,b) \times (a,b)$.

Moreover, for each $s \in (a,b)$, we obtain the following limits:

$$\ell_1(s) = \lim_{t \to a^+} \frac{(-1)^{n-k}\, g_{\bar{M}}(t,s)}{(t-a)^\alpha\, (b-t)^\beta} = \frac{(-1)^{n-k}\, \dfrac{\partial^\alpha}{\partial t^\alpha}\, g_{\bar{M}}(t,s)_{|t=a}}{\alpha!\, (b-a)^\beta} < +\infty,$$

$$\ell_2(s) = \lim_{t \to b^-} \frac{(-1)^{n-k}\, g_{\bar{M}}(t,s)}{(t-a)^\alpha\, (b-t)^\beta} = \frac{(-1)^{n-k-\beta}\, \dfrac{\partial^\beta}{\partial t^\beta}\, g_{\bar{M}}(t,s)_{|t=a}}{\beta!\, (b-a)^\alpha} < +\infty.$$

For each $s \in (a,b)$, let us construct the continuous extension on I of the following function:

$$\tilde{u}_s(t) = \frac{(-1)^{n-k}\, g_{\bar{M}}(t,s)}{(t-a)^\alpha\, (b-t)^\beta}.$$

Since $u_s > 0$ and $(t-a)^\alpha\, (b-t)^\beta > 0$ on (a,b), we have that $\tilde{u}_s > 0$ on (a,b).

Moreover, using Theorems 3.2.5 and 3.2.6, we can affirm that $\ell_1(s) > 0$ and $\ell_2(s) > 0$ for a.e. $s \in (a,b)$. Hence, for a.e. $s \in (a,b)$, $\tilde{u}_s(a) > 0$ and $\tilde{u}_s(b) > 0$.

Furthermore, since $g_{\bar{M}}(t,s)$ is the related Green's function of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we can also affirm that there exists $K > 0$ such that $\tilde{u}_s \leqslant K$ for every $(t,s) \in I \times (a,b)$. Hence, we construct the following functions:

$$\tilde{k}_1(s) = \min_{t \in I} \tilde{u}_s(t), \quad s \in (a,b),$$
$$\tilde{k}_2(s) = \max_{t \in I} \tilde{u}_s(t), \quad s \in (a,b),$$

which are continuous on (a,b) and they are positive a.e. in (a,b).

Taking $\phi(t) = (t-a)^\alpha (b-t)^\beta > 0$ on (a,b), condition (P_g) is trivially satisfied if $n-k$ is even with $k_1(s) = \tilde{k}_1(s)$ and $k_2(s) = \tilde{k}_2(s)$ and condition (N_g) if $n-k$ is odd with $k_1(s) = -\tilde{k}_1(s)$ and $k_2(s) = -\tilde{k}_2(s)$. $\square$

Remark 3.3.12. *Realise that, from Theorem 3.3.10, if $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then Theorem 1.2.10, if $n-k$ is even, or Theorem 1.2.11, if $n-k$ is odd, can be applied to operator $T_n[\bar{M}]$ on such a space to obtain one of the extremes of the intervals of constant sign Green's function.*

Along this chapter, we illustrate the different results with some particular examples. We follow a fourth order example coupled with the choice of boundary conditions given by $\{\sigma_1,\sigma_2\} = \{0,2\}$ and $\{\varepsilon_1,\varepsilon_2\} = \{1,2\}$.

Example 3.3.13. *We have already noted in the beginning of the section that the operator $T_n[M]u(t) = u^{(n)} + M\,u(t)$ satisfies property (T_d) for $M=0$ for every boundary conditions, in particular in $X_{\{0,2\}}^{\{1,2\}}$. Moreover, $\{0,2\} - \{1,2\}$ satisfy (N_a).*

Now, let us see directly that this operator fulfils the theses of Theorem 3.3.10. To do that, let us consider $I \equiv [0,1]$.

In this case, $n-k=2$ is even, so let us study the strongly inverse positive character. If $u^{(4)} \gneqq 0$, then u'' is a convex function. Since $u''(0) = u''(1) = 0$, we have that $u'' \lneqq 0$ (if $u'' \equiv 0$, then $u^{(4)} \equiv 0$ which is a contradiction).

Hence, u' is a decreasing function on I satisfying $u'(1) = 0$, so $u' \gneqq 0$. In particular, $u'(0) > 0$.

Finally, taking into account that $u(0) = 0$, u is an increasing function on I and it cannot have a non-numerable number of zeros, we have that $u(t) > 0$ for all $t \in (0,1]$.

Now, let us study the related Green's function, given by the expression:

$$g_0(t,s) = \begin{cases} \dfrac{1}{6}s\Big(t\left(t^2-3t+3\right)-s^2\Big), & 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{1}{6}(s-1)t\left(t^2-3s\right), & 0 < t < s \leqslant 1. \end{cases}$$

Let us see that it satisfies the condition (P_g).

Directly, we have $g_0(1,s) = \dfrac{1}{6}\,s\,\left(2-s^2\right) > 0$ for all $s \in (0,1)$.

Moreover,

$$\frac{\partial}{\partial t}g_0(t,s) = \begin{cases} \dfrac{1}{6}s\left(1-t^2\right), & 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{1}{2}\left(s-s^2-t^t+s\,t^2\right), & 0 < t < s \leqslant 1, \end{cases}$$

in particular, $\dfrac{\partial}{\partial t}g_0(t,s)_{|t=0} = \dfrac{1}{2}\left(s-s^2\right) > 0$ for all $s \in (0,1)$.

Now, let us verify that $g_0(t,s) > 0$ on $(0,1)\times(0,1)$.

If $t < s$, we have that $s - 1 < 0$ and $t^2 - 3s < s^2 - 3s < 0$ for all $s \in (0,1)$.

If $t \geqslant s$, we have $t\left(t^2 - 3t + 3\right) - s^2 \geqslant s\left(s^2 - 3s + 3\right) - s^2 = 3s - 4s^2 + s^3 > 0$ for all $s \in (0,1)$.

Hence, $g_0(t,s) > 0$ on $(0,1) \times (0,1)$.

We have,

$$\tilde{u}_s(t) = \frac{g_0(t,s)}{t} = \begin{cases} \dfrac{1}{6}\dfrac{s}{t}\Big(t\left(t^2 - 3t + 3\right) - s^2\Big), & 0 \leqslant s \leqslant t \leqslant 1, \\ \\ \dfrac{1}{6}(s-1)\left(t^2 - 3s\right), & 0 < t < s \leqslant 1. \end{cases}$$

Thus,

$$\begin{aligned} \phi(t) &= t, \\ k_1(s) &= \tilde{k}_1(s) = \min_{t \in I} \tilde{u}_s(t) = \frac{1}{6}\, s\,(1 - s^2), \\ k_2(s) &= \tilde{k}_2(s) = \max_{t \in I} \tilde{u}_s(t) = \frac{s}{2}\,(1 - s), \end{aligned}$$

and

$$t\,\frac{1}{6}\,s\,(1 - s^2) \leqslant g_0(t,s) \leqslant t\,\frac{s}{2}\,(1-s), \textit{ for all } (t,s) \in I \times I.$$

3.4 Adjoint operator

In order to study Green's function sign, we need to study the adjoint operator, $T_n^*[M]$, defined in (1.3.1). So, this section is devoted to make an analysis of such an operator and some of its properties in relation with the hypotheses on operator $T_n[M]$ given in previous section.

So, we describe the space $D\big(T_n^*[M]\big)$, defined in (1.3.2), for the previously given boundary conditions (3.0.2)-(3.0.3), by taking into account that, in our case, $D\big(T_n[M]\big) = X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Let us denote $D\big(T_n^*[M]\big) = X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Let us consider the following sets $\{\delta_1,\dots,\delta_k\}, \{\tau_1,\dots,\tau_{n-k}\} \subset \{0,\dots,n-1\}$, such that:

$$\begin{aligned} \big\{\sigma_1,\dots,\sigma_k, n-1-\tau_1,\dots,\, n-1-\tau_{n-k}\big\} &= \{0,\dots,n-1\}, \\ \big\{\varepsilon_1,\dots,\varepsilon_{n-k}, n-1-\delta_1,\dots,\, n-1-\delta_k\big\} &= \{0,\dots,n-1\}. \end{aligned}$$

Remark 3.4.1. *Realise that, by the definition of α and β given in* (3.2.1) *and* (3.2.2)*, respectively, we have that $\alpha = n - 1 - \tau_{n-k}$ and $\beta = n - 1 - \delta_k$.*

Having in mind the purpose of describing $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, we choose $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, such that:

$$\begin{cases} u^{(n-1-\tau_1)}(a) = 1, & \\ u^{(i)}(a) = 0, & \forall\, i = 0,\dots,n-1, \quad i \neq n-1-\tau_1, \\ u^{(i)}(b) = 0, & \forall\, i = 0,\dots,n-1. \end{cases}$$

Thus, from (1.3.2) we can conclude that every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, satisfies:

$$v^{(\tau_1)}(a) + \sum_{j=n-\tau_1}^{n-1} (-1)^{n-j}\, (p_{n-j}\, v)^{(\tau_1+j-n)}(a) = 0.$$

Proceeding analogously for $\tau_2,\dots,\tau_{n-k}$, we can obtain the boundary conditions for the adjoint operator at $t = a$. Working at $t = b$ for $\delta_1,\dots,\delta_k$, we are able to complete the boundary conditions related to the adjoint operator.

So, we conclude that every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ is a $C^n(I)$ function that satisfies the following conditions:

$$v^{(\tau_1)}(a) + \sum_{j=n-\tau_1}^{n-1} (-1)^{n-j}\, (p_{n-j}\, v)^{(\tau_1+j-n)}(a) = 0, \tag{3.4.1}$$

$$\vdots$$

$$v^{(\tau_{n-k-1})}(a) + \sum_{j=n-\tau_{n-k-1}}^{n-1} (-1)^{n-j}\, (p_{n-j}\, v)^{(\tau_{n-k-1}+j-n)}(a) = 0, \tag{3.4.2}$$

$$v^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j}\, (p_{n-j}\, v)^{(\tau_{n-k}+j-n)}(a) = 0, \tag{3.4.3}$$

$$v^{(\delta_1)}(b) + \sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\, (p_{n-j}\, v)^{(\delta_1+j-n)}(b) = 0, \tag{3.4.4}$$

$$\vdots$$

$$v^{(\delta_{k-1})}(b) + \sum_{j=n-\delta_{k-1}}^{n-1} (-1)^{n-j}\, (p_{n-j}\, v)^{(\delta_{k-1}+j-n)}(b) = 0, \tag{3.4.5}$$

$$v^{(\delta_k)}(b) + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j}\, (p_{n-j}\, v)^{(\delta_k+j-n)}(b) = 0. \tag{3.4.6}$$

Notation 3.4.2. *Let us denote η, $\gamma \in \{0,\dots,n-1\}$ as follows:*

$$\eta \notin \{\tau_1,\dots,\tau_{n-k}\}, \text{ and if } \eta \neq 0, \text{ then } \{0,\dots,\eta-1\} \subset \{\tau_1,\dots,\tau_{n-k}\}, \tag{3.4.7}$$

$$\gamma \notin \{\delta_1,\dots,\delta_k\}, \text{ and if } \gamma \neq 0, \text{ then } \{0,\dots,\gamma-1\} \subset \{\delta_1,\dots,\delta_k\}. \tag{3.4.8}$$

Remark 3.4.3. *As in Remark 3.4.1, we have that $\eta = n-1-\sigma_k$ and $\gamma = n-1-\varepsilon_{n-k}$.*

From the boundary conditions (3.4.1)–(3.4.6), since $p_{n-j} \in C^{n-j}(I)$, the following assertions are fulfilled.

- If $\eta \neq 0$, for all $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ it is satisfied $v(a) = \cdots = v^{(\eta-1)}(a) = 0$.
- If $\gamma \neq 0$, for all $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ it is satisfied $v(b) = \cdots = v^{(\gamma-1)}(b) = 0$.

Example 3.4.4. *Let us consider the fourth order operator $T_4[M]$ coupled with the boundary conditions*

$$u(a) = u''(a) = u'(b) = u''(b) = 0. \tag{3.4.9}$$

Now, we describe the domain of definition of the adjoint operator, $T_4^*[M]$.

In this case, $\{\tau_1, \tau_2\} = \{0, 2\}$ and $\{\delta_1, \delta_2\} = \{0, 3\}$. Thus, from (3.4.1)–(3.4.6)*, we deduce that:*

$$\begin{aligned} X^{*\{1,2\}}_{\{0,2\}} = \Big\{ v \in C^4(I) \mid v(a) = v''(a) - p_1(a)\, v'(a) = v(b) = 0, \\ v^{(3)}(b) - p_1(b)\, v''(b) + \big(p_2(b) - 2p_1'(b)\big) v'(b) = 0 \Big\}. \end{aligned} \tag{3.4.10}$$

Definition 3.4.5. *Let us say that the operator $T_n^*[M]$ satisfies property (T_d^*) in $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ if, and only if, there exists a decomposition:*

$$T_0^*\, v(t) = w_0(t)\, v(t), \quad T_k^*\, v(t) = \frac{-1}{w_k(t)} \frac{\mathrm{d}}{\mathrm{d}\, t} \left(T_{k-1}^*\, v(t)\right), \; k = 1, \dots, n, \quad t \in I, \tag{3.4.11}$$

where $w_k > 0$ on I, $w_k \in C^n(I)$ and:

$$T_n^*[M]\, v(t) = T_n^*\, v(t), \quad t \in I.$$

Moreover, this decomposition satisfies that for every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$:

$$T_{\tau_1}^*\, v(a) = \dots = T_{\tau_{n-k}}^*\, v(a) = 0, \tag{3.4.12}$$

$$T_{\delta_1}^*\, v(b) = \dots = T_{\delta_k}^*\, v(b) = 0. \tag{3.4.13}$$

We have the following result.

Lemma 3.4.6. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, then the adjoint operator $T_n^*[\bar{M}]$ satisfies property (T_d^*) on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*

Proof. From [42, Chapter 3, Theorem 10], it is fulfilled that if $T_n[\bar{M}]$ has the decomposition (3.3.1)-(3.3.2), then $T_n^*[\bar{M}]$ can be decomposed as:

$$T_n^*[\bar{M}]\, v(t) = \frac{(-1)^n}{v_1(t)} \frac{\mathrm{d}}{\mathrm{d}\, t} \left(\frac{1}{v_2(t)} \frac{\mathrm{d}}{\mathrm{d}\, t} \left(\cdots \frac{\mathrm{d}}{\mathrm{d}\, t} \left(\frac{1}{v_n(t)} \frac{\mathrm{d}}{\mathrm{d}\, t} \big(v_1(t) \dots v_n(t)\, v(t) \big) \right) \right) \right). \tag{3.4.14}$$

Hence,

$$T_0^*\, v(t) = v_1(t) \dots v_n(t)\, v(t), \quad \text{and} \quad T_k^* v(t) = \frac{-1}{v_{n+1-k}(t)} \frac{\mathrm{d}}{\mathrm{d}\, t} \left(T_{k-1}^*\, v(t)\right),$$

so, the existence of the decomposition given in (3.4.11) is proved by taking:

$$w_0(t) = v_1(t) \, \dots \, v_n(t) \quad \text{and} \quad w_k(t) = v_{n+1-k}(t) \text{ for } k = 1, \dots, n.$$

Let us see that for every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, the boundary conditions (3.4.12)-(3.4.13) are satisfied.

Obviously, the expression of the n^{th}- order scalar problem (3.0.1)–(3.0.3) as a first order vectorial problem, given in (1.2.4)-(1.2.5), does not depend on property (T_d) of $T_n[\bar{M}]$.

But, in our case, using the decomposition given by (T_d), we can transform the n^{th}-order problem $T_n[\bar{M}]\,u(t)=0$ into a first order vectorial problem in an alternative way as follows:

$$U_u'(t) = A_1(t)\,U_u(t), \quad t\in I, \quad B\,U_u(a)+C\,U_u(b)=0, \tag{3.4.15}$$

with $B, C \in \mathcal{M}_{n\times n}$ defined in (1.2.5), where:

$$\begin{cases} \omega_j^{1+\sigma_j}=1, & j=1,\dots,k,\\ \\ \omega_i^j=0, & \text{otherwise,}\end{cases} \qquad \begin{cases} \nu_{j+k}^{1+\varepsilon_j}=1, & j=1,\dots,n-k,\\ \\ \nu_i^j=0, & \text{otherwise,}\end{cases} \tag{3.4.16}$$

and $U_u(t)\in\mathbb{R}^n$, $A_1(t)\in\mathcal{M}_{n\times n}$, defined by:

$$U_u(t)=\begin{pmatrix} u_{1u}(t)\\ u_{2u}(t)\\ \vdots \\ u_{nu}(t)\end{pmatrix}, \quad A_1(t)=\begin{pmatrix} 0 & v_2(t) & 0 & \dots & 0\\ 0 & 0 & v_3(t) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & v_n(t)\\ 0 & 0 & 0 & \dots & 0\end{pmatrix}, \tag{3.4.17}$$

where $u_{\ell u}(t) := \dfrac{T_{\ell-1}\,u(t)}{v_\ell(t)}$ for $\ell=1,\dots,n$ and $u\in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Indeed, if $1\leqslant \ell \leqslant n-1$:

$$u_{\ell u}'(t)=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{T_{\ell-1}\,u(t)}{v_\ell(t)}\right)=\frac{T_\ell\,u(t)}{v_{\ell+1}(t)}\,v_{\ell+1}(t)=v_{\ell+1}(t)\,u_{\ell+1\,u}(t),$$

and, if $\ell=n$:

$$u_n'(t)=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{T_{n-1}\,u(t)}{v_{n-1}(t)}\right)=T_n\,u(t)=\frac{T_n[\bar{M}]\,u(t)}{v_1(t)\dots v_n(t)}=0.$$

Taking into account that $T_n[\bar{M}]$ satisfies property (T_d) on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, we have that:

$$u_{\sigma_1+1\,u}(a)=\dots=u_{\sigma_k+1\,u}(a)=u_{\varepsilon_1+1\,u}(b)=\dots=u_{\varepsilon_{n-k}+1\,u}(b)=0. \tag{3.4.18}$$

Moreover, using similar arguments, by means of the decomposition (3.4.14), we can transform the n^{th}- order scalar problem:

$$T_n^*[\bar{M}]\,v(t)=0, \quad t\in I, \tag{3.4.19}$$

coupled with the boundary conditions (3.4.1)–(3.4.6) on the following equivalent first order vectorial problem:

$$Z_v'(t)=-A_1^T(t)\,Z_v(t), \quad t\in I, \tag{3.4.20}$$

where $A_1(t) \in \mathcal{M}_{n\times n}$ is defined in (3.4.17) and $Z_v(t) \in \mathbb{R}^n$ is given by:

$$Z_v(t) = \begin{pmatrix} {z_1}_v(t) \\ {z_2}_v(t) \\ \vdots \\ {z_n}_v(t) \end{pmatrix},$$

with ${z_\ell}_v(t) := T^*_{n-\ell}\, v(t)$ for $\ell = 0, \dots, n-1$ and $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Indeed, if $2 \leqslant \ell \leqslant n$:

$${z_\ell}'_v(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(T^*_{n-\ell} v(t)\right) = T^*_{n-\ell+1} v(t)\left(-\, v_{n+1-(n-\ell+1)}(t)\right) = -v_\ell(t)\, {z_{\ell-1}}_v(t),$$

and, if $\ell = 1$:

$${z_1}'_v(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(T^*_{n-1}\, v(t)\right) = -v_1(t)\, T^*_n v(t) = -v_1(t)\, T^*_n[\bar{M}] v(t) = 0.$$

Consider the n^{th}-order linear differential operators $T_n[\bar{M}]$ and $T^*_n[\bar{M}]$ in a vectorial way:

$$\begin{aligned} {T_n}^v[\bar{M}]\, U_u(t) &= U'_u(t) - A_1(t)\, U_u(t), \\ {T^*_n}^v[\bar{M}]\, Z_v(t) &= -Z'_v(t) - A_1^T(t)\, Z_v(t), \end{aligned}$$

with $U_u(t)$, $Z_v(t) \in \mathbb{R}^n$ and $A_1(t) \in \mathcal{M}_{n\times n}$ previously defined.

As we can see in [16, Section 1.3], ${T^*_n}^v[\bar{M}]$ is the adjoint operator of ${T_n}^v[\bar{M}]$ and vice-versa.

As a direct consequence, by the definition of adjoint operator, we conclude that for every $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, the following equality is fulfilled:

$$\Big\langle {T_n}^v[\bar{M}]\, U_u(t), Z_v(t) \Big\rangle = \Big\langle U_u(t), {T^*_n}^v[\bar{M}]\, Z_v(t) \Big\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{L}^2(I, \mathbb{R}^n)$.

Moreover, from [16, Section 1.3], we have that:

$$\Big\langle U_u(a), Z_v(a) \Big\rangle = \Big\langle U_u(b), Z_v(b) \Big\rangle, \quad \forall u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}} \text{ and } v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}.$$

Now, taking into account the boundary conditions (3.4.18), we conclude that for every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ it is satisfied:

$${z_{n-\tau_1}}_v(a) = \cdots = {z_{n-\tau_n}}_v(a) = {z_{n-\delta_1}}_v(b) = \cdots = {z_{n-\delta_k}}_v(b) = 0,$$

which implies that:

$$T^*_{\tau_1} v(a) = \cdots = T^*_{\tau_{n-k}} v(a) = T^*_{\delta_1} v(b) = \cdots = T^*_{\delta_k} v(b) = 0.$$

Or, which is the same, $T^*_n[\bar{M}]$ satisfies property (T^*_d) on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. $\square$

Example 3.4.7. *Let us consider a fourth order operator $T_4[M]$. Moreover, let us assume that $T_4[M]$ satisfies property (T_d) in $X_{\{0,2\}}^{\{1,2\}}$. That is,*

$$T_0u(a) = T_2u(a) = T_1u(b) = T_2u(b) = 0, \textit{ for all } u \in X_{\{0,2\}}^{\{1,2\}}.$$

From (3.3.8) *and* (3.3.9)*, taking into account the boundary conditions* (3.4.9)*, we obtain that the following equalities are fulfilled for every $u \in X_{\{0,2\}}^{\{1,2\}}$:*

$$\begin{aligned}
T_0u(a) &= 0,\\
T_2u(a) &= -u'(a)\frac{2v_2(a)v_1'(a)+v_1(a)v_2'(a)}{v_1^2(a)\,v_2^2(a)},\\
T_1u(b) &= -u(b)\frac{v_1'(b)}{v_1^2(b)},\\
T_2u(b) &= u(b)\frac{v_1(b)v_1'(b)v_2'(b)+v_2(b)\left(2{v_1'}^2(b)-v_1(b)v_1''(b)\right)}{v_1^3(b)\,v_2^2(b)}.
\end{aligned}$$

So, $T_4[M]$ satisfies property (T_d) in $X_{\{0,2\}}^{\{1,2\}}$ if, and only if, there exists a decomposition as (3.3.1)-(3.3.2)*, where $v_1,\ v_2 \in C^4(I)$ are such that:*

$$\frac{2v_1'(a)}{v_1(a)} = -\frac{v_2'(a)}{v_2(a)}, \tag{3.4.21}$$

$$v_1'(b) = v_1''(b) = 0. \tag{3.4.22}$$

Let us verify that in such a case, the operator $T_4^*[M]$ satisfies property (T_d^*) in ${X^*}_{\{0,2\}}^{\{1,2\}}$.
In order to do that, we express p_1 and p_2 as functions of v_1, v_2, v_3 and v_4.
Expanding the related expression (3.3.1)-(3.3.2) *for $n = 4$, we obtain that:*

$$p_1 \equiv -\frac{4v_1'}{v_1} - \frac{3v_2'}{v_2} - \frac{2v_3'}{v_3} - \frac{v_4'}{v_4},$$

and,

$$\begin{aligned}
p_2 \equiv &\frac{12{v_1'}^2}{v_1^2} + \frac{6{v_2'}^2}{v_2} + \frac{2{v_3'}^2}{v_3^2} + \frac{9v_1'\,v_2'}{v_1\,v_2} + \frac{6v_1'\,v_3'}{v_1\,v_3} + \frac{4v_2'\,v_3'}{v_2\,v_3} + \frac{3v_1'\,v_4'}{v_1\,v_4} + \frac{2v_2'\,v_4'}{v_2\,v_4} + \frac{v_3'\,v_4'}{v_3\,v_4}\\
&- \frac{6v_1''}{v_1} - \frac{3v_2''}{v_2} - \frac{v_3''}{v_3}.
\end{aligned}$$

Moreover,

$$p_1' \equiv \frac{4{v_1'}^2}{v_1^2} + \frac{3{v_2'}^2}{v_2^2} + \frac{2{v_3'}^2}{v_3^2} + \frac{{v_4'}^2}{v_4^2} - \frac{4v_1''}{v_1} - \frac{3v_2''}{v_2} - \frac{2v_3''}{v_3} - \frac{v_4''}{v_4}.$$

Taking into account (3.4.21)-(3.4.22)*, the boundary conditions for the adjoint operator, given in Example 3.4.4, can be expressed, in terms of v_1, v_2, v_3 and v_4, as follows:*

$$v(a) = v''(a) + \left(\frac{v_2'(a)}{v_2(a)} + \frac{2v_3'(a)}{v_3(a)} + \frac{v_4'(a)}{v_4(a)}\right)v'(a) = v(b) = 0, \tag{3.4.23}$$

$$v^{(3)}(b)+\left(\frac{3v_2'(b)}{v_2(b)}+\frac{2v_3'(b)}{v_3(b)}+\frac{v_4'(b)}{v_4(b)}\right)v''(b)+\left(\frac{4v_2'(b)v_3'(b)}{v_2(b)\,v_3(b)}-\frac{2{v_3'}^2(b)}{v_3^2(b)}\right. \tag{3.4.24}$$
$$\left.+\frac{2\,v_2'(b)v_4'(b)}{v_2(b)\,v_4(b)}+\frac{v_3'(b)v_4'(b)}{v_3(b)\,v_4(b)}-\frac{2{v_4'}^2(b)}{v_4^2(b)}+\frac{3v_2''(b)}{v_2(b)}+\frac{3v_3''(b)}{v_3(b)}+\frac{2v_4''(b)}{v_4(b)}\right)v'(b)=0.$$

Now, let us see that $T_0^*v(a)=T_2^*v(a)=T_0^*v(b)=T_3^*v(b)=0$ for all $v\in X^{*\{1,2\}}_{\{0,2\}}$.
Trivially, $T_0^*v(a)=v(a)=0$ and $T_0^*v(b)=v(b)=0$.
Using the decomposition (3.4.11)*, we have:*

$$T_2^*v(t)=-\frac{1}{v_3(t)}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{-1}{v_4(t)}\frac{\mathrm{d}}{\mathrm{d}\,t}\big(v_1(t)\,v_2(t)\,v_3(t)\,v_4(t)\,v(t)\big)\right),$$

from which, considering (3.4.21) *and* (3.4.23)*, we obtain:*

$$T_2^*v(a)=v_1(a)\,v_2(a)\left(v''(a)+\left(\frac{v_2'(a)}{v_2(a)}+\frac{2v_3'(a)}{v_3(a)}+\frac{v_4'(a)}{v_4(a)}\right)v'(a)\right)=0,$$

and, finally,

$$T_3^*v(t)=-\frac{1}{v_2(t)}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{-1}{v_3(t)}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{-1}{v_4(t)}\frac{\mathrm{d}}{\mathrm{d}\,t}\big(v_1(t)\,v_2(t)\,v_3(t)\,v_4(t)\,v(t)\big)\right)\right).$$

Combining the previous expression with (3.4.22) –(3.4.24)*, we obtain:*

$$T_3^*v(b)=-\,v_1(b)\left(v^{(3)}(b)+\left(\frac{3v_2'(b)}{v_2(b)}+\frac{2v_3'(b)}{v_3(b)}+\frac{v_4'(b)}{v_4(b)}\right)v''(b)+\left(\frac{4v_2'(b)v_3'(b)}{v_2(b)\,v_3(b)}-\frac{2{v_3'}^2(b)}{v_3^2(b)}\right.\right.$$
$$\left.\left.+\frac{2\,v_2'(b)v_4'(b)}{v_2(b)\,v_4(b)}+\frac{v_3'(b)v_4'(b)}{v_3(b)\,v_4(b)}-\frac{2{v_4'}^2(b)}{v_4^2(b)}+\frac{3v_2''(b)}{v_2(b)}+\frac{3v_3''(b)}{v_3(b)}+\frac{2v_4''(b)}{v_4(b)}\right)v'(b)\right)=0.$$

As a particular case of Lemma 3.4.6, we have proved that if $T_4[M]$ satisfies property (T_d) in $X^{\{1,2\}}_{\{0,2\}}$, then $T_4^*[M]$ satisfies property (T_d^*) in $X^{*\{1,2\}}_{\{0,2\}}$.

It is obvious that we can enunciate an analogous result to Lemma 3.4.6 referring to operator $\widehat{T}_n\big[(-1)^n\bar{M}\big]$ defined in (1.3.4).

Lemma 3.4.8. *Let $\bar{M}\in\mathbb{R}$ be such that operator $T_n[\bar{M}]$ satisfies property (T_d) in the space $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$, then $\widehat{T}_n\big[(-1)^n\bar{M}\big]$ also satisfies property (T_d^*) in $X^{*\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$.*

Proof. We only have to consider $\widehat{T}_\ell v(t)=(-1)^\ell T_\ell^*v(t)$, for $\ell=0,\ldots,1$, and the result follows directly from Lemma 3.4.6. □

Arguing as in Section 3.3 to deduce the equality (3.3.3), let us see that the expression of $\widehat{T}_\ell v(t)$ is given by:

$$\widehat{T}_\ell v(t)=v_1(t)\,\ldots\,v_{n-\ell}(t)\,v^{(\ell)}(t)+\widehat{p}_{\ell_1}(t)\,v^{(\ell-1)}(t)+\cdots+\widehat{p}_{\ell_\ell}(t)\,v(t),\quad t\in I, \tag{3.4.25}$$

where $\widehat{p}_{\ell_i} \in C^{n-\ell}(I)$.

For $\ell = 0$, we have that $\widehat{T}_0 v(t) = v_1(t) \dots v_n(t)\, v(t)$.

Let us assume that (3.4.25) is true for a given $\ell \geqslant 0$, then, by (3.4.11), we have:

$$\widehat{T}_{\ell+1} v(t) = \frac{1}{v_{n-\ell}(t)} \frac{\mathrm{d}}{\mathrm{d}\,t} \left(\widehat{T}_\ell v(t) \right), \quad t \in I.$$

Thus, using the induction hypothesis:

$$\widehat{T}_{\ell+1} v(t) = \frac{1}{v_{n-\ell}(t)} \frac{\mathrm{d}}{\mathrm{d}\,t} \left(v_1(t) \, \dots \, v_{n-\ell}(t)\, v^{(\ell)}(t) + \widehat{p}_{\ell_1}(t)\, v^{(\ell-1)}(t) + \cdots + \widehat{p}_{\ell_\ell}(t)\, v(t) \right),$$

for all $t \in I$, which follows the expression (3.4.25) for $\ell + 1$.

As a consequence of (3.4.25), we are able to obtain analogous results to Lemmas 3.3.6 and 3.3.7 for $\widehat{T}_n\big[(-1)^n M\big]$.

Lemma 3.4.9. *Let $\bar{M} \in \mathbb{R}$ be such that operator $T_n\big[(-1)^n \bar{M}\big]$ satisfies property (T_d^*) in $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. If $v \in C^n\big([a,c)\big)$, with $c > a$, is a function that satisfies* (3.4.1)–(3.4.2)*, then*

$$\widehat{T}_{\tau_1}\, v(a) = \cdots = \widehat{T}_{\tau_{n-k-1}}\, v(a) = 0,$$

and:

$$\widehat{T}_{\tau_{n-k}}\, v(b) = \widehat{f}(a) \left(v^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \left(p_{n-j}\, v\right)^{(\tau_{n-k}+j-n)}(a) \right),$$

where $\widehat{f}(t) = v_1(t) \dots v_{n-\tau_{n-k}}(t) > 0$ on I.

In particular, if v satisfies (3.4.3)*, then $\widehat{T}_{\tau_{n-k}} v(a) = 0$.*

Proof. The proof is analogous to the one given in Lemma 3.3.6, but in this case we have that:

$$\widehat{T}_{\tau_{n-k}} v(a) = v_1(a) \, \dots \, v_{n-\tau_{n-k}}(a)\, v^{(\tau_{n-k})}(a) + \widehat{p}_{\tau_{n-k\,1}}(a)\, v^{(\tau_{n-k}-1)}(a) + \cdots + \widehat{p}_{\tau_{n-k}\,\tau_{n-k}}(a)\, v(a).$$

If (3.4.3) is satisfied, then $\widehat{T}_{\tau_{n-k}\, v(a)} = 0$, and the result is true. □

Lemma 3.4.10. *Let $\bar{M} \in \mathbb{R}$ be such that operator $T_n\big[(-1)^n \bar{M}\big]$ satisfies property (T_d^*) in $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. If $v \in C^n\big((c,b]\big)$, with $c < b$, is a function that satisfies* (3.4.4)–(3.4.5)*, then*

$$\widehat{T}_{\delta_1}\, v(b) = \cdots = \widehat{T}_{\delta_{k-1}}\, v(b) = 0,$$

and

$$\widehat{T}_{\delta_k}\, v(b) = \widehat{g}(a) \left(v^{(\delta_k)}(b) + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \left(p_{n-j}\, v\right)^{(\delta_k+j-n)}(b) \right),$$

where $\widehat{g}(t) = v_1(t) \dots v_{n-\delta_k}(t) > 0$ on I.

In particular, if v satisfies (3.4.6)*, then $\widehat{T}_{\delta_k} v(b) = 0$.*

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Proof. The proof is analogous to the one given in Lemma 3.4.9. $\square$

To finish this section, let us construct, as for $T_n[M]$, the associated vectorial problem to $\widehat{T}_n\left[(-1)^n M\right]$ in $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

From the expression of $\widehat{T}_n\left[(-1)^n\, M\right]$ given in (1.3.4) and (1.3.1), we construct the associated vectorial problem (1.2.4) tacking, in this case:

$$\begin{aligned}
\widehat{p}_{n-j}(t) &= (-1)^{n+j} p_{n-j}(t) + (-1)^{(n+j+1)}(j+1)\, p'_{n-j-1}(t) + \cdots - \begin{pmatrix} n-1 \\ j \end{pmatrix} p_1^{(n-j-1)}(t), \\
& \qquad j = 1, \dots, n-1, \\
\widehat{p}_n(t) &= (-1)^n p_n(t) + (-1)^{n+1} p'_{n-1}(t) + (-1)^{n+2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} p''_{n-2}(t) + \cdots - p_1^{(n-1)}(t).
\end{aligned}$$

The correspondent boundary conditions (1.2.5) are given by the matrices $\widehat{B}, \widehat{C} \in \mathcal{M}_{n\times n}$, defined as follows:

$$\begin{aligned}
&\left.\begin{aligned}
&\left(\widehat{B}\right)_{i\,\tau_i+1} = 1, \\
&\left(\widehat{B}\right)_{i\,j} = 0, \quad \tau_i + 1 < j \leqslant n, \\
&\left(\widehat{B}\right)_{i\,\tau_i-h} = \widehat{p}_{h+1}(a),\; h = 0, \dots \tau_i - 1,
\end{aligned}\right\} i = 1, \dots, n-k, \\
&\left(\widehat{B}\right)_{i\,j} = 0, \quad j = 0, \dots, n, \quad i = n-k+1, \dots n \\
&\left(\widehat{C}\right)_{i\,j} = 0, \quad j = 0, \dots, n, \quad i = 0, \dots, n-k, \\
&\left.\begin{aligned}
&\left(\widehat{C}\right)_{i\,\delta_{i-(n-k)}+1} = 1, \\
&\left(\widehat{C}\right)_{i\,j} = 0, \quad \delta_{i-(n-k)} + 1 < j \leqslant n, \\
&\left(\widehat{C}\right)_{i\,\delta_{i-(n-k)}-h} = \widehat{p}_{h+1}(b),\; h = 0, \dots \delta_{i-(n-k)} - 1,
\end{aligned}\right\} i = n-k+1, \dots, n,
\end{aligned} \tag{3.4.26}$$

that is, for every $v \in C^n(I)$, we have:

$$\widehat{B} \begin{pmatrix} v(a) \\ \vdots \\ v^{(n-1)}(a) \end{pmatrix} + \widehat{C} \begin{pmatrix} v(b) \\ \vdots \\ v^{(n-1)}(b) \end{pmatrix} = \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix},$$

where:

$$\begin{aligned}
W_1 &= v^{(\tau_1)}(a) + \sum_{j=n-\tau_1}^{n-1} (-1)^{n-j} \left(p_{n-j}\, v\right)^{(\tau_1+j-n)}(a), \\
&\;\;\vdots \\
W_{n-k} &= v^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \left(p_{n-j}\, v\right)^{(\tau_{n-k}+j-n)}(a),
\end{aligned}$$

$$W_{n-k+1} = v^{(\delta_1)}(b) + \sum_{j=n-\delta_1}^{n-1} (-1)^{n-j}\,(p_{n-j}\,v)^{(\delta_1+j-n)}(b),$$

$$\vdots$$

$$W_n = v^{(\delta_k)}(b) + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j}\,(p_{n-j}\,v)^{(\delta_k+j-n)}(b).$$

Now, the related Green's function is:

$$\widehat{G}(t,s) = \begin{pmatrix} \widehat{g}_1(t,s) & \cdots & \widehat{g}_{n-1}(t,s) & \widehat{g}_{(-1)^n\,M}(t,s) \\ \dfrac{\partial}{\partial t}\,\widehat{g}_1(t,s) & \cdots & \dfrac{\partial}{\partial t}\,\widehat{g}_{n-1}(t,s) & \dfrac{\partial}{\partial t}\,\widehat{g}_{(-1)^n\,M}(t,s) \\ \vdots & \cdots & \vdots & \vdots \\ \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_1(t,s) & \cdots & \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{n-1}(t,s) & \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{(-1)^n\,M}(t,s) \end{pmatrix}, \tag{3.4.27}$$

and, repeating the arguments done with $T_n[M]$, we obtain:

$$\widehat{g}_{n-j}(t,s) = (-1)^j\,\frac{\partial^j}{\partial\, s^j}\,\widehat{g}_{(-1)^n M}(t,s) + \sum_{i=0}^{j-1} \widehat{\alpha}_i^j(s)\,\frac{\partial^i}{\partial s^i}\widehat{g}_{(-1)^n M}(t,s), \tag{3.4.28}$$

where $\widehat{\alpha}_i^j(s)$ follow the recurrence formula (3.1.3)–(3.1.3) for this problem with the obvious notation.

3.5 Derivatives of the Green's function

This section is devoted to describe the behaviour of several partial derivatives of the Green's function at a suitable set of points.

Let us denote:

$$g_M(t,s) = \begin{cases} g_M^1(t,s), & a \leqslant s \leqslant t \leqslant b, \\ g_M^2(t,s), & a < t < s < b, \end{cases}$$

as the related Green's function of $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

First, let us introduce a result which describes the Green's function at $s=a$.

Lemma 3.5.1. *Let us consider* $w_M(t) = \dfrac{\partial^\eta}{\partial s^\eta} g_M^1(t,s)_{|s=a}$, *where* η *has been defined in* (3.4.7). *Then, the following assertions are fulfilled.*

- w_M *is a solution of* $T_n[M]\,w_M(t) = 0$ *for all* $t \in (a,b]$.
- w_M *satisfies the following boundary conditions:*

$$\begin{aligned} w_M^{(\sigma_1)}(a) = \cdots = w_M^{(\sigma_{k-1})}(a) = 0, \\ w_M^{(\varepsilon_1)}(b) = \cdots = w_M^{(\varepsilon_{n-k})}(b) = 0, \end{aligned} \tag{3.5.1}$$

coupled with:

$$w_M^{(\sigma_k)}(a) = (-1)^{n-1-\sigma_k}. \tag{3.5.2}$$

Moreover, if $\eta > 0$, then

$$g_M^1(t,a) = \frac{\partial}{\partial s} g_M^1(t,s)_{|s=a} = \cdots = \frac{\partial^{\eta-1}}{\partial s^{\eta-1}} g_M^1(t,s)_{|s=a} = 0.$$

Proof. The last assertion follows directly from the relation (1.3.3) and the boundary conditions of the adjoint operator given in (3.4.1)–(3.4.6).

Now, let us study w_M. From item (g_4) in the definition of Green's function, Definition 1.2.3, we have that $T_n[M]\, g_M(t,a) = 0$ for all $t \in (a,b]$. Hence:

$$\frac{\partial^\eta}{\partial s^\eta}\big(T_n[M]\, g_M(t,s)\big)_{|s=a} = T_n[M]\, w_M(t) = 0, \quad t \in (a,b].$$

Now, let us see which boundary conditions are fulfilled by w_M.

To this end, we use the Green's matrix related to the vectorial problem (1.2.4)-(1.2.5), where B and C have been defined in (3.4.16). We have obtained the expression of the Green's matrix in Section 3.1, it is given by the expression (3.1.8).

If $k > 1$, considering the first row of (1.2.5), for the considered B and C, we have:

$$\left.\begin{aligned} \frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}} g_M^2(t,s)_{|t=a} &= 0, \\ -\frac{\partial^{\sigma_1+1}}{\partial t^{\sigma_1}\partial s} g_M^2(t,s)_{|t=a} + \alpha_0^1(s)\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}} g_M^2(t,s)_{|t=a} &= 0, \\ &\vdots \\ (-1)^\eta \frac{\partial^{\sigma_1+\eta}}{\partial t^{\sigma_1}\partial s^\eta} g_M^2(t,s)_{|t=a} + \sum_{i=0}^{\eta-1}\alpha_i^\eta(s)\frac{\partial^{i+\sigma_1}}{\partial t^{\sigma_1}\partial s^i} g_M^2(t,s)_{|t=a} &= 0. \end{aligned}\right\}$$

This system is satisfied in particular for $s = a$. Since $\eta + \sigma_1 < n-1$ we do not reach any diagonal element of $G(t,s)$, hence we obtain by continuity:

$$\left.\begin{aligned} \frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}} g_M^1(t,s)_{|(t,s)=(a,a)} &= 0, \\ -\frac{\partial^{\sigma_1+1}}{\partial t^{\sigma_1}\partial s} g_M^1(t,s)_{|(t,s)=(a,a)} + \alpha_0^1(a)\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}} g_M^1(t,s)_{|(t,s)=(a,a)} &= 0, \\ &\vdots \\ (-1)^\eta \frac{\partial^{\sigma_1+\eta}}{\partial t^{\sigma_1}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} + \sum_{i=0}^{\eta-1}\alpha_i^\eta(a)\frac{\partial^{i+\sigma_1}}{\partial t^{\sigma_1}\partial s^i} g_M^1(t,s)_{|(t,s)=(a,a)} &= 0. \end{aligned}\right\}$$

Taking into account that $\alpha_i^j \in C(I)$, we have:

$$w_M^{(\sigma_1)}(a) = \frac{\partial^{\sigma_1+\eta}}{\partial t^{\sigma_1}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = 0.$$

Proceeding analogously for $\sigma_2, \dots, \sigma_{k-1}$, we obtain:

$$w_M^{(\sigma_2)}(a) = \cdots = w_M^{(\sigma_{k-1})}(a) = 0.$$

Now, let us choose the row σ_k of $G(t,s)$. From (1.2.5) with B and C described in (3.4.16), we have:

$$\left.\begin{aligned} \frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^2(t,s)_{|t=a} &= 0, \\ -\frac{\partial^{\sigma_k+1}}{\partial t^{\sigma_k}\partial s} g_M^2(t,s)_{|t=a} + \alpha_0^1(s)\frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^2(t,s)_{|t=a} &= 0, \\ &\vdots \\ (-1)^\eta \frac{\partial^{\sigma_k+\eta}}{\partial t^{\sigma_k}\partial s^\eta} g_M^2(t,s)_{|t=a} + \sum_{i=0}^{\eta-1} \alpha_i^\eta(s) \frac{\partial^{i+\sigma_k}}{\partial t^{\sigma_k}\partial s^i} g_M^2(t,s)_{|t=a} &= 0. \end{aligned}\right\}$$

This system is satisfied in particular for $s=a$. However, since $\sigma_k + \eta = n-1$, we reach a diagonal element of $G(t,s)$. Hence, in order to express the previous system by means of $g_M^1(t,s)$, we have to take into account Remark 1.2.2 to obtain:

$$\left.\begin{aligned} \frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^1(t,s)_{|(t,s)=(a,a)} &= 0, \\ -\frac{\partial^{\sigma_k+1}}{\partial t^{\sigma_k}\partial s} g_M^1(t,s)_{|(t,s)=(a,a)} + \alpha_0^1(a)\frac{\partial^{\sigma_k}}{\partial t^{\sigma_k}} g_M^1(t,s)_{|(t,s)=(a,a)} &= 0, \\ &\vdots \\ (-1)^\eta \frac{\partial^{\sigma_k+\eta}}{\partial t^{\sigma_k}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} + \sum_{i=0}^{\eta-1} \alpha_i^\eta(a) \frac{\partial^{i+\sigma_k}}{\partial t^{\sigma_k}\partial s^i} g_M^1(t,s)_{|(t,s)=(a,a)} &= 1. \end{aligned}\right\}$$

So, since $\alpha_i^j \in C(I)$, we have:

$$w_M^{(\sigma_k)}(a) = \frac{\partial^{\sigma_k+\eta}}{\partial t^{\sigma_k}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = (-1)^\eta = (-1)^{(n-1-\sigma_k)}.$$

Analogously, if $k=1$, then $w_M^{(\sigma_1)}(a) = (-1)^{n-1-\sigma_1}$.

Now, let us see what happens at $t=b$. If we consider the $(k+1)^{\text{th}}$ row of (1.2.5), we have:

$$\left.\begin{aligned} \frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^1(t,s)_{|t=b} &= 0, \\ -\frac{\partial^{\varepsilon_1+1}}{\partial t^{\varepsilon_1}\partial s} g_M^1(t,s)_{|t=b} + \alpha_0^1(s)\frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^1(t,s)_{|t=b} &= 0, \\ &\vdots \\ (-1)^\eta \frac{\partial^{\varepsilon_1+\eta}}{\partial t^{\varepsilon_1}\partial s^\eta} g_M^1(t,s)_{|t=b} + \sum_{i=0}^{\eta-1} \alpha_i^\eta(s) \frac{\partial^{i+\varepsilon_1}}{\partial t^{\varepsilon_1}\partial s^i} g_M^1(t,s)_{|t=b} &= 0. \end{aligned}\right\}$$

Since $b \neq a$, this system is satisfied in particular at $s = a$. Thus, using that $\alpha_i^j \in C(I)$, we conclude:

$$w_M^{(\varepsilon_1)}(b) = \frac{\partial^{\varepsilon_1+\eta}}{\partial t^{\varepsilon_1}\partial s^{\eta}} g_M^1(t,s)_{|(t,s)=(b,a)} = 0.$$

Proceeding analogously we obtain:

$$w_M^{(\varepsilon_2)}(b) = \cdots = w_M^{(\varepsilon_{n-k})}(b) = 0. \qquad \square$$

Now, let us introduce an analogous result for the behaviour of the Green's function at $s = b$.

Lemma 3.5.2. *Let us consider* $y_M(t) = \dfrac{\partial^\gamma}{\partial s^\gamma} g_M^2(t,s)_{|s=b}$, *where* γ *has been defined in* (3.4.8). *Then, the following assertions are fulfilled.*

- y_M *is a solution of* $T_n[M]\, y_M(t) = 0$ *for all* $t \in [a,b)$.
- y_M *satisfies the following boundary conditions:*

$$\begin{aligned} y_M^{(\sigma_1)}(a) = \cdots = y_M^{(\sigma_k)}(a) = 0,\\ y_M^{(\varepsilon_1)}(b) = \cdots = y_M^{(\varepsilon_{n-k-1})}(b) = 0, \end{aligned} \tag{3.5.3}$$

coupled with:

$$y_M^{(\varepsilon_{n-k})}(b) = (-1)^{n-\varepsilon_{n-k}}. \tag{3.5.4}$$

Moreover, if $\gamma > 0$, *then:*

$$g_M^2(t,b) = \frac{\partial}{\partial s} g_M^2(t,s)_{|s=b} = \cdots = \frac{\partial^{\gamma-1}}{\partial s^{\gamma-1}} g_M^2(t,s)_{|s=b} = 0.$$

Proof. The last assertion follows directly, as in Lemma 3.5.1, from the relation (1.3.3) and the boundary conditions of the adjoint operator given in (3.4.1)–(3.4.6).

Now, let us study y_M. Again, from item (g_4) in Definition 1.2.3, we have that:

$$T_n[M]\, g_M(t,b) = 0, \text{ for all } t \in [a,b).$$

Thus:

$$\frac{\partial^\gamma}{\partial s^\gamma}\big(T_n[M]\, g_M(t,s)\big)_{|s=b} = T_n[M]\, y_M(t) = 0, \quad t \in [a,b).$$

Now, let us see which are the boundary conditions satisfied by y_M. Studying the row σ_1 of $G(t,s)$, taking into account the boundary conditions (1.2.5), where B and C have been introduced in (3.4.16), we have:

$$\left.\begin{aligned} \frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}} g_M^2(t,s)_{|t=a} = 0,\\ -\frac{\partial^{\sigma_1+1}}{\partial t^{\sigma_1}\partial s} g_M^2(t,s)_{|t=a} + \alpha_0^1(s)\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}} g_M^2(t,s)_{|t=a} = 0,\\ \vdots\\ (-1)^\gamma \frac{\partial^{\sigma_1+\gamma}}{\partial t^{\sigma_1}\partial s^\gamma} g_M^2(t,s)_{|t=a} + \sum_{i=0}^{\gamma-1}\alpha_i^\eta(s)\frac{\partial^{i+\sigma_1}}{\partial t^{\sigma_1}\partial s^i} g_M^2(t,s)_{|t=a} = 0. \end{aligned}\right\}$$

This system is satisfied, in particular, for $s=b$. Hence, taking into account the fact that $\alpha_i^j \in C(I)$, we conclude:

$$y_M^{(\sigma_1)}(a) = \frac{\partial^{\sigma_1+\gamma}}{\partial t^{\sigma_1}\partial s^{\gamma}} g_M^2(t,s)_{|(t,s)=(a,b)} = 0.$$

Repeating these arguments, we conclude that:

$$y_M^{(\sigma_2)}(a) = \cdots = y_M^{(\sigma_k)}(a) = 0.$$

Now, let us see what happens at $t=b$.

If $k < n-1$, considering the row ε_1 of $G(t,s)$, from (1.2.5), we have:

$$\left.\begin{aligned} \frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^1(t,s)_{|t=b} &= 0,\\ -\frac{\partial^{\varepsilon_1+1}}{\partial t^{\varepsilon_1}\partial s} g_M^1(t,s)_{|t=b} + \alpha_0^1(s)\frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^1(t,s)_{|t=b} &= 0,\\ &\vdots\\ (-1)^{\gamma}\frac{\partial^{\varepsilon_1+\gamma}}{\partial t^{\varepsilon_1}\partial s^{\gamma}} g_M^1(t,s)_{|t=b} + \sum_{i=0}^{\gamma-1}\alpha_i^{\gamma}(s)\frac{\partial^{i+\varepsilon_1}}{\partial t^{\varepsilon_1}\partial s^{i}} g_M^1(t,s)_{|t=b} &= 0. \end{aligned}\right\}$$

This system is satisfied for $s=b$, and taking into account that $\gamma+\varepsilon_1 < n-1$, we do not reach any diagonal element of $G(t,s)$. Thus, since $G(t,s)$ is continuous on $I \times I$, for every non diagonal element we obtain:

$$\left.\begin{aligned} \frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^2(t,s)_{|(t,s)=(b,b)} &= 0,\\ -\frac{\partial^{\varepsilon_1+1}}{\partial t^{\varepsilon_1}\partial s} g_M^2(t,s)_{|(t,s)=(b,b)} + \alpha_0^1(b)\frac{\partial^{\varepsilon_1}}{\partial t^{\varepsilon_1}} g_M^2(t,s)_{|(t,s)=(b,b)} &= 0,\\ &\vdots\\ (-1)^{\gamma}\frac{\partial^{\varepsilon_1+\gamma}}{\partial t^{\varepsilon_1}\partial s^{\gamma}} g_M^2(t,s)_{|(t,s)=(b,b)} + \sum_{i=0}^{\gamma-1}\alpha_i^{\gamma}(b)\frac{\partial^{i+\varepsilon_1}}{\partial t^{\varepsilon_1}\partial s^{i}} g_M^2(t,s)_{|(t,s)=(b,b)} &= 0. \end{aligned}\right\}$$

Hence, by the continuity of α_i^j, we have:

$$y_M^{(\varepsilon_1)}(b) = \frac{\partial^{\varepsilon_1+\gamma}}{\partial t^{\varepsilon_1}\partial s^{\gamma}} g_M^2(t,s)_{|(t,s)=(b,b)} = 0.$$

Analogously, we can prove:

$$y_M^{(\varepsilon_2)}(b) = \cdots = y_M^{(\varepsilon_{n-k-1})}(b) = 0.$$

Now, for ε_{n-k}, we have:

$$\left.\begin{aligned} \frac{\partial^{\varepsilon_{n-k}}}{\partial t^{\varepsilon_{n-k}}} g_M^1(t,s)_{|t=b} &= 0,\\ -\frac{\partial^{\varepsilon_{n-k}+1}}{\partial t^{\varepsilon_{n-k}}\partial s} g_M^1(t,s)_{|t=b} + \alpha_0^1(s)\frac{\partial^{\varepsilon_{n-k}}}{\partial t^{\varepsilon_{n-k}}} g_M^1(t,s)_{|t=b} &= 0,\\ &\vdots\\ (-1)^\gamma \frac{\partial^{\varepsilon_{n-k}+\gamma}}{\partial t^{\varepsilon_{n-k}}\partial s^\gamma} g_M^1(t,s)_{|t=b} + \sum_{i=0}^{\gamma-1}\alpha_i^\gamma(s)\frac{\partial^{i+\varepsilon_{n-k}}}{\partial t^{\varepsilon_{n-k}}\partial s^i} g_M^1(t,s)_{|t=b} &= 0. \end{aligned}\right\}$$

Since $\gamma + \varepsilon_{n-k} = n-1$, we achieve a diagonal element of $G(t,s)$. So, taking into account Remark 1.2.2, we have:

$$\left.\begin{aligned} \frac{\partial^{\varepsilon_{n-k}}}{\partial t^{\varepsilon_{n-k}}} g_M^2(t,s)_{|(t,s)=(b,b)} &= 0,\\ -\frac{\partial^{\varepsilon_{n-k}+1}}{\partial t^{\varepsilon_{n-k}}\partial s} g_M^2(t,s)_{|(t,s)=(b,b)} + \alpha_0^1(b)\frac{\partial^{\varepsilon_{n-k}}}{\partial t^{\varepsilon_{n-k}}} g_M^2(t,s)_{|(t,s)=(b,b)} &= 0,\\ &\vdots\\ (-1)^\gamma \frac{\partial^{\varepsilon_{n-k}+\gamma}}{\partial t^{\varepsilon_{n-k}}\partial s^\gamma} g_M^2(t,s)_{|(t,s)=(b,b)} + \sum_{i=0}^{\gamma-1}\alpha_i^\gamma(b)\frac{\partial^{i+\varepsilon_{n-k}}}{\partial t^{\varepsilon_{n-k}}\partial s^i} g_M^2(t,s)_{|(t,s)=(b,b)} &= -1. \end{aligned}\right\}$$

Since $\alpha_i^j \in C(I)$, we conclude:

$$y_M^{(\varepsilon_{n-k})}(b) = \frac{\partial^{\varepsilon_{n-k}+\gamma}}{\partial t^{\varepsilon_{n-k}}\partial s^\gamma} g_M^2(t,s)_{|(t,s)=(b,b)} = (-1)^{\gamma+1} = (-1)^{n-\varepsilon_{n-k}}.$$

Analogously, if $k = n-1$, then $y_M^{(\varepsilon_1)}(b) = (-1)^{n-\varepsilon_1}$. □

Now, to study the behaviour of the Green's function at $t=a$ and $t=b$, we need to study the adjoint operator, more specifically, the operator $\widehat{T}_n\big[(-1)^n M\big]$ introduced in (1.3.4). In Section 3.4, we have obtained the expression of the related Green's matrix in (3.4.27) and the boundary conditions related to the associated first order vectorial problem, defined in (3.4.26).

Thus, repeating similar arguments for the following function:

$$\widehat{g}_{(-1)^n M}(t,s) = \begin{cases} \widehat{g}^1_{(-1)^n M}(t,s), & a \leqslant s \leqslant t \leqslant b,\\ \\ \widehat{g}^2_{(-1)^n M}(t,s), & a < t < s < b, \end{cases}$$

which denotes the related Green's function of $\widehat{T}_n\big[(-1)^n M\big]$ in $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$, we obtain similar properties for the Green's function at $t=a$ and $t=b$.

We have the following result, which is proved by using the relation between $g_M(t,s)$ and $\widehat{g}_{(-1)^n M}(t,s)$ given in (1.3.5).

Lemma 3.5.3. *Let us consider* $\tilde{w}_M(s) = \dfrac{\partial^\alpha}{\partial t^\alpha} g_M^2(t,s)_{|t=a}$, *where* α *has been defined in* (3.2.1). *Then, the following assertions are fulfilled.*

- $\tilde{w}_M$ *is a solution of* $\widehat{T}_n\left[(-1)^n M\right]\tilde{w}_M(s) = 0$ *for all* $s \in (a,b]$.
- $\tilde{w}_M$ *satisfies the boundary conditions* (3.4.1)–(3.4.2) *and* (3.4.4)–(3.4.6) *coupled with:*

$$\widehat{w}_M^{(\tau_{n-k})}(a) + \sum_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j}\,\left(p_{n-j}\widehat{w}_M\right)^{(\tau_{n-k}+j-n)}(a) = (-1)^{n-\alpha} = (-1)^{1+\tau_{n-k}}. \quad (3.5.5)$$

Moreover, if $\alpha > 0$, *then:*

$$g_M^2(a,t) = \frac{\partial}{\partial t} g_M^2(t,s)_{|t=a} = \cdots = \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}} g_M^2(t,s)_{|t=a} = 0.$$

Proof. The last assertion follows from Definition 1.2.3.

From (1.3.5), we can consider:

$$\tilde{w}_M(t) = (-1)^n \frac{\partial^\alpha}{\partial s^\alpha}\widehat{g}_M^1(t,s)_{|s=a}.$$

Now, let us repeat the arguments done in Lemma 3.5.3 for the operator $\widehat{T}_n\left[(-1)^n M\right]$. By definition, $\widehat{g}^1_{(-1)^n M}(t,a)$ is a solution of $\widehat{T}_n\left[(-1)^n M\right]\widehat{g}_{(-1)^n M}(t,a) = 0$ for all $t \in (a,b]$ as a function of t. Thus, the first item is directly fulfilled.

Moreover, $\widehat{G}(t,s)$ satisfies:

$$\widehat{B}\,\widehat{G}(a,s) + \widehat{C}\,\widehat{G}(b,s) = 0, \quad \forall s \in (a,b). \quad (3.5.6)$$

If $k < n-1$, let us consider the first row of (3.5.6) to deduce:

$$\left.\begin{array}{r}
\dfrac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} + \displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a} = 0, \\
-\left(\dfrac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} + \displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n+1}}{\partial t^{\tau_1+j-n}\partial s}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a}\right) \\
+\widehat{\alpha}_0^1(s)\left(\dfrac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} + \displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a}\right) = 0, \\
\vdots \\
(-1)^\alpha\displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n+\alpha}}{\partial t^{\tau_1+j-n}\partial s^\alpha}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a} \\
+(-1)^\alpha\left(\dfrac{\partial^{\tau_1+\alpha}}{\partial t^{\tau_1}\partial s^\alpha}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a}\right) + \displaystyle\sum_{i=0}^{\alpha-1}\widehat{\alpha}_i^\alpha(s)\left(\dfrac{\partial^{\tau_1+i}}{\partial t^{\tau_1}\partial s^i}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a}\right. \\
\left. + \displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n+i}}{\partial t^{\tau_1+j-n}\partial s^i}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a}\right) = 0.
\end{array}\right\}$$

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Since $\tau_1 + \alpha < n-1$, we do not reach any diagonal element, hence the previous system is satisfied for $s = a$, and we obtain:

$$\left.\begin{array}{r}
\frac{\partial^{\tau_1}}{\partial t^{\tau_1}} \widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} + \sum\limits_{j=n-\tau_1}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)} = 0, \\
- \sum\limits_{j=n-\tau_1}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_1+j-n+1}}{\partial t^{\tau_1+j-n}\partial s} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)} \quad\quad \\
- \frac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s} \widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} + \widehat{\alpha}^1_0(a) \left(\frac{\partial^{\tau_1}}{\partial t^{\tau_1}} \widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} \right. \quad\quad \\
\left. + \sum\limits_{j=n-\tau_1}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)} \right) = 0, \\
\vdots \quad \\
(-1)^{\alpha} \sum\limits_{j=n-\tau_1}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_1+j-n+\alpha}}{\partial t^{\tau_1+j-n}\partial s^{\alpha}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)} \quad\quad \\
+(-1)^{\alpha} \left(\frac{\partial^{\tau_1+\alpha}}{\partial t^{\tau_1}\partial s^{\alpha}} \widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} \right) + \sum\limits_{i=0}^{\alpha-1} \widehat{\alpha}^{\alpha}_i(a) \left(\frac{\partial^{\tau_1+i}}{\partial t^{\tau_1}\partial s^{i}} \widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} \right. \quad\quad \\
\left. + \sum\limits_{j=n-\tau_1}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_1+j-n+i}}{\partial t^{\tau_1+j-n}\partial s^{i}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)} \right) = 0.
\end{array}\right\}$$

Since $\widehat{\alpha}^j_i \in C(I)$, we conclude that:

$$\widehat{w}_M^{(\tau_1)}(a) + \sum_{j=n-\tau_1}^{n-1} (-1)^{n-j} \left(p_{n-j}\widehat{w}_M\right)^{(\tau_1+j-n)}(a) = 0.$$

Proceeding analogously with $\tau_2, \dots, \tau_{n-k-1}$, we can ensure that $\widehat{w}_M$ satisfies the boundary conditions (3.4.1)–(3.4.2).

Now, let us see what happens for τ_{n-k}. From (3.5.6), we obtain that for all $s \in (a,b)$ the following equalities hold:

$$\left.\begin{array}{r}
\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} + \sum\limits_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a} = 0, \\
- \sum\limits_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+1}}{\partial t^{\tau_{n-k}+j-n}\partial s} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a} \quad\quad \\
- \frac{\partial^{\tau_{n-k}+1}}{\partial t^{\tau_{n-k}}\partial s} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} + \widehat{\alpha}^1_0(s) \left(\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} \right. \quad\quad \\
\left. + \sum\limits_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a} \right) = 0, \\
\vdots \quad \\
(-1)^{\alpha} \sum\limits_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+\alpha}}{\partial t^{\tau_{n-k}+j-n}\partial s^{\alpha}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a} \quad\quad \\
+(-1)^{\alpha} \frac{\partial^{\tau_{n-k}+\alpha}}{\partial t^{\tau_{n-k}}\partial s^{\alpha}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} + \sum\limits_{i=0}^{\alpha-1} \widehat{\alpha}^{\alpha}_i(s) \left(\frac{\partial^{\tau_{n-k}+i}}{\partial t^{\tau_{n-k}}\partial s^{i}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} \right. \quad\quad \\
\left. + \sum\limits_{j=n-\tau_{n-k}}^{n-1} (-1)^{n-j} \frac{\partial^{\tau_{n-k}+j-n+i}}{\partial t^{\tau_{n-k}+j-n}\partial s^{i}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|t=a} \right) = 0.
\end{array}\right\}$$

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In this case, since $\tau_{n-k}+\alpha=n-1$, we reach a diagonal element of $\widehat{G}(t,s)$, hence by Remark 1.2.2, we obtain the following system for $s=a$:

$$\left.\begin{array}{r}
\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} \\
+\sum_{j=n-\tau_{n-k}}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)}=0, \\
-\sum_{j=n-\tau_{n-k}}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_{n-k}+j-n+1}}{\partial t^{\tau_{n-k}+j-n}\partial s}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)} \\
-\frac{\partial^{\tau_{n-k}+1}}{\partial t^{\tau_{n-k}}\partial s}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)}+\widehat{\alpha}^1_0(a)\left(\frac{\partial^{\tau_{n-k}}}{\partial t^{\tau_{n-k}}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)}\right. \\
\left.+\sum_{j=n-\tau_{n-k}}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_{n-k}+j-n}}{\partial t^{\tau_{n-k}+j-n}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)}\right)=0, \\
\vdots \\
(-1)^{\alpha}\sum_{j=n-\tau_{n-k}}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_{n-k}+j-n+\alpha}}{\partial t^{\tau_{n-k}+j-n}\partial s^{\alpha}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)} \\
+(-1)^{\alpha}\frac{\partial^{\tau_{n-k}+\alpha}}{\partial t^{\tau_{n-k}}\partial s^{\alpha}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)}+\sum_{i=0}^{\alpha-1}\widehat{\alpha}^{\alpha}_i(s)\left(\frac{\partial^{\tau_{n-k}+i}}{\partial t^{\tau_{n-k}}\partial s^{i}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)}\right. \\
\left.+\sum_{j=n-\tau_{n-k}}^{n-1}(-1)^{n-j}\frac{\partial^{\tau_{n-k}+j-n+i}}{\partial t^{\tau_{n-k}+j-n}\partial s^{i}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|(t,s)=(a,a)}\right)=1.
\end{array}\right\}$$

Since $\widehat{\alpha}^j_i\in C(I)$, from the definition of $\widehat{w}_M$, we deduce that (3.5.5) is fulfilled.

Now, let us study the behaviour of $\widehat{w}_M$ at $t=b$. Studying the $(n-k+1)^{\text{th}}$ row of (3.5.6), we have for all $s\in(a,b)$:

$$\left.\begin{array}{r}
\frac{\partial^{\delta_1}}{\partial t^{\delta_1}}\widehat{g}^1_{(-1)^n M}(t,s)_{|t=b}+\sum_{j=n-\delta_1}^{n-1}(-1)^{n-j}\frac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}=0, \\
-\left(\frac{\partial^{\delta_1+1}}{\partial t^{\delta_1}\partial s}\widehat{g}^1_{(-1)^n M}(t,s)_{|t=b}+\sum_{j=n-\delta_1}^{n-1}(-1)^{n-j}\frac{\partial^{\delta_1+j-n+1}}{\partial t^{\delta_1+j-n}\partial s}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right) \\
+\widehat{\alpha}^1_0(s)\left(\frac{\partial^{\delta_1}}{\partial t^{\delta_1}}\widehat{g}^1_{(-1)^n M}(t,s)_{|t=b}+\sum_{j=n-\delta_1}^{n-1}(-1)^{n-j}\frac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right)=0, \\
\vdots \\
(-1)^{\alpha}\sum_{j=n-\delta_1}^{n-1}(-1)^{n-j}\frac{\partial^{\delta_1+j-n+\alpha}}{\partial t^{\delta_1+j-n}\partial s^{\alpha}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b} \\
+(-1)^{\alpha}\frac{\partial^{\delta_1+\alpha}}{\partial t^{\delta_1}\partial s^{\alpha}}\widehat{g}^1_{(-1)^n M}(t,s)_{|t=b}+\sum_{i=0}^{\alpha-1}\widehat{\alpha}^{\alpha}_i(s)\left(\frac{\partial^{\delta_1+i}}{\partial t^{\delta_1}\partial s^{i}}\widehat{g}^1_{(-1)^n M}(t,s)_{|t=b}\right. \\
\left.+\sum_{j=n-\delta_1}^{n-1}(-1)^{n-j}\frac{\partial^{\delta_1+j-n+i}}{\partial t^{\delta_1+j-n}\partial s^{i}}\left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right)=0.
\end{array}\right\}$$

Since $b\neq a$, this system is satisfied for $s=a$. Taking into account that $\widehat{\alpha}^j_i\in C(I)$, we conclude:

$$\widehat{w}_M^{(\delta_1)}(b)+\sum_{j=n-\delta_1}^{n-1}(-1)^{n-j}\left(p_{n-j}\widehat{w}_M\right)^{(\delta_1+j-n)}(b)=0.$$

Proceeding analogously with $\delta_2,\dots,\delta_k$, we can affirm that $\widehat{w}_M$ satisfies the boundary conditions (3.4.4)–(3.4.6) and the result is proved. $\square$

Finally, let us introduce an analogous result which describe the behaviour of the Green's function at $t=b$.

Lemma 3.5.4. *Let us consider* $\widehat{y}_M(s)=\dfrac{\partial^\beta}{\partial t^\beta} g_M^2(t,s)_{|t=b}$*, where* β *has been defined in* (3.2.2). *Then, the following assertions are fulfilled.*

- $\tilde{y}_M$ *is a solution of* $\widehat{T}_n\left[(-1)^n M\right]\tilde{y}_M(s)=0$ *for all* $s\in[a,b)$.
- $\tilde{y}_M$ *satisfies the boundary conditions* (3.4.1)–(3.4.3) *and* (3.4.4)–(3.4.5) *coupled with:*

$$\widehat{y}_M^{(\delta_k)}(b)+\sum_{j=n-\delta_k}^{n-1}(-1)^{n-j}\left(p_{n-j}\widehat{y}_M\right)^{(\delta_k+j-n)}(b)=(-1)^{n-\beta+1}=(-1)^{\delta_k}. \tag{3.5.7}$$

Moreover, if $\beta>0$*, then:*

$$g_M^1(b,s)=\frac{\partial}{\partial t}g_M^1(t,s)_{|t=b}=\cdots=\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}g_M^1(t,s)_{|t=b}=0.$$

Proof. The proof is analogous to the one given in Lemma 3.5.3. The last assertion follows from Definition 1.2.3 and the boundary conditions (3.0.2)-(3.0.3).

From (1.3.5), we can consider:

$$\widehat{y}_M(t)=(-1)^n\frac{\partial^\beta}{\partial s^\beta}\widehat{g}_M^2(t,s)_{|s=b}.$$

As in Lemma 3.5.3, first item is fulfilled.
Moreover, let us consider the first row of (3.5.6) to obtain:

$$\left.\begin{array}{r}
\dfrac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^2_{(-1)^nM}(t,s)_{|t=a}+\displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^nM}(t,s)\right)_{|t=a}=0,\\
-\left(\dfrac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s}\widehat{g}^2_{(-1)^nM}(t,s)_{|t=a}+\displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n+1}}{\partial t^{\tau_1+j-n}\partial s}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^nM}(t,s)\right)_{|t=a}\right)\\
+\widehat{\alpha}_0^1(s)\left(\dfrac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^2_{(-1)^nM}(t,s)_{|t=a}+\displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n}}{\partial t^{\tau_1+j-n}}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^nM}(t,s)\right)_{|t=a}\right)=0,\\
\vdots\\
(-1)^\beta\displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n+\beta}}{\partial t^{\tau_1+j-n}\partial s^\beta}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^nM}(t,s)\right)_{|t=a}\\
+(-1)^\beta\dfrac{\partial^{\tau_1+\beta}}{\partial t^{\tau_1}\partial s^\beta}\widehat{g}^2_{(-1)^nM}(t,s)_{|t=a}+\displaystyle\sum_{i=0}^{\beta-1}\widehat{\alpha}_i^\beta(s)\left(\dfrac{\partial^{\tau_1+i}}{\partial t^{\tau_1}\partial s^i}\widehat{g}^2_{(-1)^nM}(t,s)_{|t=a}\right.\\
\left.+\displaystyle\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\dfrac{\partial^{\tau_1+j-n+i}}{\partial t^{\tau_1+j-n}\partial s^i}\left(p_{n-j}(t)\widehat{g}^2_{(-1)^nM}(t,s)\right)_{|t=a}\right)=0.
\end{array}\right\}$$

Since $b\neq a$, this system is satisfied for $s=b$, and taking into account that $\widehat{\alpha}_i^j\in C(I)$, we conclude that:

$$\widehat{y}_M^{(\tau_1)}(a)+\sum_{j=n-\tau_1}^{n-1}(-1)^{n-j}\left(p_{n-j}\widehat{y}_M\right)^{(\tau_1+j-n)}(a)=0.$$

Proceeding analogously with $\tau_2, \ldots, \tau_{n-k}$; we deduce that $\widehat{y}_M$ satisfies the boundary conditions (3.4.1)–(3.4.3).

Now, let us study the behaviour of $\widehat{y}_M$ at $t=b$.

If $k>1$, studying the $(n-k+1)^{\text{th}}$ row of (3.5.6), we have for all $s \in (a,b)$:

$$
\left.\begin{array}{r}
\dfrac{\partial^{\delta_1}}{\partial t^{\delta_1}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b} = 0, \\
-\left(\dfrac{\partial^{\delta_1+1}}{\partial t^{\delta_1}\partial s} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n+1}}{\partial t^{\delta_1+j-n}\partial s} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right) \\
+\widehat{\alpha}^1_0(s)\left(\dfrac{\partial^{\delta_1}}{\partial t^{\delta_1}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right) = 0, \\
\vdots \\
(-1)^{\beta} \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n+\beta}}{\partial t^{\delta_1+j-n}\partial s^{\beta}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b} \\
+(-1)^{\beta} \dfrac{\partial^{\delta_1+\beta}}{\partial t^{\delta_1}\partial s^{\beta}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \displaystyle\sum_{i=0}^{\beta-1} \widehat{\alpha}^{\beta}_i(s)\left(\dfrac{\partial^{\delta_1+i}}{\partial t^{\delta_1}\partial s^{i}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b}\right. \\
\left.+ \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n+i}}{\partial t^{\delta_1+j-n}\partial s^{i}} \left(p_{n-j}(t)\widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right) = 0.
\end{array}\right\}
$$

Since $\delta_1+\beta<n-1$, we do not reach any diagonal element, hence we can express the previous system for $s=b$.

$$
\left.\begin{array}{r}
\dfrac{\partial^{\delta_1}}{\partial t^{\delta_1}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} + \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} = 0, \\
- \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n+1}}{\partial t^{\delta_1+j-n}\partial s} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} \\
-\dfrac{\partial^{\delta_1+1}}{\partial t^{\delta_1}\partial s} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} + \widehat{\alpha}^1_0(b)\left(\dfrac{\partial^{\delta_1}}{\partial t^{\delta_1}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)}\right. \\
\left.+ \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n}}{\partial t^{\delta_1+j-n}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)}\right) = 0, \\
\vdots \\
(-1)^{\beta} \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n+\beta}}{\partial t^{\delta_1+j-n}\partial s^{\beta}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} \\
+(-1)^{\beta} \dfrac{\partial^{\delta_1+\beta}}{\partial t^{\delta_1}\partial s^{\beta}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} + \displaystyle\sum_{i=0}^{\beta-1} \widehat{\alpha}^{\beta}_i(b)\left(\dfrac{\partial^{\delta_1+i}}{\partial t^{\delta_1}\partial s^{i}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)}\right. \\
\left.+ \displaystyle\sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \dfrac{\partial^{\delta_1+j-n+i}}{\partial t^{\delta_1+j-n}\partial s^{i}} \left(p_{n-j}(t)\widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)}\right) = 0.
\end{array}\right\}
$$

By the continuity of α^j_i, we conclude that:

$$
\widehat{y}_M^{(\delta_1)}(b) + \sum_{j=n-\delta_1}^{n-1} (-1)^{n-j} \left(p_{n-j}\widehat{y}_M\right)^{(\delta_1+j-n)}(b) = 0.
$$

Repeating this argument for $\delta_2, \ldots, \delta_{k-1}$,we obtain that $\widehat{y}_M$ satisfies the boundary conditions (3.4.4)–(3.4.5).

Let us see what happens for δ_k. From (3.5.6), we have for all $s \in (a,b)$:

$$\left.\begin{array}{r}
\frac{\partial^{\delta_k}}{\partial t^{\delta_k}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n}}{\partial t^{\delta_k+j-n}} \left(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b} = 0, \\
-\left(\frac{\partial^{\delta_k+1}}{\partial t^{\delta_k}\partial s} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n+1}}{\partial t^{\delta_k+j-n}\partial s} \left(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right) \\
+\widehat{\alpha}^1_0(s) \left(\frac{\partial^{\delta_k}}{\partial t^{\delta_k}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n}}{\partial t^{\delta_k+j-n}} \left(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right) = 0, \\
\vdots \\
(-1)^{\beta} \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n+\beta}}{\partial t^{\delta_k+j-n}\partial s^{\beta}} \left(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b} \\
+(-1)^{\beta} \frac{\partial^{\delta_k+\beta}}{\partial t^{\delta_k}\partial s^{\beta}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} + \sum_{i=0}^{\beta-1} \widehat{\alpha}^{\beta}_i(s) \left(\frac{\partial^{\delta_k+i}}{\partial t^{\delta_k}\partial s^{i}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} \right. \\
\left. + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n+i}}{\partial t^{\delta_k+j-n}\partial s^{i}} \left(p_{n-j}(t) \widehat{g}^1_{(-1)^n M}(t,s)\right)_{|t=b}\right) = 0.
\end{array}\right\}$$

Since $\delta_k+\beta = n-1$, we reach a diagonal element of $\widehat{G}(t,s)$, hence, from Remark 1.2.2, we can express the previous system for $s=b$ as follows:

$$\left.\begin{array}{r}
\frac{\partial^{\delta_k}}{\partial t^{\delta_k}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n}}{\partial t^{\delta_k+j-n}} \left(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} = 0, \\
- \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n+1}}{\partial t^{\delta_k+j-n}\partial s} \left(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} \\
- \frac{\partial^{\delta_k+1}}{\partial t^{\delta_k}\partial s} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} + \widehat{\alpha}^1_0(b) \left(\frac{\partial^{\delta_k}}{\partial t^{\delta_k}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} \right. \\
\left. + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n}}{\partial t^{\delta_k+j-n}} \left(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} \right) = 0, \\
\vdots \\
(-1)^{\beta} \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n+\beta}}{\partial t^{\delta_k+j-n}\partial s^{\beta}} \left(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} \\
(-1)\beta \frac{\partial^{\delta_k+\beta}}{\partial t^{\delta_k}\partial s^{\beta}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} + \sum_{i=0}^{\beta-1} \widehat{\alpha}^{\beta}_i(b) \left(\frac{\partial^{\delta_k+i}}{\partial t^{\delta_k}\partial s^{i}} \widehat{g}^2_{(-1)^n M}(t,s)_{|(t,s)=(b,b)} \right. \\
\left. + \sum_{j=n-\delta_k}^{n-1} (-1)^{n-j} \frac{\partial^{\delta_k+j-n+i}}{\partial t^{\delta_k+j-n}\partial s^{i}} \left(p_{n-j}(t) \widehat{g}^2_{(-1)^n M}(t,s)\right)_{|(t,s)=(b,b)} \right) = -1.
\end{array}\right\}$$

Since $\widehat{\alpha}^j_i \in C(I)$, we conclude that (3.5.7) is fulfilled and the proof is complete. □

From these results, we obtain directly a conclusion about the sign of the Green's function in some cases:

Theorem 3.5.5. *Let us denote $g_M(t,s)$ as the related Green's function of the boundary value problem* (3.0.1)-(3.0.2)

If either $\sigma_k = k-1$ or $\varepsilon_{n-k} = n-k-1$, we have the following properties:

- *If $n-k$ is even, then there is not any $M \in \mathbb{R}$ such that $g_M(t,s) \leqslant$ on $I \times I$.*
- *If $n-k$ is odd, then there is not any $M \in \mathbb{R}$ such that $g_M(t,s) \geqslant 0$ on $I \times I$.*

Proof. If $\sigma_k = k-1$, then $\{\sigma_1, \dots, \sigma_k\} = \{0, \dots, k-1\}$.

We consider w_M introduced in Lemma 3.5.1. Applying that result for this particular case, we have

$$w_M(a) = \cdots = w_M^{(k-2)}(a) = 0, \quad w_M^{(k-1)}(a) = (-1)^{n-\sigma_k-1} = (-1)^{n-k}.$$

Hence, if $n-k$ is even, then there exists $\rho > 0$, such that $w_M(t) > 0$ for all $t \in (a, a+\rho)$. So, there exist $t_0 \in (a, a+\rho)$ and $\omega > 0$, such that $g_M(t_0, s) > 0$ for all $s \in (a, a+\omega)$.

Now, if $n-k$ is odd, then there exists $\rho > 0$, such that $w_M(t) < 0$ for all $t \in (a, a+\rho)$. Thus, there are $t_0 \in (a, a+\rho)$ and $\omega > 0$, such that $g_M(t_0, s) < 0$ for all $s \in (a, a+\omega)$.

Analogously, if $\varepsilon_{n-k} = n-k-1$, then $\{\varepsilon_1, \dots, \varepsilon_{n-k}\} = \{0, \dots, n-k-1\}$ and $\gamma = n - \varepsilon_{n-k} - 1 = k$.

We consider now y_M introduced in Lemma 3.5.2. Applying that result to this particular case, we conclude that for all $M \in \mathbb{R}$, y_M satisfies the following boundary conditions:

$$y_M(b) = \cdots = y_M^{(n-k-2)}(b) = 0, \quad y_M^{(n-k-1)}(b) = (-1)^{n-\varepsilon_{n-k}} = (-1)^{k+1}.$$

Hence, if $n-k$ and k are even, then there exists $\rho > 0$, such that $y_M(t) > 0$ for all $t \in (b-\rho, b)$. So, there exist $t_0 \in (b-\rho, b)$ and $\omega > 0$, such that $g_M(t_0, s) > 0$ for all $s \in (b-\omega, b)$.

Moreover, if $n-k$ is even and k odd, then there exists $\rho > 0$, such that $y_M(t) < 0$ for all $t \in (b-\rho, b)$. Thus, there are $t_0 \in (b-\rho, b)$ and $\omega > 0$, such that $g_M(t_0, s) > 0$ for all $s \in (b-\omega, b)$.

Now, if both $n-k$ and k are odd, then there exists $\rho > 0$, such that $y_M(t) > 0$ for all $t \in (b-\rho, b)$. Hence, there are $t_0 \in (b-\rho, b)$ and $\omega > 0$, such that $g_M(t_0, s) < 0$ for all $s \in (b-\omega, b)$.

Finally, if $n-k$ is odd and k even, then there exists $\rho > 0$, such that $y_M(t) < 0$ for all $t \in (b-\rho, b)$. As a consequence, there exist $t_0 \in (b-\rho, b)$ and $\omega > 0$, such that $g_M(t_0, s) < 0$ for all $s \in (b-\omega, b)$. $\square$

Remark 3.5.6. *Realise that along this section we have not imposed any hypotheses neither on the operator nor in the boundary conditions.*

3.6 Constant sign solutions

In Section 3.5, we have obtained the boundary conditions satisfied for certain derivatives of the Green's function at some subsets of $I \times I$. Is this section we study the solutions of $T_n[M]\, u(t) = 0$ and $\widehat{T}_n\big[(-1)^n M\big]\, v(t) = 0$ which satisfy the same boundary conditions as the obtained in the previous section.

Our aim is to characterise the parameter set where these solutions are of constant sign. Such a characterisation is given by means of spectral theory.

Firstly, we prove the existence of eigenvalues of $T_n[\bar{M}]$ in different sets of definition. In a second subsection we study the solutions of:

$$T_n[M]\, u(t) = 0, \quad t \in I,$$

coupled with the same boundary conditions as w_M or y_M, respectively. After that, we study the solutions of:

$$\widehat{T}_n\big[(-1)^n M\big]v(t) = 0, \quad t \in I,$$

coupled with the same boundary conditions as $\widehat{w}_M$ or $\widehat{y}_M$, respectively.

3.6.1 Eigenvalues and related eigenfunctions of $T_n[\bar{M}]$

In the sequel we introduce the sets which we will study:

$$X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}},\ X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}},\ X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}},\ X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}} \text{ and } X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}.$$

Firstly, let us see a result which allows us to affirm that, by assuming that property (T_d) is fulfilled on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, operator $T_n[\bar{M}]$ satisfies such a property in all these spaces.

Remark 3.6.1. *Remember that the bar in the set of indices indicates that they are not necessarily placed in cardinal order.*

Lemma 3.6.2. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Then the following properties are fulfilled.*

- *$T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *$T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
- *If $\sigma_k \neq k-1$, $T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
- *If $\varepsilon_{n-k} \neq n-k-1$, $T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*

Proof. The proof follows trivially from Lemmas 3.3.6 and 3.3.7, taking into account that under our hypotheses, from (3.3.3) we have

$$T_\alpha\, u(a) = \frac{u^{(\alpha)}(a)}{v_1(a)\dots v_\alpha(a)}, \quad T_\beta\, u(b) = \frac{u^{(\beta)}(b)}{v_1(b)\dots v_\beta(b)}. \tag{3.6.1}$$

□

Remark 3.6.3. *Realise that if $\sigma_k = k-1$ ($\varepsilon_{n-k} = n-k-1$) then $\alpha = k$ ($\beta = n-k$). So, if $u \in X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ $\left(u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}\right)$, then (3.6.1) can be false.*

Example 3.6.4. *Let us consider the fourth order operator $T_4[M]$. In Example 3.4.7, we have seen that if $T_4[M]$ satisfies (T_d) in $X_{\{0,2\}}^{\{1,2\}}$, then (3.4.21)-(3.4.22) are fulfilled. Let us see that, in such a case, (T_d) also holds in $X_{\{0,1,2\}}^{\{1\}}$, $X_{\{0\}}^{\{0,1,2\}}$, $X_{\{0,1\}}^{\{1,2\}}$ and $X_{\{0,2\}}^{\{0,1\}}$.*

- $X^{\{1\}}_{\{0,1,2\}}$

  Trivially, since $T_\ell u(t)$ is a linear combination of $u(t), \ldots, u^{(\ell)}(t)$, we have that

  $$T_0 u(a) = T_1 u(a) = T_2 u(a) = 0.$$

  Moreover, from (3.3.8), $T_1 u(b) = -\dfrac{v_1'(b)}{v_1^2(b)} u(b) = 0.$

- $X^{\{0,1,2\}}_{\{0\}}$

  Clearly, due to the fact that $T_\ell u(t)$ is a linear combination of $u(t), \ldots, u^{(\ell)}(t)$, we have that $T_0 u(a) = T_0 u(b) = T_1 u(b) = T_2 u(b) = 0$.

- $X^{\{1,2\}}_{\{0,1\}}$

  Directly, $T_0 u(a) = T_1 u(a) = 0$.

  From (3.3.8) *and Example 3.3.4, $T_1 u(b) = \dfrac{-v_1'(b)}{v_1^2(b)} u(b) = 0$ and*

  $$T_2 u(b) = \frac{v_1(b)\, v_1'(b)\, v_2'(b) + v_2(b)\, \left(2{v_1'}^2(b) - v_1(b)\, v_1''(b)\right)}{v_1^3(b)\, v_2^2(b)} u(b) = 0.$$

- $X^{\{0,1\}}_{\{0,2\}}$

  Straightforward, $T_0 u(a) = T_0 u(b) = T_1 u(b) = 0$.

  Finally, from Example 3.3.4, $T_2 u(a) = -\dfrac{2 v_2(a)\, v_1'(a) + v_1(a)\, v_2'(a)}{v_1^2(a)\, v_2^2(a)} u'(a) = 0.$

As a consequence we arrive to the following corollary.

Corollary 3.6.5. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$, and $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ satisfy (N_a). Then the following assertions are fulfilled.*

- *If $n-k$ is even:*

  * *$T_n[\bar{M}]$ is strongly inverse positive and satisfies condition (P_g) in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$. If, in addition, $\sigma_k \neq k-1$, these properties are satisfied in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_{k-1}|\alpha\}}$. Moreover, if $\varepsilon_{n-k} \neq n-k-1$, these properties are also fulfilled in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\ldots,\sigma_k\}}$.*

  * *$T_n[\bar{M}]$ is strongly inverse negative and satisfies condition (N_g) in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\ldots,\sigma_{k-1}\}}$, provided $k > 1$, and $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\ldots,\sigma_k|\alpha\}}$, whenever $k < n-1$.*

- *If $n-k$ is odd:*

* $T_n[\bar{M}]$ *is strongly inverse negative and satisfies condition* (N_g) *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *If, in addition,* $\sigma_k \neq k-1$, *these properties are satisfied in* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *Moreover, if* $\varepsilon_{n-k} \neq n-k-1$, *these properties are also fulfilled in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.

* $T_n[\bar{M}]$ *is strongly inverse positive and satisfies condition* (P_g) *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$, *provided* $k > 1$, *and* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$, *whenever* $k < n-1$.

Proof. It is obvious that if $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a), then the sets of indices $\{\sigma_1,\dots,\sigma_k|\alpha\} - \{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}$and $\{\sigma_1,\dots,\sigma_{k-1}\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}$ also do.

Moreover, if $\sigma_k \neq k-1$, then $\alpha < \sigma_k$ and if $\varepsilon_{n-k} \neq n-k-1$, then $\beta < \varepsilon_{n-k}$. So, if $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a), then $\{\sigma_1,\dots,\sigma_{k-1}|\alpha\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}$ also do.

Thus, using Theorem 3.3.10 and Lemma 3.6.2, the result is proved. □

Now, from the previous Corollary and the first assertion on Theorems 1.2.10 and 1.2.11, we obtain, as a direct consequence, the following result.

Corollary 3.6.6. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a). *Then the following assertions are fulfilled.*

- *If* $n-k$ *is even:*
  - *There is* $\lambda_1 > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_1 *on* (a,b).
  - *If* $k > 1$, *there exists* $\lambda_2' < 0$, *the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_2' *on* (a,b).
  - *If* $k < n-1$, *there is* $\lambda_2'' < 0$, *the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_2'' *on* (a,b).
  - *If* $\sigma_k \neq k-1$, *there is* $\lambda_3' > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_3' *on* (a,b).
  - *If* $\varepsilon_{n-k} \neq n-k-1$, *there is* $\lambda_3'' > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_3'' *on* (a,b).
- *If* $n-k$ *is odd:*
  - *There exists* $\lambda_1 < 0$, *the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *Moreover, there is a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_1 *on* (a,b).

– *If* $k > 1$, *there exists* $\lambda_2' > 0$, *the least positive eigenvalue of operator* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_2' *on* (a,b).

– *If* $k < n-1$, *there is* $\lambda_2'' > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_2'' *on* (a,b).

– *If* $\sigma_k \neq k-1$, *there is* $\lambda_3' < 0$, *the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_3' *on* (a,b).

– *If* $\varepsilon_{n-k} \neq n-k-1$, *there is* $\lambda_3'' < 0$, *the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$. *Moreover, there exists a non-trivial constant sign eigenfunction corresponding to the eigenvalue* λ_3'' *on* (a,b).

Example 3.6.7. *Continuing the study of the particular operator* $T_4^0[M]u(t) = u^{(4)} + M\,u(t)$ *introduced in Example 3.3.13, we can affirm the existence of the eigenvalues of* $T_4^0[0]$ *in the different spaces introduced in Example 3.6.4 and the related constant sign eigenfunctions.*

In the sequel, we obtain those eigenvalues and related eigenfunctions.

- *The eigenvalues of* $T_4^0[0]$ *in* $X_{\{0,2\}}^{\{1,2\}}$ *are given by* $\lambda = m^4$, *where* m *is a positive solution of the following equation:*

$$\tan(m) + \tanh(m) = 0. \tag{3.6.2}$$

The least positive eigenvalue is $\lambda_1 = m_1^4 \approx 2,36502^4$, *where* m_1 *is the least positive solution of* (3.6.2). *The related constant sign eigenfunctions are given by:*

$$u(t) = K\left(\frac{\sinh(m_1\,t)}{\cosh(m_1)} - \frac{\sin(m_1\,t)}{\cos(m_1)}\right),$$

where $K \in \mathbb{R}\backslash\{0\}$.

- *The biggest negative eigenvalue of* $T_4^0[0]$ *in* $X_{\{0,1,2\}}^{\{1\}}$ *is* $\lambda_2'' = -4\,\pi^4$. *The related constant sign eigenfunctions are:*

$$u(t) = K\left(\cosh(\pi\,t)\,\sin(\pi\,t) - \cos(\pi\,t)\,\sinh(\pi\,t)\right),$$

where $K \in \mathbb{R}\backslash\{0\}$.

- *The eigenvalues of* $T_4^0[0]$ *in* $X_{\{0\}}^{\{0,1,2\}}$ *are given by* $\lambda = -m^4$, *where* m *is a positive solution of the following equation:*

$$\tan\left(\frac{m}{\sqrt{2}}\right) - \tanh\left(\frac{m}{\sqrt{2}}\right) = 0. \tag{3.6.3}$$

The biggest negative eigenvalue is $\lambda_2' = -m_2^4 \approxeq -5,550305^4$*, where* m_2 *is the least positive solution of* (3.6.3)*. The related constant sign eigenfunctions are given by:*

$$u(t) = K\left(\cosh\left(\frac{m_2}{\sqrt{2}}(1-t)\right)\sin\left(\frac{m_2}{\sqrt{2}}(1-t)\right) - \cos\left(\frac{m_2}{\sqrt{2}}(1-t)\right)\sinh\left(\frac{m_2}{\sqrt{2}}(1-t)\right)\right),$$

where $K \in \mathbb{R}\backslash\{0\}$*.*

- *The eigenvalues of* $T_4^0[0]$ *in* $X_{\{0,2\}}^{\{0,1\}}$ *are given by* $\lambda = m^4$*, where* m *is a positive solution of the following equation:*

$$\tan(m) - \tanh(m) = 0. \tag{3.6.4}$$

The least positive eigenvalue is $\lambda_3'' = m_3^4 \approxeq 3,9266^4$*, where* m_3 *is the least positive solution of* (3.6.4)*. The related constant sign eigenfunctions are given by:*

$$u(t) = K\left(\frac{\sinh(m_3\, t)}{\cosh(m_3)} - \frac{\sin(m_3\, t)}{\cos(m_3)}\right),$$

where $K \in \mathbb{R}\backslash\{0\}$*.*

- *The least positive eigenvalue of* $T_4^0[0]$ *in* $X_{\{0,1\}}^{\{1,2\}}$ *is* $\lambda_3' = \pi^4$*. The related constant sign eigenfunctions are given by:*

$$u(t) = K\, e^{-\pi(t+1)}\left(e^{2\pi t} + e^{\pi t}\left((e^\pi - 1)\sin(\pi t) + (-1 - e^\pi)\cos(\pi t)\right) + e^\pi\right),$$

where $K \in \mathbb{R}\backslash\{0\}$*.*

3.6.2 Constant sign solutions of $T_n[M]\, u(t) = 0$

Now, we introduce some results that provide sufficient conditions to ensure that suitable solutions of equation (1.0.3) are of constant sign.

Proposition 3.6.8. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a)*. If* $u \in C^n(I)$ *is a solution of* (1.0.3) *on* (a,b)*, satisfying the boundary conditions* (3.5.1)*, then it does not have any zero on* (a,b) *provided that one of the following assertions is satisfied.*

- *Let* $n-k$ *be even:*
  - *If* $k > 1$*,* $\sigma_k \neq k-1$ *and* $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2']$*, where:*
    - $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$*.*
    - $\lambda_2' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$*.*
  - *If* $k = 1$*,* $\sigma_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_3', +\infty)$*, where:*
    - $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$*, where* $\alpha = 0$*.*

  - *If $k > 1$, $\sigma_k = k-1$ and $M \in [\bar{M} - \lambda_1, \bar{M} - \lambda_2']$, where:*
    - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\lambda_2' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,k-2\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  - *If $k = 1$, $\sigma_1 = 0$ and $M \in [\bar{M} - \lambda_1, +\infty)$, where:*
    - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.

- *Let $n-k$ be odd:*
  - *If $k > 1$, $\sigma_k \neq k-1$ and $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_3']$, where:*
    - $\lambda_3' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  - *If $k = 1$, $\sigma_1 \neq 0$ and $M \in (-\infty, \bar{M} - \lambda_3']$, where:*
    - $\lambda_3' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$, *where* $\alpha = 0$.
  - *If $k > 1$, $\sigma_k = k-1$ and $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_1]$, where:*
    - $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,k-2\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  - *If $k = 1$, $\sigma_1 = 0$ and $M \in (-\infty, \bar{M} - \lambda_1]$, where:*
    - $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. At first, let us see that for $M = \bar{M}$ every solution of (1.0.3) on (a,b) satisfying the boundary conditions (3.5.1) does not have any zero on (a,b). On the proof of Lemma 3.3.9 we have seen that, without taking into account the boundary conditions, every solution of (1.0.3) for $M = \bar{M}$ has at most $n-1$ zeros on (a,b). Let us prove that these $n-1$ possible zeros are not attained because of the boundary conditions.

Let us denote $u_M \in C^n(I)$ as a solution of (1.0.3) satisfying the boundary conditions (3.5.1).

Each time that either $T_{n-\ell}\, u_M(a) = 0$ or $T_{n-\ell}\, u_M(b) = 0$ for $\ell = 1, \dots n$, a possible oscillation is lost.

Since $T_n[\bar{M}]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, by applying Lemmas 3.3.6 and 3.3.7 we conclude that for every $M \in \mathbb{R}$:

$$T_{\sigma_1}\, u_M(a) = \cdots = T_{\sigma_{k-1}}\, u_M(a) = 0, \tag{3.6.5}$$

$$T_{\varepsilon_1}\, u_M(b) = \cdots = T_{\varepsilon_{n-k}}\, u_M(b) = 0. \tag{3.6.6}$$

In particular, this property holds for $M = \bar{M}$. Hence, we lose the $n-1$ possible zeros and we can affirm that $u_{\bar{M}}$ does not vanish on (a,b).

Now, in order to prove the result, let us move u_M in a continuous way with M in a neighbourhood of $\bar{M}$. We have that u_M is a solution of (1.0.3) on (a,b), hence:

$$T_n[\bar{M}]\, u_M(t) = (\bar{M} - M)\, u_M(t), \quad t \in (a,b). \tag{3.6.7}$$

First, let us see that while u_M is of constant sign it cannot have any double zero on (a,b).

Let us assume that $u_{\bar{M}} > 0$ on (a,b) (if $u_{\bar{M}} < 0$ on (a,b) the arguments are applicable by multiplying by -1). Thus, in equation (3.6.7) we have:

$$\begin{cases} T_n[\bar{M}]\, u_M(t) \geqslant 0, \quad \forall\, t \in (a,b), \quad \text{if } M < \bar{M}, \\ T_n[\bar{M}]\, u_M(t) \leqslant 0, \quad \forall\, t \in (a,b), \quad \text{if } M > \bar{M}. \end{cases} \tag{3.6.8}$$

In both cases, $T_n[\bar{M}]\, u_M$ is a constant sign function. Then, since $v_1 \dots v_n > 0$, $T_{n-1}\, u_M$ is a monotone function with, at most, one zero.

Under analogous arguments, we conclude that $T_{n-\ell}\, u_M$ has at most ℓ zeros, for every $\ell = 1, \dots, n$. In particular, u_M can have at most n zeros.

But, u_M satisfies (3.5.1), i.e., $n-1$ possible oscillations are lost. Thus u_M is only allowed to have a simple zero on (a,b), but this is not possible while it is of constant sign.

Let us assume that $k > 1$ and $\sigma_k \neq k-1$. In such a case, we can affirm that u_M is of constant sign up to one of the two following boundary conditions is satisfied:

$$u_M^{(\alpha)}(a) = 0 \quad \text{or} \quad u_M^{(\beta)}(b) = 0.$$

Now, in order to see when the sign change begins, let us study the problem with different signs of M. Since we are considering $u_M \geqslant 0$, it is obvious that:

$$u_M^{(\alpha)}(a) \geqslant 0, \quad \text{and} \quad \begin{cases} u_M^{(\beta)}(b) \geqslant 0, \quad \text{if } \beta \text{ is even}, \\ u_M^{(\beta)}(b) \leqslant 0, \quad \text{if } \beta \text{ is odd}. \end{cases} \tag{3.6.9}$$

Let us study the behaviour of $u_M^{(\alpha)}(a)$ and $u_M^{(\beta)}(b)$, to keep the maximal oscillation, considered as in Notation 3.3.11. In this case, the maximum number of zeros which u can have, taking into account the boundary conditions (3.5.1) is 1. Then, a zero on the boundary is allowed without implying that $u \equiv 0$. If $T_{n-\ell}\, u_M(a) = 0$ for $\ell \neq n-\alpha$ and $n-\ell \notin \{\sigma_1, \dots, \sigma_{k-1}\}$ or $T_{n-\ell}\, u_M(b) = 0$ for $\ell \neq n-\beta$ and $n-\ell \notin \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$, then the maximum number of zeros which u can have is 0 and we cannot have more zeros on the boundary for any non-trivial solution of (1.0.3). Therefore, let us assume that we find the only zero which is allowed either at $T_\alpha\, u_M(a)$ or $T_\beta\, u_M(b)$.

At first, consider $M < \bar{M}$, we have that $T_n[\bar{M}]\, u_M \geqslant 0$, hence, with maximal oscillation, if $T_{n-\ell} u_M(a) \neq 0$ and $T_{n-\ell} u_M(b) \neq 0$ for all $\ell = 1, \dots n$, then (3.3.11) is satisfied.

However, each time that $T_{n-\ell} u_M(a) = 0$, the sign change come on the next $\tilde{\ell}$ for which $T_{n-\tilde{\ell}} u_M(a) \neq 0$. And, if $T_{n-\ell} u_M(b) = 0$, it changes its sign on the next $\tilde{\ell}$ for which $T_{n-\tilde{\ell}} u_M(b) \neq 0$ many times as it has vanished, see Figure 3.3.2. From $\ell = 1$ to $n-\alpha$ there are $k-1-\alpha$ zeros for $T_{n-\ell} u_M(a)$ and from $\ell = 1$ to $n-\beta$ there are $n-k-\beta$ zeros for $T_{n-\ell} u_M(b)$. Hence, to allow the maximal oscillation it is necessary that:

$$\begin{cases} T_\alpha\, u_M(a) \geqslant 0, \quad \text{if } n-\alpha-(k-\alpha-1) = n-k-1 \text{ is even}, \\ T_\alpha\, u_M(a) \leqslant 0, \qquad\qquad \text{if } n-k-1 \text{ is odd}, \end{cases} \tag{3.6.10}$$

and,

$$\begin{cases} T_\beta \, u_M(b) \geqslant 0, & \text{if } n-k-\beta \text{ is even,} \\ \\ T_\beta \, u_M(b) \leqslant 0, & \text{if } n-k-\beta \text{ is odd.} \end{cases} \tag{3.6.11}$$

From (3.6.1), we can affirm that with maximal oscillation:

$$\begin{cases} u_M^{(\alpha)}(a) \geqslant 0, & \text{if } n-k \text{ is odd,} \\ \\ u_M^{(\alpha)}(a) \leqslant 0, & \text{if } n-k \text{ is even,} \end{cases}$$

and,

- If $n-k$ is even:
$$\begin{cases} u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is even,} \\ \\ u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$
- If $n-k$ is odd:
$$\begin{cases} u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is even,} \\ \\ u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

Hence, we arrive at the following conclusions, taking into account (3.6.9):

- If $n-k$ is even, the maximal oscillation is not allowed for u_M whenever $u_N^{(\alpha)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for all M which belongs to $[\bar{M} - \lambda_3', \bar{M}]$.
- If $n-k$ is odd, the maximal oscillation is not allowed for u_M whenever $u_N^{(\beta)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for all M in $[\bar{M} - \lambda_2', \bar{M}]$.

Now, considering $M > \bar{M}$, we have that $T_n[\bar{M}]\, u_M \leqslant 0$ on I, hence with maximal oscillation, if $T_{n-\ell} u_M(a) \neq 0$ and $T_{n-\ell} u_M(b) \neq 0$, for all $\ell = 1, \dots n$, the following inequalities are satisfied:

$$\begin{cases} T_{n-\ell}\, u_M(a) < 0, & \text{if } \ell \text{ is even,} \\ \\ T_{n-\ell}\, u_M(a) > 0, & \text{if } \ell \text{ is odd,} \end{cases} \qquad T_{n-\ell}\, u_M(b) < 0. \tag{3.6.12}$$

In this case, since we have contrary signs to the previous case, where $M < \bar{M}$, to allow the maximal oscillation, the following inequalities must be satisfied:

$$\begin{cases} T_\alpha \, u_M(a) \leqslant 0, & \text{if } n-k-1 \text{ is even,} \\ \\ T_\alpha \, u_M(a) \geqslant 0, & \text{if } n-k-1 \text{ is odd,} \end{cases} \tag{3.6.13}$$

and,

$$\begin{cases} T_\beta\, u_M(b) \leqslant 0, & \text{if } n-k-\beta \text{ is even,} \\ T_\beta\, u_M(b) \geqslant 0, & \text{if } n-k-\beta \text{ is odd.} \end{cases} \tag{3.6.14}$$

Hence, from (3.6.1), we can affirm that with maximal oscillation:

$$\begin{cases} u_M^{(\alpha)}(a) \leqslant 0, & \text{if } n-k \text{ is odd,} \\ u_M^{(\alpha)}(a) \geqslant 0, & \text{if } n-k \text{ is even,} \end{cases}$$

and,

- If $n-k$ is even:

$$\begin{cases} u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is even,} \\ u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

- If $n-k$ is odd:

$$\begin{cases} u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is even,} \\ u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

Hence, we arrive at the following conclusions, taking into account (3.6.9):

- If $n-k$ is even, the maximal oscillation is not allowed for u_M whenever $u_N^{(\beta)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for all M in $[\bar{M}, \bar{M}-\lambda_2']$.

- If $n-k$ is odd, the maximal oscillation is not allowed for u_M whenever $u_N^{(\alpha)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for all M which is in $[\bar{M}, \bar{M}-\lambda_3']$.

The proof is complete since if $k=1$, $u_M^{(\beta)}(b) \neq 0$ for every $M \neq \bar{M}$, because the contrary will imply that u_M is a non-trivial solution of the linear differential equation (1.0.3) with a zero of multiplicity n at $t=b$ and this is not possible.

And, if $\sigma_k = k-1$, consider $u_M^{(k-1)}(a)$ instead of $u_M^{(\alpha)}(a) = u^{(k)}(a)$, since it is the first non-null derivative at $t=a$. Since $u_M \geqslant 0$, then $u_M^{(k-1)}(a) \geqslant 0$. But, with maximal oscillation, $T_{k-1}\, u_M(a)$ fulfils (3.3.11) if $M < \bar{M}$ and (3.6.12) if $M > \bar{M}$ for $\ell = n-k-1$. Hence, from (3.6.1), we can affirm that, with maximal oscillation, the following inequalities must be fulfilled:

- If $M < \bar{M}$:

$$\begin{cases} u_M^{(k-1)}(a) \geqslant 0, & \text{if } n-k \text{ is odd,} \\ u_M^{(k-1)}(a) \leqslant 0, & \text{if } n-k \text{ is even.} \end{cases}$$

- If $M > \bar{M}$:
$$\begin{cases} u_M^{(k-1)}(a) \leqslant 0, & \text{if } n-k \text{ is odd,} \\ \\ u_M^{(k-1)}(a) \geqslant 0, & \text{if } n-k \text{ is even.} \end{cases}$$

And, we conclude the proof as follows.

- If $n-k$ is even and $M < \bar{M}$, the maximal oscillation is not allowed for u_M whenever $u_N^{(k-1)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for all $M \in [\bar{M} - \lambda_1, \bar{M}]$.

- If $n-k$ is odd and $M > \bar{M}$, the maximal oscillation is not allowed for u_M whenever $u_N^{(n-k-1)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $u_M > 0$ on (a,b) for all $M \in [\bar{M}, \bar{M} - \lambda_1]$.

□

Example 3.6.9. *From Proposition 3.6.8 and Example 3.6.7, we can affirm that any non-trivial solution of $T_4^0[M] \equiv u^{(4)}(t) + M\,u(t) = 0$ on $[0,1]$, satisfying the boundary conditions:*

$$u(0) = u'(1) = u''(1) = 0,$$

does not have any zero on $(0,1)$ for $M \in [-\pi^4, m_2^4]$, where $m_2^4 = -\lambda_2'$ with λ_2' the first negative eigenvalue of $T_4^0[0]$ in $X_{\{0\}}^{\{0,1,2\}}$ and m_2 has been introduced in Example 3.6.7 as the least positive solution of (3.6.3)*.*

Such functions are given as multiples of the following expression:

$$\begin{cases} \begin{array}{c} \cos(m-mt)\big(\sin(m)-\sinh(m)\big)+\sin(m-mt)\big(-\cos(m)-\cosh(m)\big) \\ +\sinh(m-mt)\big(\cos(m)+\cosh(m)\big)+\cosh(m-mt)\big(\sin(m)-\sinh(m)\big), \end{array} & M=-m^4<0, \\ \\ t^3-3t^2+3t, & M=0, \\ \\ \begin{array}{c} e^{-\frac{mt}{\sqrt{2}}}\left(-\left(e^{\sqrt{2}m(t-1)}+e^{\sqrt{2}mt}+e^{\sqrt{2}m}+1\right)\sin\left(\frac{mt}{\sqrt{2}}\right)\right. \\ \left.+\left(e^{\sqrt{2}mt}-1\right)\cos\left(\frac{m(t-2)}{\sqrt{2}}\right)+\left(e^{\sqrt{2}mt}-1\right)\cos\left(\frac{mt}{\sqrt{2}}\right)\right), \end{array} & M=m^4>0. \end{cases}$$

Now, we enunciate a similar result, which refers to the eigenvalues in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$ and $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.

Proposition 3.6.10. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). If $u \in C^n(I)$ is a solution of* (1.0.3) *on (a,b) satisfying the boundary conditions* (3.5.3)*, then it does not have any zero on (a,b) provided that one of the following assertions is satisfied.*

- *Let $n-k$ be even:*
  - *If $\varepsilon_{n-k} \neq n-k-1$ and $M \in [\bar{M} - \lambda_3'', \bar{M} - \lambda_2'']$, where:*

  * $\lambda_3'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.
  * $\lambda_2'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
- *If* $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M}-\lambda_1, \bar{M}-\lambda_2'']$, *where:*
  * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.
  * $\lambda_2'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{0,\dots,k-2\}}$.

- *Let* $n-k$ *be odd:*
  - *If* $k < n-1$, $\varepsilon_{n-k} \neq n-k-1$ *and* $M \in [\bar{M}-\lambda_2'', \bar{M}-\lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.
    * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
  - *If* $k = n-1$, $\varepsilon_1 \neq 0$ *and* $M \in (-\infty, \bar{M}-\lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\beta\}}$, *where* $\beta = 0$.
  - *If* $k < n-1$, $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M}-\lambda_2'', \bar{M}-\lambda_1]$, *where:*
    * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,1,\dots,n-k-1\}}$.
    * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
  - *If* $k = n-1$, $\varepsilon_1 = 0$ *and* $M \in (-\infty, \bar{M}-\lambda_1]$, *where:*
    * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{0\}}$.

Proof. The proof is analogous to the one of Proposition 3.6.8. □

Example 3.6.11. *Now, consider the fourth order differential equation* $u^{(4)}(t) + M\,u(t) = 0$ *coupled with the boundary conditions* $u(0) = u''(0) = u'(1) = 0$. *Using Proposition 3.6.10 and Example 3.6.7, we conclude that such functions do not have any zero on* $(0,1)$ *if* M *belongs to* $[-m_3^4, 4\,\pi^4]$, *where* $m_3^4 = -\lambda_2''$, *with* λ_2'' *the first negative eigenvalue of* $T_4^0[0]$ *in* $X_{\{0,1,2\}}^{\{1\}}$ *and* m_3 *has been introduced in Example 3.6.7 as the least positive solution of* (3.6.4).

It is not difficult to verify that the solutions of this problem are given as multiples of the following expression:

$$\begin{cases} \dfrac{\sin(m\,t)}{\cos(m)} - \dfrac{\sinh(m\,t)}{\cosh(m)}, & M = -m^4 < 0, \\ t^3 - 3t, & M = 0, \\ e^{-\frac{mt}{\sqrt{2}}}\left(\left(e^{\sqrt{2}m(t+1)}+1\right)\sin\left(\dfrac{m(t-1)}{\sqrt{2}}\right) + \left(e^{\sqrt{2}mt}+e^{\sqrt{2}m}\right)\sin\left(\dfrac{m(t+1)}{\sqrt{2}}\right)\right. & \\ \left. + \left(1-e^{\sqrt{2}m(t+1)}\right)\cos\left(\dfrac{m(t-1)}{\sqrt{2}}\right) + \left(e^{\sqrt{2}m}-e^{\sqrt{2}mt}\right)\cos\left(\dfrac{m(t+1)}{\sqrt{2}}\right)\right), & M = m^4 > 0. \end{cases}$$

To finish this section, we show a result which gives an order on the previously obtained eigenvalues λ_1, λ_3' and λ_3''. First, let us introduce some notation.

Notation 3.6.12. *Let us denote $\alpha_1 \in \{1, \ldots, n-1\}$ such that $\alpha_1 \notin \{\sigma_1, \ldots, \sigma_{k-1}|\alpha\}$ and $\{0, \ldots, \alpha_1 - 1\} \subset \{\sigma_1, \ldots, \sigma_{k-1}|\alpha\}$.*

Let us denote $\beta_1 \in \{1, \ldots, n-1\}$ such that $\beta_1 \notin \{\varepsilon_1, \ldots, \varepsilon_{n-k-1}|\beta\}$ and, moreover, $\{0, \ldots, \beta_1 - 1\} \subset \{\varepsilon_1, \ldots, \varepsilon_{n-k-1}|\beta\}$.

Proposition 3.6.13. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$ and $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ satisfy (N_a). Then the following assertions are fulfilled.*

- *Let $n-k$ be even, we have:*
  - *If $\sigma_k \neq k-1$, then $\lambda_3' > \lambda_1 > 0$, where:*
    - *$\lambda_3' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*

    Moreover if there exists $\lambda_1' > \lambda_1$ another eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$, then $\lambda_1' > \lambda_3'$.
  - *If $\varepsilon_{n-k} \neq n-k-1$, then $\lambda_3'' > \lambda_1 > 0$, where:*
    - *$\lambda_3'' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k-1}|\beta\}}$.*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*

    Moreover if there exists $\lambda_1' > \lambda_1$ another eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$, then $\lambda_1' > \lambda_3''$.
- *Let $n-k$ be odd, we have:*
  - *If $\sigma_k \neq k-1$, then $\lambda_3' < \lambda_1 < 0$, where:*
    - *$\lambda_3' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*

    Moreover if there exists $\lambda_1' < \lambda_1$ another eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$, then $\lambda_1' < \lambda_3'$.
  - *If $\varepsilon_{n-k} \neq n-k-1$, then $\lambda_3'' < \lambda_1 < 0$, where:*
    - *$\lambda_3'' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k-1}|\beta\}}$.*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*

    Moreover if there exists $\lambda_1' < \lambda_1$ another eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$, then $\lambda_1' < \lambda_3''$.

Proof. First, we focus on the relation between λ_1 and λ_3'.

We have seen in Proposition 3.6.8 that a function u_M, solution of (1.0.3), satisfying the boundary conditions (3.5.1) cannot have any zero on (a, b) for $M \in [\bar{M} - \lambda_3', \bar{M}]$ if $n-k$ is even and for $M \in [\bar{M}, \bar{M} - \lambda_3']$ if $n-k$ is odd.

Moreover, we have proved that for $M = \bar{M}$, without taking into account the boundary conditions, $u_{\bar{M}}$ has at most $n-1$ zeros, moreover, conditions (3.5.1) are satisfied by $u_{\bar{M}}$. Hence, we lose the $n-1$ possible zeros. So, for $M = \bar{M}$ with the given boundary conditions, $u_{\bar{M}}$ achieves the maximal oscillation for the boundary conditions (3.5.1) .

Let us assume that $u_{\bar{M}} \geqslant 0$ (if $u_{\bar{M}} \leqslant 0$ the arguments are applicable by multiplying by -1), hence $T_\alpha\, u_{\bar{M}}(a) = \dfrac{u_{\bar{M}}^{(\alpha)}(a)}{v_1(a)\dots v_\alpha(a)} > 0$.

As we have proved before, $T_h\, u(a)$ changes its sign for every $h = 0, \dots, n-1$ if it is non-null. From $h = \alpha$ to σ_k, taking into account (3.6.5), we find $k-1-\alpha$ zeros for $T_h u(a)$. Hence, with maximal oscillation:

$$\begin{cases} T_{\sigma_k}\, u_{\bar{M}}(a) > 0, & \text{if } (\sigma_k-\alpha)-(k-1-\alpha) = \sigma_k - k + 1 \text{ is even,} \\ T_{\sigma_k}\, u_{\bar{M}}(a) < 0, & \text{if } \sigma_k - k + 1 \text{ is odd.} \end{cases}$$

On the other hand, by means of Lemma 3.3.6, we have that:

$$\begin{cases} u_{\bar{M}}^{(\sigma_k)}(a) > 0, & \text{if } \sigma_k - k \text{ is odd,} \\ u_{\bar{M}}^{(\sigma_k)}(a) < 0, & \text{if } \sigma_k - k \text{ is even.} \end{cases} \tag{3.6.15}$$

Let us move u_M continuously on M up to $M = \bar{M} - \lambda_3'$. In Proposition 3.6.8 we have proved that u_M has at most n zeros for every $M \in [\bar{M} - \lambda_3', \bar{M}]$ (resp. $[\bar{M}, \bar{M} - \lambda_3]$ if $n-k$ is odd) if $u_M \geqslant 0$, without taking into account the boundary conditions.

Since λ_3' is an eigenvalue of $T_n[\bar{M}]$ on $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we have that $u_{\bar{M}-\lambda_3'}^{(\alpha)}(a) = 0$. Thus, $T_\alpha\, u_{\bar{M}-\lambda_3'}(a) = 0$. This fact, coupled with the boundary conditions (3.5.1), allows us to affirm that $u_{\bar{M}-\lambda_3'}$ cannot lose more zeros if it is a non-trivial solution. Hence, $u_{\bar{M}-\lambda_3'}$ achieves the maximal oscillation.

Since we have moved continuously from $\bar{M}$ to $\bar{M} - \lambda_3'$ and it was assumed $u_{\bar{M}} \geqslant 0$ on I, we conclude that $u_{\bar{M}-\lambda_3'} \geqslant 0$, hence $T_{\alpha_1}\, u_{\bar{M}}(a) = \dfrac{u_{\bar{M}}^{(\alpha_1)}(a)}{v_1(a)\dots v_{\alpha_1}(a)} > 0$, where α_1 has been introduced in Notation 3.6.12.

As for $M = \bar{M}$, provided it is non-null, $T_h\, u(a)$ changes its sign for all $h = 0, \dots, n-1$. From $h = \alpha_1$ to σ_k, taking into account (3.6.5), we find $k - \alpha_1$ zeros. Hence, with maximal oscillation:

$$\begin{cases} T_{\sigma_k}\, u_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } (\sigma_k-\alpha_1)-(k-\alpha_1) = \sigma_k - k \text{ is even,} \\ T_{\sigma_k}\, u_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } \sigma_k - k \text{ is odd.} \end{cases}$$

From Lemma 3.3.6 again, we have:

$$\begin{cases} u_{\bar{M}-\lambda_3'}^{(\sigma_k)}(a) > 0, & \text{if } \sigma_k - k \text{ is even,} \\ u_{\bar{M}-\lambda_3'}^{(\sigma_k)}(a) < 0, & \text{if } \sigma_k - k \text{ is odd.} \end{cases} \tag{3.6.16}$$

So, since we have been moving with continuity, from (3.6.15) and (3.6.16), we can ensure the existence of a $\tilde{M}$ between $\bar{M}$ and $\bar{M} - \lambda_3'$ such that $u_{\tilde{M}}^{(\sigma_k)}(a) = 0$. As a consequence:

- If $n-k$ is even, $0 < \lambda_1 = \bar{M} - \tilde{M} < \lambda_3'$.
- If $n-k$ is odd, $0 > \lambda_1 = \bar{M} - \tilde{M} > \lambda_3'$.

The relation between λ_1 and λ_3'' is proved analogously by using Proposition 3.6.10.

The assertion referring to λ_1' is due to the fact that, if $0 < \lambda_1 < \lambda_1' < \lambda_3'$ on the case where $n-k$ is even, then, by Proposition 3.6.8, the eigenfunctions related to λ_1 and λ_1' are of constant sign and this is not possible for an strongly inverse positive (negative) operator $\big($see [98, Corollary 7.27] and [16, Section 1.8]$\big)$. The same happens when $n-k$ is odd and $0 > \lambda_1 > \lambda_1' > \lambda_3$.

Similarly, if either $n-k$ is even and $0 < \lambda_1 < \lambda_1' < \lambda_3''$ or, on the contrary, $n-k$ is odd and $0 > \lambda_1 > \lambda_1' > \lambda_3''$, then, by Proposition 3.6.10, the eigenfunctions related to λ_1 and λ_1' are of constant sign.

Thus, the result is proved. □

Example 3.6.14. *Let us return to Example 3.6.7, where we have obtained the different eigenvalues for operator $T_4^0[0]$. Let us see that the theses of Proposition 3.6.13 are fulfilled.*

- $\lambda_1 = m_1^4 \approxeq 2.36502^4 < \lambda_3' = \pi^4$.
- $\lambda_1 < \lambda_3'' = m_3^4 \approxeq 3.9266^4$.

Moreover, we have seen in Example 3.6.7 that the eigenvalues of $T_4^0[0]$ in $X_{\{0,2\}}^{\{1,2\}}$ are given by $\lambda = m^4$, where m is a positive solution of (3.6.2)*. So $\lambda_1' \approxeq 5.497^4 > \lambda_3'$ and $\lambda_1' > \lambda_3''$.*

3.6.3 Constant sign solutions of $\widehat{T}_n\big[(-1)^n M\big]\, u(t) = 0$

This section is devoted to characterise the parameter set of constant sign solutions for the equation $\widehat{T}_n\big[(-1)^n M\big]\, u(t) = 0$ coupled with suitable boundary conditions. Such a characterisation is done by means of the eigenvalues of the adjoint operator $T_n^*[\bar{M}]$ in the different spaces:

$$X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}},\ X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}},\ X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}},\ X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}} \text{ and } X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}.$$

In Section 3.4, we have already proved that the boundary conditions satisfied for every $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ are given by (3.4.1)–(3.4.6). Proceeding analogously in the different spaces, taking into account that $\eta = n-1-\sigma_k$, $\gamma = n-1-\varepsilon_{n-k}$, $\alpha = n-1-\tau_{n-k}$ and $\beta = n-1-\delta_k$, we have the following assertions:

- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$, then it satisfies (3.4.1)–(3.4.2) and (3.4.4)–(3.4.6) coupled with $v^{(\gamma)}(b) = 0$.

- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$, then it satisfies (3.4.1)–(3.4.3) and (3.4.4)–(3.4.5) coupled with $v^{(\eta)}(a) = 0$.

- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$, then it satisfies (3.4.1)–(3.4.2) and (3.4.4)–(3.4.6) coupled with $v^{(\eta)}(a) = 0$.

- If $v \in X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$, then it satisfies (3.4.1)–(3.4.3) and (3.4.4)–(3.4.5) coupled with $v^{(\gamma)}(b) = 0$.

Example 3.6.15. *Arguing in an analogous way to Example 3.4.4, we obtain:*

$$
\begin{aligned}
X^{*\{1\}}_{\{0,1,2\}} &= \Big\{ v \in C^4(I) \mid v(a) = v(b) = v'(b) = v^{(3)}(b) - p_1(b)\, v''(b) = 0 \Big\},\\
X^{*\{0,1,2\}}_{\{0\}} &= \Big\{ v \in C^4(I) \mid v(a) = v'(a) = v''(a) = v(b) = 0 \Big\} = X^{\{0\}}_{\{0,1,2\}},\\
X^{*\{0,1\}}_{\{0,2\}} &= \Big\{ v \in C^4(I) \mid v(a) = v''(a) - p_1(a)\, v'(a) = v(b) = v'(b) = 0 \Big\},\\
X^{*\{1,2\}}_{\{0,1\}} &= \Big\{ v \in C^4(I) \mid v^{(3)}(b) - p_1(b)\, v''(b) + \big(p_2(b) - 2p_1'(b)\big) v'(b) = 0,\\
&\qquad v(a) = v'(a) = v(b) = 0 \Big\}.
\end{aligned}
$$

In the sequel, we prove analogous results to those of the previous section referring to functions defined in these spaces.

Remark 3.6.16. *In this case, taking into account that the eigenvalues of one operator and those of its adjoint are the same, we do not need to prove the existence of the eigenvalues. Such existence follows from the one of the eigenvalues of $T_n[\bar M]$ in the correspondent spaces.*

First, we prove two results which refer to the operator $T_n^*[M]$ and then we will be able to extrapolate them for $\widehat{T}_n\big[(-1)^n M\big]$.

Proposition 3.6.17. *Let $\bar M \in \mathbb{R}$ be such that $T_n[\bar M]$ satisfies property (T_d) on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). Then every solution of $T_n^*[M]\, v(t) = 0$, for $t \in (a,b)$, satisfying the boundary conditions* (3.4.1)–(3.4.3) *and* (3.4.4)–(3.4.5) *does not have any zero on (a,b) provided that one of the following assertions is fulfilled.*

- *Let $n-k$ be even:*
  - *If $k > 1$, $\varepsilon_{n-k} \neq n-k-1$ and $M \in [\bar M - \lambda_3'', \bar M - \lambda_2']$, where:*
    - *$\lambda_3'' > 0$ is the least positive eigenvalue of $T_n[\bar M]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*
    - *$\lambda_2' < 0$ is the biggest negative eigenvalue of $T_n[\bar M]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.*
  - *If $k = 1$, $\varepsilon_{n-1} \neq n-2$ and $M \in [\bar M - \lambda_3'', +\infty)$, where:*
    - *$\lambda_3'' > 0$ is the least positive eigenvalue of $T_n[\bar M]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}_{\{\sigma_1\}}$.*
  - *If $k > 1$, $\varepsilon_{n-k} = n-k-1$ and $M \in [\bar M - \lambda_1, \bar M - \lambda_2']$, where:*

    * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.
    * $\lambda_2' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{0,\dots,n-k\}}$.
  - *If* $k = 1$, $\varepsilon_{n-1} = n-2$ *and* $M \in [\bar{M} - \lambda_1, +\infty)$, *where:*
    * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{0,\dots,n-2\}}$.

- *Let* $n-k$ *be odd:*
  - *If* $1 < k < n-1$, $\varepsilon_{n-k} \neq n-k-1$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.
    * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  - *If* $1 < k = n-1$, $\varepsilon_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the least biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\beta\}}$, *where* $\beta = 0$.
    * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-2}\}}^{\{\varepsilon_1|\beta\}}$, *where* $\beta = 0$.
  - *If* $k = 1 < n-1$, $\varepsilon_{n-1} \neq n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}$.
  - *If* $k = 1$, $n = 2$, $\varepsilon_1 \neq 0$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
    * $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\beta\}} = X_{\{0\}}^{\{0\}}$.
  - *If* $1 < k$, $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_1]$, *where:*
    * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.
    * $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{0,\dots,n-k\}}$.
  - *If* $k = 1$, $\varepsilon_{n-1} = n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_1]$, *where:*
    * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{0,\dots,n-2\}}$.

Proof. The proof follows the same steps as Proposition 3.6.8.

Let us denote $v_M \in C^n(I)$ a solution of:

$$T_n^*[M]\, v(t) = 0, \quad t \in (a,b), \tag{3.6.17}$$

satisfying the boundary conditions (3.4.1)–(3.4.3) and (3.4.4)–(3.4.5).

First, let us see that $v_{\bar{M}}$ does not have any zero on (a,b). In order to see this, we consider the decomposition (3.4.11) whose existence is ensured by Lemma 3.4.6.

Analogously to the proof of Lemma 3.3.9, since $w_0, \dots, w_n > 0$, we can conclude that, without taking into account the boundary conditions, every solution of equation (3.6.17) for $M = \bar{M}$ can have at most $n-1$ zeros. However, as we have said before, each time that either $T_\ell^* v_M(a) = 0$ or $T_\ell^* v_M(b) = 0$, a possible oscillation is lost. From the boundary conditions and Lemmas 3.4.9 and 3.4.10, taking into account that $T_\ell^* v_M(t) = (-1)^\ell \widehat{T}_\ell v_M(t)$, we can affirm that for every $M \in \mathbb{R}$:

$$T^*_{\tau_1} v_M(a) = \cdots = T^*_{\tau_{n-k}} v_M(a) = 0, \tag{3.6.18}$$

$$T^*_{\delta_1} v_M(b) = \cdots = T^*_{\delta_{k-1}} v_M(b) = 0. \tag{3.6.19}$$

Thus, every non-trivial solution of (3.6.17) for $M = \bar{M}$ does not have any zero on (a, b).

Now, let us move v_M continuously as a function of M on a neighbourhood of $M = \bar{M}$. We have that v_M is a solution of (3.6.17), hence:

$$T_n^*[\bar{M}] v_M(t) = (\bar{M} - M)\, v_M, \quad t \in (a, b). \tag{3.6.20}$$

Analogously to the proof of Proposition 3.6.8, we will see that, while v_M is of constant sign, it cannot have any double zero on (a, b).

We can assume that $v_{\bar{M}} > 0$ on I (if $v_{\bar{M}} < 0$, then the arguments are applicable by multiplying by -1). So, in equation (3.6.20) we have:

$$\begin{cases} T_n^*[\bar{M}]\, v_M(t) \geqslant 0, & t \in I, \quad \text{if } M < \bar{M}, \\ T_n^*[\bar{M}]\, v_M(t) \leqslant 0, & t \in I, \quad \text{if } M > \bar{M}. \end{cases} \tag{3.6.21}$$

In both cases, since $\dfrac{-1}{w_n} < 0$, $T^*_{n-1} v_M$ is a monotone function, with at most one zero. Studying the maximal oscillation of $T^*_{n-\ell} v_M$ for $\ell = 2, \ldots, n$, we conclude that $T^*_{n-\ell} v_M$ has at most ℓ zeros.

In particular, $T_0^*\, v_M$ has no more than n zeros. Since $w_0 > 0$, we can affirm that v_M has at most n zeros.

However, v_M satisfies (3.6.18)-(3.6.19), hence $n - 1$ possible oscillation are lost. Thus, v_M can have at most a simple zero on (a, b) which is not possible if it is of constant sign.

Let us assume that $k \neq 1$ and that $\delta_k \neq k - 1$ (this is equivalent to $\varepsilon_{n-k} \neq n - k - 1$). Under these assumptions, we can affirm that v_M is of constant sign up to one of the following boundary conditions is fulfilled:

$$v_M^{(\eta)}(a) = 0 \quad \text{or} \quad v_M^{(\gamma)}(b) = 0.$$

Let us study what happens by moving M. Since we are considering $v_M \geqslant 0$, we have:

$$v_M^{(\eta)}(a) \geqslant 0, \quad \text{and} \quad \begin{cases} v_M^{(\gamma)}(b) \geqslant 0, & \text{if } \gamma \text{ is even}, \\ v_M^{(\gamma)}(b) \leqslant 0, & \text{if } \gamma \text{ is odd}. \end{cases} \tag{3.6.22}$$

Now, let us see the behaviour of $v_M^{(\eta)}(a)$ and $v_M^{(\gamma)}(b)$ with maximal oscillation. As before, with maximal oscillation only one zero on the boundary is allowed. If $T_\ell^* v_M(a) = 0$ for all $\ell \notin \{\tau_1, \ldots, \tau_{n-k}, \eta\}$ or $T_\ell^* v_M(b) = 0$ for every $\ell \notin \{\delta_1, \ldots, \delta_{k-1}, \gamma\}$, we have that

$T^*_\eta v_M(a) \neq 0$ and $T^*_\gamma v_M(b) \neq 0$. Because, otherwise, $v_M \equiv 0$ on I and we are looking for non-trivial solutions.

From (3.4.25), taking into account that $T^*_\ell v_M(t) = (-1)^\ell \widehat{T}_\ell\, v_M(t)$, we obtain

$$\begin{aligned} T^*_\eta v_M(a) &= (-1)^\eta v_1(a)\, \ldots\, v_{n-\eta}(a)\, v^{(\eta)}(a), \\ T^*_\gamma v_M(b) &= (-1)^\gamma v_1(b)\, \ldots\, v_{n-\gamma}(b)\, v^{(\gamma)}(b), \end{aligned} \tag{3.6.23}$$

where $v_1\, \ldots, v_n > 0$ are given in (3.3.1).

Hence, if $T^*_\eta v_M(a) \neq 0$ and $T^*_\gamma v_M(b) \neq 0$, then $v_M^{(\eta)}(a) \neq 0$ and $v_M^{(\gamma)}(b) \neq 0$, thus the function v_M remains of constant sign.

So, we can assume that the unique zero, which is allowed with maximal oscillation, is found either in $T^*_\eta v_M(a)$ or $T^*_\gamma v_M(b)$.

In this case, $T^*_k v_M = \dfrac{-1}{w_k} \dfrac{\mathrm{d}}{\mathrm{d}\,t}\left(T^*_{k-1} v_M\right)$ with $w_k > 0$. On the contrary that in the case $T_k\, u_M$, see Figure 3.3.2, to allow the maximal oscillation, $T^*_{n-\ell} v_M(a)$ remains of constant sign if it is non-null and, if it vanishes for $\tilde{\ell}, \tilde{\ell}+1, \ldots, \tilde{\ell}+h \in \{0, \ldots, n-1\}$ with $h \geqslant 0$, it changes it sign for the next ℓ where it is non-null, or which is the same:

$$\operatorname{sign}\left(T^*_{n-\tilde{\ell}-h-1} v_M(a)\right) = (-1)^{h+1} \operatorname{sign}\left(T^*_{n-\tilde{\ell}+1} v_M(a)\right).$$

On the other hand, $T^*_{n-\ell} v_M(b)$ changes its sign each time that it is non-null.

At first, let us focus on the case $M < \bar{M}$, we have that $T^*_n[\bar{M}] v_M = T^*_n v_M \geqslant 0$ on I.

In particular, $T^*_n v_M(a) \geqslant 0$ and $T^*_n v_M(b) \geqslant 0$. Using (3.6.18)-(3.6.19), from $\ell = 0$ to $n-\eta$, we have that $T_{n-\ell} v_M(a)$ vanishes $n-k-\eta$ times and from $\ell = 0$ to $n-\gamma$, $T_{n-\ell} v_M(b)$ gets the value zero $k-1-\gamma$ times.

Hence, to allow the maximal oscillation:

$$\begin{cases} T^*_\eta\, v_M(a) \geqslant 0, & \text{if } n-k-\eta \text{ is even}, \\[2ex] T^*_\eta\, v_M(a) \leqslant 0, & \text{if } n-k-\eta \text{ is odd}, \end{cases} \tag{3.6.24}$$

and

$$\begin{cases} T^*_\gamma\, v_M(b) \geqslant 0, & \text{if } n-\gamma-(k-1-\gamma) = n-k+1 \text{ is even}, \\[2ex] T^*_\gamma\, v_M(b) \leqslant 0, & \text{if } n-k+1 \text{ is odd}. \end{cases} \tag{3.6.25}$$

Using (3.6.23) and (3.6.24)-(3.6.25), we can affirm that to get maximal oscillation:

$$\begin{cases} v_M^{(\eta)}(a) \geqslant 0, & \text{if } n-k \text{ is even}, \\[2ex] v_M^{(\eta)}(a) \leqslant 0, & \text{if } n-k \text{ is odd}, \end{cases} \tag{3.6.26}$$

and,

- If $n-k$ is even:
$$\begin{cases} v_M^{(\gamma)}(b) \geqslant 0, & \text{if } \gamma \text{ is odd}, \\ v_M^{(\gamma)}(b) \leqslant 0, & \text{if } \gamma \text{ is even}. \end{cases} \tag{3.6.27}$$

- If $n-k$ is odd:
$$\begin{cases} v_M^{(\gamma)}(b) \leqslant 0, & \text{if } \gamma \text{ is odd}, \\ v_M^{(\gamma)}(b) \geqslant 0, & \text{if } \gamma \text{ is even}. \end{cases} \tag{3.6.28}$$

Hence, taking into account (3.6.22), we arrive at the following conclusions:

- If $n-k$ is even, the maximal oscillation is not allowed for v_M if $v_N^{(\gamma)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for every M in $[\bar{M}-\lambda_3^{*\prime\prime}, \bar{M}]$, where $\lambda_3^{*\prime\prime} > 0$ is the least positive eigenvalue of $T_n^*[\bar{M}]$ on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\bar{\beta}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

- If $n-k$ is odd, the maximal oscillation is not allowed for v_M whenever $v_N^{(\eta)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for every M which belongs to $[\bar{M}-\lambda_2^{*\prime}, \bar{M}]$, where $\lambda_2^{*\prime} > 0$ is the least positive eigenvalue of $T_n^*[\bar{M}]$ on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\bar{\beta}\}}_{\{\sigma_1,\dots,\sigma_{k-1}\}}$.

Moreover, since the eigenvalues of an operator and its adjoint are the same, we can affirm that $\lambda_3'' = \lambda_3^{*\prime\prime}$ and $\lambda_2' = \lambda_2^{*\prime}$.

Consider now the other case, i.e. $M > \bar{M}$. From (3.6.21), we have that $T_n^* v_M \leqslant 0$. Thus, to obtain the maximal oscillation, the inequalities (3.6.24)-(3.6.25) must be reversed. So, taking into account (3.6.23), we can affirm that, to get maximal oscillation:

$$\begin{cases} v_M^{(\eta)}(a) \leqslant 0, & \text{if } n-k \text{ is even}, \\ v_M^{(\eta)}(a) \geqslant 0, & \text{if } n-k \text{ is odd}, \end{cases} \tag{3.6.29}$$

and,

- If $n-k$ is even:
$$\begin{cases} v_M^{(\gamma)}(b) \leqslant 0, & \text{if } \gamma \text{ is odd}, \\ v_M^{(\gamma)}(b) \geqslant 0, & \text{if } \gamma \text{ is even}. \end{cases} \tag{3.6.30}$$

- If $n-k$ is odd:
$$\begin{cases} v_M^{(\gamma)}(b) \geqslant 0, & \text{if } \gamma \text{ is odd}, \\ v_M^{(\gamma)}(b) \leqslant 0, & \text{if } \gamma \text{ is even}. \end{cases} \tag{3.6.31}$$

Hence, we arrive at the following conclusions, taking into account (3.6.22):

- If $n-k$ is even, the maximal oscillation is not allowed for v_M whenever $v_N^{(\eta)}(a) \neq 0$ for all N between $\bar{M}$ and M; this fact implies that $v_M > 0$ on (a,b) for every M which belongs to $[\bar{M}, \bar{M} - \lambda_2^{*\prime}] = [\bar{M}, \bar{M} - \lambda_2']$.

- If $n-k$ is odd, the maximal oscillation is not allowed for v_M whenever $v_N^{(\gamma)}(b) \neq 0$ for all N between $\bar{M}$ and M; this fact implies that $v_M > 0$ on (a,b) for every M which belongs to $[\bar{M}, \bar{M} - \lambda_3^{*\prime\prime}] = [\bar{M}, \bar{M} - \lambda_3']$.

Now, we realise that if $k=1$, $v_M^{(\eta)}(a) \neq 0$ for all $M \in \mathbb{R}$, since the contrary implies that a non-trivial solution of the homogeneous linear differential equation (3.6.17) has a zero at $t=a$ of multiplicity n, which is not possible.

Finally, if $\varepsilon_{n-k} = n-k-1$ or, which is the same, $\delta_k = k-1$, we consider $v_M^{(k-1)}(b)$ instead of $v_M^{(\gamma)}(b) = v_M^{(k)}(b)$ and, taking into account that, from $\ell = 0$ to $n-(k-1)$, $T_{n-\ell} v_M(b) \neq 0$, we obtain that to allow maximal oscillation the following properties hold:

- If $M < \bar{M}$:

$$\begin{cases} T_{k-1}^* v_M(b) \geqslant 0, & \text{if } n-k \text{ is odd,} \\ \\ T_{k-1}^* v_M(b) \leqslant 0, & \text{if } n-k \text{ is even.} \end{cases} \tag{3.6.32}$$

- If $M > \bar{M}$:

$$\begin{cases} T_{k-1}^* v_M(b) \leqslant 0, & \text{if } n-k \text{ is odd,} \\ \\ T_{k-1}^* v_M(b) \geqslant 0, & \text{if } n-k \text{ is even.} \end{cases} \tag{3.6.33}$$

From (3.4.25), since $T_\ell^* v_M(t) = (-1)^\ell \widehat{T}_\ell v_M(t)$, we have that:

$$T_{k-1}^* v_M(b) = (-1)^{k-1} v_1(b) \cdots v_{n-k-1}(b)\, v_M^{(k-1)}(b).$$

So, we obtain:

- If $M < \bar{M}$, then:
  - If $n-k$ is even:

$$\begin{cases} v_M^{(k-1)}(b) \geqslant 0, & \text{if } k-1 \text{ is odd,} \\ \\ v_M^{(k-1)}(b) \leqslant 0, & \text{if } k-1 \text{ is even.} \end{cases} \tag{3.6.34}$$

  - If $n-k$ is odd:

$$\begin{cases} v_M^{(k-1)}(b) \leqslant 0, & \text{if } k-1 \text{ is odd,} \\ \\ v_M^{(k-1)}(b) \geqslant 0, & \text{if } k-1 \text{ is even.} \end{cases} \tag{3.6.35}$$

- If $M > \bar{M}$, then:

– If $n-k$ is even:

$$\begin{cases} v_M^{(k-1)}(b) \leqslant 0, & \text{if } k-1 \text{ is odd,} \\ \\ v_M^{(k-1)}(b) \geqslant 0, & \text{if } k-1 \text{ is even.} \end{cases} \tag{3.6.36}$$

– If $n-k$ is odd:

$$\begin{cases} v_M^{(k-1)}(b) \geqslant 0, & \text{if } k-1 \text{ is odd,} \\ \\ v_M^{(k-1)}(b) \leqslant 0, & \text{if } k-1 \text{ is even.} \end{cases} \tag{3.6.37}$$

And, from (3.6.22) we are able to finish the proof.

- If $n-k$ is even and $M < \bar{M}$, the maximal oscillation is not allowed for v_M whenever $v_N^{(k-1)}(b) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for all $M \in [\bar{M} - \lambda_1^*, \bar{M}]$, where $\lambda_1^* > 0$ is the least positive eigenvalue of $T_n^*[\bar{M}]$ on $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
- If $n-k$ is odd and $M > \bar{M}$, the maximal oscillation is not allowed for v_M whenever $v_N^{(n-k-1)}(a) \neq 0$ for all N between $\bar{M}$ and M; which implies that $v_M > 0$ on (a,b) for all $M \in [\bar{M}, \bar{M} - \lambda_1^*]$.

Due to the coincidence of the eigenvalues of an operator and the ones of its adjoint, we can affirm that $\lambda_1 = \lambda_1^*$ and the proof is complete. □

Now, we obtain an analogous result for different boundary conditions:

Proposition 3.6.18. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). Then every solution of $T_n^*[M]\, v(t) = 0$ for $t \in (a,b)$, satisfying the boundary conditions* (3.4.1)–(3.4.2) *and* (3.4.4)–(3.4.6)*, does not have any zero on (a,b) provided that one of the following assertions holds.*

- *Let $n-k$ be even:*
  - *If $k > 1$, $\sigma_k \neq k-1$ and $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, where:*
    - $*$ *$\lambda_3' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.*
    - $*$ *$\lambda_2'' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.*
  - *If $k = 1$, $\sigma_1 \neq 0$ and $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, where:*
    - $*$ *$\lambda_3' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\alpha\}}$, where $\alpha = 0$.*
    - $*$ *$\lambda_2'' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}_{\{\sigma_1|\alpha\}}$, where $\alpha = 0$.*
  - *If $\sigma_k = k-1$ and $M \in [\bar{M} - \lambda_1, \bar{M} - \lambda_2'']$, where:*
    - $*$ *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{1,\dots,k-1\}}$.*

  * $\lambda_2'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.

- *Let* $n-k$ *be odd:*

  – *If* $1 < k < n-1$, $\sigma_k \neq k-1$ *and* $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, *where:*

    * $\lambda_3' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.

  – *If* $1 = k < n-1$, $\sigma_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, *where:*

    * $\lambda_3' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$, *where* $\alpha = 0$.
    * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$, *where* $\alpha = 0$.

  – *If* $1 < k = n-1$, $\sigma_{n-1} \neq n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_3']$, *where:*

    * $\lambda_3' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-2}|\alpha\}}^{\{\varepsilon_1\}}$.

  – *If* $k = 1$, $n = 2$, $\sigma_1 \neq 0$ *and* $M \in (-\infty, \bar{M} - \lambda_3']$, *where:*

    * $\lambda_3' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\alpha\}}^{\{\varepsilon_1\}} = X_{\{0\}}^{\{0\}}$.

  – *If* $k < n-1$, $\sigma_k = k-1$ *and* $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_1]$, *where:*

    * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    * $\lambda_2'' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.

  – *If* $k = n-1$, $\sigma_{n-1} = n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_1]$, *where:*

    * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{0,\dots,n-2\}}^{\{\varepsilon_1\}}$.

Proof. The proof is analogous to Proposition 3.6.17. □

Example 3.6.19. *Returning to our problem, introduced in Example 3.6.7, we have that operator* $T_4^{0^*}[M]\,v(t) = v^{(4)}(t) + M\,v(t) = T_4^0[M]\,v(t)$ *is defined in:*

$$X^{*\{1,2\}}_{\{0,2\}} = \Big\{ v \in C^4\big([0,1]\big) \;\mid\; v(0) = v''(0) = v(1) = v^{(3)}(1) = 0 \Big\},$$

as we have proved in (3.4.10)*, because, in this case,* $p_1(t) = p_2(t) = p_3(t) = p_4(t) = 0$ *for all* $t \in [0,1]$.

From Proposition 3.6.17, we conclude that each solution of $v^{(4)}(t) + M\,v(t) = 0$ *on* $[0,1]$ *satisfying the boundary conditions* $v(0) = v''(0) = v(1) = 0$ *does not have any zero on* $(0,1)$ *for* $M \in \left[-m_3^4, m_2^4\right]$*, where* m_3 *and* m_4 *have been introduced in Example 3.6.7.*

We note that such functions follow the expressions:

$$\begin{cases} K\big(\sin(m\,t)\sinh(m) - \sinh(m\,t)\sin(m)\big), & M = -m^4 < 0, \\[2mm] K\,(t - t^3), & M = 0, \\[2mm] K\,e^{-\frac{mt}{\sqrt{2}}}\left(\left(e^{\sqrt{2}m(t+1)} - 1\right)\sin\left(\frac{m(t-1)}{\sqrt{2}}\right) + \left(e^{\sqrt{2}m} - e^{\sqrt{2}mt}\right)\sin\left(\frac{m(t+1)}{\sqrt{2}}\right)\right), & M = m^4 > 0, \end{cases}$$

where $K \in \mathbb{R}\backslash\{0\}$.

Moreover, from Proposition 3.6.18, we can affirm that any solution of:

$$v^{(4)}(t) + M\, v(t) = 0, \quad t \in [0,1],$$

satisfying the boundary conditions $v(0) = v(1) = v^{(3)}(1) = 0$, *does not have any zero on* $(0,1)$ *for* $M \in [-\pi^4, 4\,\pi^4]$. *One can show that such solutions are given as multiples of:*

$$\begin{cases} \begin{array}{c} \cos(m-mt)\big(\sin(m)+\sinh(m)\big)+\sin(m-mt)\big(\cosh(m)-\cos(m)\big) \\ +\sinh(m-mt)\big(\cosh(m)-\cos(m)\big)-\cosh(m-mt)\big(\sin(m)+\sinh(m)\big), \end{array} & M=-m^4<0, \\[2ex] t-t^2, & M=0, \\[2ex] \begin{array}{c} e^{-\frac{mt}{\sqrt{2}}}\left(-\left(e^{\sqrt{2}m(t-1)}-e^{\sqrt{2}mt}+e^{\sqrt{2}m}-1\right)\sin\left(\frac{mt}{\sqrt{2}}\right)\right. \\ \left.+\left(e^{\sqrt{2}mt}-1\right)\cos\left(\frac{m(t-2)}{\sqrt{2}}\right)-\left(e^{\sqrt{2}mt}-1\right)\cos\left(\frac{mt}{\sqrt{2}}\right)\right), \end{array} & M=m^4>0, \end{cases}$$

Taking into account that if v_M is a solution of (3.6.17), then $(-1)^n v_M$ is a solution of $\widehat{T}_n\big[(-1)^n M\big]v(t) = 0$ for all $t \in I$, we obtain the analogous results for $\widehat{T}_n$.

Proposition 3.6.20. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a). *Then every solution of:*

$$\widehat{T}_n\big[(-1)^n M\big]\, v(t) = 0, \quad t \in (a,b),$$

satisfying the boundary conditions (3.4.1)–(3.4.3) *and* (3.4.4)–(3.4.5)*, does not have any zero on* (a,b) *provided that one of the following assertions is fulfilled.*

- *Let* k *be even:*
  - *If* $k>1$, $\varepsilon_{n-k} \neq n-k-1$ *and* $M \in [\bar{M}-\lambda_3'', \bar{M}-\lambda_2']$, *where:*
    - $\lambda_3''>0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.
    - $\lambda_2'<0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  - *If* $k=1$, $\varepsilon_{n-1} \neq n-2$ *and* $M \in [\bar{M}-\lambda_3'', +\infty)$, *where:*
    - $\lambda_3''>0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}$.
  - *If* $k>1$, $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M}-\lambda_1, \bar{M}-\lambda_2']$, *where:*
    - $\lambda_1>0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.
    - $\lambda_2'<0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{0,\dots,n-k\}}$.
  - *If* $k=1$, $\varepsilon_{n-1} = n-2$ *and* $M \in [\bar{M}-\lambda_1, +\infty)$, *where:*
    - $\lambda_1>0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{0,\dots,n-2\}}$.
- *Let* k *be odd:*

- *If* $1 < k < n-1$, $\varepsilon_{n-k} \neq n-k-1$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_3'']$, *where:*
  - $*$ $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.
  - $*$ $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
- *If* $1 < k = n-1$, $\varepsilon_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_3'']$, *where:*
  - $*$ $\lambda_3'' < 0$ *is the least biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\beta\}}$, *where* $\beta = 0$.
  - $*$ $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-2}\}}^{\{\varepsilon_1|\beta\}}$, *where* $\beta = 0$.
- *If* $k = 1 < n-1$, $\varepsilon_{n-1} \neq n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
  - $*$ $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}|\beta\}}$.
- *If* $k = 1$, $n = 2$, $\varepsilon_1 \neq 0$ *and* $M \in (-\infty, \bar{M} - \lambda_3'']$, *where:*
  - $*$ $\lambda_3'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\beta\}} = X_{\{0\}}^{\{0\}}$.
- *If* $1 < k$, $\varepsilon_{n-k} = n-k-1$ *and* $M \in [\bar{M} - \lambda_2', \bar{M} - \lambda_1]$, *where:*
  - $*$ $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{0,\dots,n-k-1\}}$.
  - $*$ $\lambda_2' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{0,\dots,n-k\}}$.
- *If* $k = 1$, $\varepsilon_{n-1} = n-2$ *and* $M \in (-\infty, \bar{M} - \lambda_1]$, *where:*
  - $*$ $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{0,\dots,n-2\}}$.

Proposition 3.6.21. *Let* $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfy* (N_a). *Then every solution of:*

$$\widehat{T}_n\big[(-1)^n M\big]\, v(t) = 0, \quad t \in (a,b),$$

satisfying the boundary conditions (3.4.1)–(3.4.2) *and* (3.4.4)–(3.4.6)*, does not have any zero on* (a,b) *provided that one of the following assertions is fulfilled.*

- *Let* k *be even:*
  - *If* $k > 1$, $\sigma_k \neq k-1$ *and* $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, *where:*
    - $*$ $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
    - $*$ $\lambda_2'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.
  - *If* $k = 1$, $\sigma_1 \neq 0$ *and* $M \in [\bar{M} - \lambda_3', \bar{M} - \lambda_2'']$, *where:*
    - $*$ $\lambda_3' > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$, *where* $\alpha = 0$.
    - $*$ $\lambda_2'' < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$, *where* $\alpha = 0$.

– *If $\sigma_k = k-1$ and $M \in [\bar{M} - \lambda_1, \bar{M} - \lambda_2'']$, where:*
  * *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{1,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  * *$\lambda_2'' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{0,\dots,k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*

- *Let k be odd:*
  – *If $1 < k < n-1$, $\sigma_k \neq k-1$ and $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, where:*
    * *$\lambda_3' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    * *$\lambda_2'' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
  – *If $1 = k < n-1$, $\sigma_1 \neq 0$ and $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_3']$, where:*
    * *$\lambda_3' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$, where $\alpha = 0$.*
    * *$\lambda_2'' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$, where $\alpha = 0$.*
  – *If $1 < k = n-1$, $\sigma_{n-1} \neq n-2$ and $M \in (-\infty, \bar{M} - \lambda_3']$, where:*
    * *$\lambda_3' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{n-2}|\alpha\}}^{\{\varepsilon_1\}}$.*
  – *If $k = 1$, $n = 2$, $\sigma_1 \neq 0$ and $M \in (-\infty, \bar{M} - \lambda_3']$, where:*
    * *$\lambda_3' < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\alpha\}}^{\{\varepsilon_1\}} = X_{\{0\}}^{\{0\}}$.*
  – *If $k < n-1$, $\sigma_k = k-1$ and $M \in [\bar{M} - \lambda_2'', \bar{M} - \lambda_1]$, where:*
    * *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{0,\dots,k-1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    * *$\lambda_2'' > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{0,\dots,k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
  – *If $k = n-1$, $\sigma_{n-1} = n-2$ and $M \in (-\infty, \bar{M} - \lambda_1]$, where:*
    * *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{0,\dots,n-2\}}^{\{\varepsilon_1\}}$.*

Remark 3.6.22. *In our recurrent example, we have that $n = 4$ is even, so:*

$$\widehat{T}_4\big[(-1)^4 M\big] \equiv T_4^*[M].$$

Then, Example 3.6.19 is also applicable to illustrate Propositions 3.6.20 and 3.6.21.

3.7 Constant sign Green's function

This section is devoted to obtain the main results of this chapter. That is, under the hypotheses introduced in Section 3.3, we study the parameter set for which the Green's function is of constant sign.

First, we characterise the parameter set for which the Green's function is of constant sign on a neighbourhood of $\bar{M} \in \mathbb{R}$ for which the hypotheses introduced in Section 3.3 are satisfied. Indeed, we obtain stronger properties than (P_g) and (N_g), respectively, which will be used in Chapter 6 to prove the existence of one or multiple solutions for several non-linear problems. Now, let us introduce such properties:

(P_{g_1}) There are three continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a,b)$ and $0 < k_1(t) < k_2(t)$ for all $t \in (a,b)$, satisfying:

$$\phi(s)\, k_1(t) \leqslant g_M(t,s) \leqslant \phi(s)\, k_2(t), \quad \text{for all } (t,s) \in I \times I.$$

(N_{g_1}) There are three continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a,b)$ and $k_2(t) < k_1(t) < 0$ for all $t \in (a,b)$, satisfying:

$$\phi(s)\, k_2(t) \leqslant g_M(t,s) \leqslant \phi(s)\, k_1(t), \quad \text{for all } (t,s) \in I \times I.$$

Once we have obtained the characterisation of the parameter set for which one of these properties is fulfilled on a neighbourhood of $\bar{M} \in \mathbb{R}$, we deduce some properties of the Green's function sign for different values of M which are not considered on the previous characterization.

3.7.1 Constant sign Green's functions on a neighbourhood of $\bar{M}$

First, we obtain a characterisation of the parameter set where the related Green's function of problem (3.0.1)-(3.0.3) satisfies either property (P_{g_1}) or (N_{g_1}). The characterisation here obtained is related to the parameter set which contains $\bar{M}$. That is, if $n-k$ is even we characterise the parameter set where $g_M(t,s)$ satisfies (P_{g_1}) and, if $n-k$ is odd we characterise the parameter set where $g_M(t,s)$ satisfies (N_{g_1}). In particular, $\bar{M}$ belongs to those intervals.

Theorem 3.7.1. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a).*

Let us denote $g_M(t,s)$ as the related Green's function of problem (3.0.1)-(3.0.3). *The following properties are fulfilled.*

- *If $n-k$ is even and $2 \leqslant k \leqslant n-1$, then $g_M(t,s)$ satisfies property (P_{g_1}) if, and only if, $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, where:*
  - * $\lambda_1 > 0$ *is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * $\lambda_2 < 0$ *is the maximum between:*
    - · $\lambda_2' < 0$, *the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
    - · $\lambda_2'' < 0$, *the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*

- *If $k=1$ and n is odd, then $g_M(t,s)$ satisfies property (P_{g_1}) if, and only if, M belongs to $(\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, where:*
  - * $\lambda_1 > 0$ *is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
  - * $\lambda_2 < 0$ *is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.*

- *If $n-k$ is odd and $2 \leqslant k \leqslant n-2$, then $g_M(t,s)$ satisfies property (N_{g_1}) if, and only if, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, where:*

* $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
* $\lambda_2 > 0$ *is the minimum between:*
  · $\lambda_2' > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  · $\lambda_2'' > 0$, *the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.

- *If* $k = 1$ *and* $n > 2$ *is even, then* $g_M(t,s)$ *satisfies property* $\left(N_{g_1}\right)$ *if, and only if,* $M \in [\bar{M} - \lambda_2, \bar{M} - \lambda_1)$, *where:*
  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.
  * $\lambda_2 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.
- *If* $k = n-1$ *and* $n > 2$, *then* $g_M(t,s)$ *satisfies property* $\left(N_{g_1}\right)$ *if, and only if,* M *belongs to* $[\bar{M} - \lambda_2, \bar{M} - \lambda_1)$, *where:*
  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.
  * $\lambda_2 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-2}\}}^{\{\varepsilon_1|\beta\}}$.
- *If* $n = 2$, *then* $g_M(t,s)$ *satisfies* $\left(N_{g_1}\right)$ *if, and only if,* $M \in (-\infty, \bar{M} - \lambda_1)$, *where:*
  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.

Proof. First, by using Theorems 1.2.8, 1.2.10 and 1.2.11 we study $g_M(t,s)$ for $M = \bar{M}$ and we obtain the extreme of the interval given by λ_1.

In order to characterise the other extreme of the intervals, we use the results obtained in Sections 3.5 and 3.6. First, we prove that, in the given intervals, $g_M(t,s)$ satisfies either $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$. Finally, we verify that such an interval cannot be increased. So, we divide the proof in several steps.

Step 1. Study of the related Green's function for $M = \bar{M}$.

Step 2. Study of the related Green's function at the boundary of $I \times I$

Step 3. Study of the related Green's function on $(a,b) \times (a,b)$.

Step 4. The intervals of constant sign Green's function cannot be increased.

Step 1. Study of the related Green's function for $M = \bar{M}$.

From Theorem 3.3.10, we know that operator $T_n[\bar{M}]$ satisfies property (P_g) and is strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if $n-k$ is even. Moreover, it satisfies (N_g) and is strongly inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if $n-k$ is odd.

In addition, in that proof we have seen that $(-1)^{n-k} g_{\bar{M}}(t,s) > 0$ for all (t,s) belonging to $(a,b) \times (a,b)$.

Now, let us consider functions $w_{\bar{M}}$ and $y_{\bar{M}}$, previously introduced in Lemmas 3.5.1 and 3.5.2. From these results, coupled with Propositions 3.6.8 and 3.6.10, we know that $w_{\bar{M}}$ and $y_{\bar{M}}$ do not have any zero on (a,b). That is, they are of constant sign on I.

Furthermore, from the positiveness of $(-1)^{n-k} g_{\bar{M}}(t,s)$ on $(a,b)\times(a,b)$ and the last assertions of Lemmas 3.5.1 and 3.5.2, we conclude that for all $t\in(a,b)$:

$$(-1)^{n-k} w_{\bar{M}}(t) > 0, \quad \text{and} \quad \begin{cases} (-1)^{n-k} y_{\bar{M}}(t) > 0, & \text{if } \gamma \text{ is even,} \\ \\ (-1)^{n-k} y_{\bar{M}}(t) < 0, & \text{if } \gamma \text{ is odd.} \end{cases} \tag{3.7.1}$$

Consider the following function:

$$v_M(t,s) = \frac{(-1)^{n-k} g_M(t,s)}{(s-a)^\eta (b-s)^\gamma}. \tag{3.7.2}$$

Clearly, $v_M > 0$ on $(a,b)\times(a,b)$. For each $t\in(a,b)$, we have the following functions:

$$\begin{aligned} h_1(s) &= \lim_{s\to a^+} \frac{(-1)^{n-k}\, g_{\bar{M}}(t,s)}{(s-a)^\eta\,(b-s)^\gamma} = \frac{(-1)^{n-k}\,\dfrac{\partial^\eta}{\partial s^\eta}\, g_{\bar{M}}(t,s)_{|s=a}}{\eta!\,(b-a)^\gamma} < \infty, \\ h_2(s) &= \lim_{s\to b^-} \frac{(-1)^{n-k}\, g_{\bar{M}}(t,s)}{(s-a)^\eta\,(b-t)^\gamma} = \frac{(-1)^{n-k-\gamma}\,\dfrac{\partial^\gamma}{\partial s^\gamma}\, g_{\bar{M}}(t,s)_{|s=a}}{\gamma!\,(b-a)^\eta} < \infty. \end{aligned} \tag{3.7.3}$$

Thus, for each $t\in(a,b)$ we consider $\tilde{v}^t_{\bar{M}}$, the continuous extension of $v_{\bar{M}}$, where v_M has been defined in (3.7.2), to I. From (3.7.3) and (3.7.1), we conclude that $\tilde{v}^t_{\bar{M}}(s) > 0$ for all $s\in I$. Thus, consider:

$$\begin{aligned} \widehat{k}_1(t) &= \min_{s\in I} \tilde{v}^t_M(s), \quad t\in(a,b), \\ \widehat{k}_2(t) &= \max_{s\in I} \tilde{v}^t_M(s), \quad t\in(a,b), \end{aligned}$$

which are continuous and positive in (a,b).

As in Theorem 3.3.10, taking $\phi(s) = (s-a)^\eta\,(b-s)^\gamma > 0$ on (a,b), we have that condition $\left(P_{g_1}\right)$ is trivially fulfilled if $n-k$ is even, with $k_1(t) = \widehat{k}_1(t)$; and $k_2(t) = \widehat{k}_2(t)$ and condition $\left(N_{g_1}\right)$ holds if $n-k$ is odd, with $k_1(s) = -\widehat{k}_1(s)$ and $k_2(s) = -\widehat{k}_2(s)$.

Now, using Theorems 1.2.8, 1.2.10 and 1.2.11, we conclude that:

- If $n-k$ is even and $M \leqslant \bar{M}$, then $g_M(t,s)$ satisfies property $\left(P_{g_1}\right)$ if, and only if, $M\in(\bar{M}-\lambda_1, \bar{M}]$.
- If $n-k$ is odd and $M \geqslant \bar{M}$, then $g_M(t,s)$ satisfies property $\left(N_{g_1}\right)$ if, and only if, $M\in[\bar{M}, \bar{M}-\lambda_1)$.

Step 2. Study of the related Green's function at the boundary of $I\times I$.

From Lemma 3.5.1 and Proposition 3.6.8, since w_M is a continuous function with respect to M, we conclude:

- If $n-k$ is even, $k>1$ and $M\in[\bar{M}, \bar{M}-\lambda_2']$, then $w_M(t) > 0$ for all $t\in(a,b)$.

- If $k=1$, n is odd and $M \geqslant \bar{M}$, then $w_M(t) > 0$ for all $t \in (a,b)$.
- If $n-k$ is odd, $k>1$ and $M \in [\bar{M}-\lambda_2', \bar{M}]$, then $w_M(t) < 0$ for all $t \in (a,b)$.
- If $k=1$, n is even and $M \leqslant \bar{M}$, then $w_M(t) < 0$ for all $t \in (a,b)$.

From Lemma 3.5.2 and Proposition 3.6.10, since y_M is a continuous function with respect to M, we conclude:

- If $n-k$ is even and $M \in [\bar{M}, \bar{M}-\lambda_2'']$, then $(-1)^\gamma y_M(t) > 0$ for all $t \in (a,b)$.
- If $n-k$ is odd, $k < n-1$ and $M \in [\bar{M}-\lambda_2'', \bar{M}]$, then $(-1)^\gamma y_M(t) < 0$ for all $t \in (a,b)$.
- If $k = n-1$ and $M \leqslant \bar{M}$, then $(-1)^\gamma y_M(t) < 0$ for all $t \in (a,b)$.

From Lemma 3.5.3 and Proposition 3.6.17, since $\widehat{w}_M$ is a continuous function with respect to M, we conclude:

- If $n-k$ is even and $M \in [\bar{M}, \bar{M}-\lambda_2'']$, then $\widehat{w}_M(s) > 0$ for all $s \in (a,b)$.
- If $n-k$ is odd, $k<n-1$ and $M \in [\bar{M}-\lambda_2'', \bar{M}]$, then $\widehat{w}_M(s) < 0$ for all $s \in (a,b)$.
- If $k=n-1$ and $M \leqslant \bar{M}$, then $\widehat{w}_M(s) < 0$ for all $s \in (a,b)$.

From Lemma 3.5.4 and Proposition 3.6.18, since $\widehat{y}_M$ is a continuous function with respect to M, we conclude:

- If $n-k$ is even, $k>1$ and $M \in [\bar{M}, \bar{M}-\lambda_2']$, then $(-1)^\beta \widehat{y}_M(s) > 0$ for all $s \in (a,b)$.
- If $k=1$, n is odd and $M \geqslant \bar{M}$, then $(-1)^\beta \widehat{y}_M(s) > 0$ for all $s \in (a,b)$.
- If $n-k$ is odd, $k>1$ and $M \in [\bar{M}-\lambda_2', \bar{M}]$, then $(-1)^\beta \widehat{y}_M(s) < 0$ for all $s \in (a,b)$.
- If $k=1$, n is even and $M \leqslant \bar{M}$, then $(-1)^\beta \widehat{y}_M(s) < 0$ for all $s \in (a,b)$.

Step 3. Study of the related Green's function on $(a,b)\times(a,b)$.

Now, we need to verify that if M belongs to the given intervals, then $(-1)^{n-k}\, g_M(t,s) > 0$ for all $(t,s) \in I \times I$.

To this end, for all $s \in (a,b)$, let us denote $u_M^s(t) = g_M(t,s)$.

By the Green's function definition, see Definition 1.2.3, we know that for all $s \in (a,b)$:

$$T_n[\bar{M}]\, u_M^s(t) = \left(\bar{M}-M\right)\, u_M^s(t), \quad \forall t \neq s,\ t \in I. \tag{3.7.4}$$

Moreover, $u_M^s \in C^{n-2}(I)$ and it satisfies the boundary conditions (3.0.2)-(3.0.3).

As we have said, from Theorem 3.3.10, we have that $(-1)^{n-k} u_{\bar{M}}^s \geqslant 0$ on I.

Now, moving continuously with M, we will verify that while u_M^s is of constant sign on I, it cannot have a double zero on (a,b), which implies that the sign change must be either at $t=a$ or $t=b$ and then the result is proved.

We study separately the cases where $n-k$ is even or odd.

First, let us assume that $n-k$ is even. In this case, from Theorem 1.2.10, we only need to study the behaviour for $M > \bar{M}$ and $u_M^s \geqslant 0$.

From (3.7.4), we have that $T_n[\bar{M}]\, u_M^s \leqslant 0$; hence, since $v_1 \,\ldots\, v_n > 0$, $\dfrac{1}{v_n} T_{n-1}\, u_M^s$ is a decreasing function, with two continuous components. Then, it has at most two zeros on I (see Figure 3.7.1).

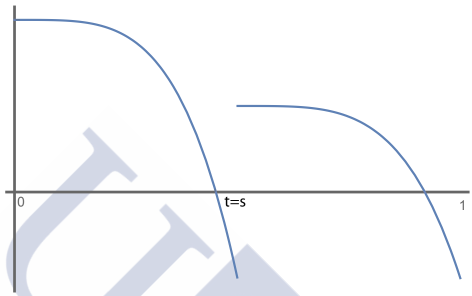

Figure 3.7.1: $\dfrac{1}{v_n(t)}\, T_{n-1} u_M^s(t)$, maximal oscillation with $t \in I = [0,1]$

Although we cannot know in which intervals $T_{n-1}u_M^s$ increases or decreases, using the fact that $v_n > 0$, then $T_{n-1}u_M^s$ has the same sign as $\dfrac{1}{v_n} T_{n-1}\, u_M^s$. Thus, $T_{n-1}u_M^s$ has at most two zeros on I.

So, $\dfrac{1}{v_{n-1}} T_{n-2}\, u_M^s$ is a continuous function, with at most four zeros on I (see Figure 3.7.2).

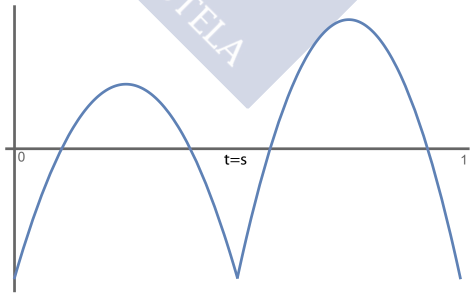

Figure 3.7.2: $\dfrac{1}{v_{n-1}(t)}\, T_{n-2} u_M^s(t)$, maximal oscillation with $t \in I = [0,1]$

Again, since $v_{n-1} > 0$, we conclude that $T_{n-2}u_M^s$ has the same sign as $\dfrac{1}{v_{n-1}}\, T_{n-2}u_M^s$. Thus, $T_{n-2}u_M^s$ has at most four zeros on I. Hence, $\dfrac{1}{v_{n-2}}\, T_{n-3}u_M^s$ has at most five zeros on I, the same as $T_{n-3}u_M^s$.

By recurrence, we conclude that $T_{n-\ell}\, u_M^s$ has, with maximal oscillation, at most $\ell+2$ zeros on I.

However, each time that either $T_{n-\ell}\, u_M^s(a) = 0$ or $T_{n-\ell}\, u_M^s(b) = 0$, a possible oscillation on (a,b) is lost. From the boundary conditions (3.0.2)-(3.0.3), coupled with Lemmas 3.3.6 and 3.3.7, we can affirm that n possible oscillations are lost. Hence, u_M^s can have at most two zeros on (a,b).

Let us see that, despite this fact does not inhibit that, with maximal oscillation, u_M^s has a double zero on (a,b), this double zero is not possible. If $T_\ell u(a) = 0$ for $\ell \notin \{\sigma_1, \ldots, \sigma_k\}$, then u_s can have only a simple zero and this is not possible while it is of constant sign.

Now, to allow this possible double zero, let us study which should be the sign of $u^{(\alpha)}(a)$. We have already said that $T_{n-\ell} u_M^s(a)$ changes its sign for two consecutive $\ell \in \{0, \ldots, n\}$ if it does not vanish. Moreover, at every time that $T_{n-\ell} u_M^s(a) = 0$ the sign change comes on the next $\tilde{\ell}$ for which $T_{n-\tilde{\ell}} u_M^s(a) \neq 0$, see Figure 3.3.2. Since, from $\ell = 0$ to $n-\alpha$ there are $k-\alpha$ zeros of $T_{n-\ell} u_M^s(a)$, to allow the maximal oscillation it is necessary to have that:

$$\begin{cases} T_\alpha\, u_M^s(a) \leqslant 0, & \text{if } n-\alpha-(k-\alpha) = n-k \text{ is even,} \\ \\ T_\alpha\, u_M^s(a) \geqslant 0, & \text{if } n-k \text{ is odd.} \end{cases}$$

As a direct consequence of (3.6.1), we can affirm that with maximal oscillation it must be satisfied:

$$\begin{cases} {u_M^s}^{(\alpha)}(a) \leqslant 0, & \text{if } n-k \text{ is even,} \\ \\ {u_M^s}^{(\alpha)}(a) \geqslant 0, & \text{if } n-k \text{ is odd.} \end{cases} \tag{3.7.5}$$

On another hand, since $u_M^s \geqslant 0$, we have that ${u_M^s}^{(\alpha)}(a) \geqslant 0$. We can assume that ${u_M^s}^{(\alpha)}(a) > 0$. Because, in other case, i.e. if ${u_M^s}^{(\alpha)}(a) = 0$, then $T_\alpha\, u_M^s(a) = 0$ and another possible oscillation is lost, so it only remains the possibility of having a simple zero on (a,b), which is not possible when u_M^s is of constant sign.

If ${u_M^s}^{(\alpha)}(a) > 0$, from (3.7.5) the maximal oscillation is not allowed. So, we have again only the possibility of a simple zero on (a,b). Hence we conclude:

- If $n-k$ even and $M > \bar{M}$, if $u_M^s \geqslant 0$, then $u_M^s > 0$ on (a,b).

Now, let us see what happens if $n-k$ is odd. In this case, $u_M^s \leqslant 0$ on I and, from Theorem 1.2.11, we only need to study the case $M < \bar{M}$. Thus from (3.7.4), we have again that $T_n[\bar{M}]\, u_M^s \leqslant 0$. Hence, we can use previous arguments to conclude that with maximal oscillation u_M^s satisfies (3.7.5).

However, in this case, since $u_M^s \leqslant 0$, ${u_M^s}^{(\alpha)}(a) \leqslant 0$. Arguing as before, we can assume that ${u_M^s}^{(\alpha)}(a) < 0$. Hence, the maximal oscillation is lost and we can affirm:

- If $n-k$ odd and $M < \bar{M}$, if $u_M^s \leqslant 0$, then $u_M^s < 0$ on (a,b).

Thus, repeating the arguments of Step 1, the sufficient condition on the result is proved.

Step 4. The intervals of constant sign Green's function cannot be increased.

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Now, let us assume that there is another M for which $g_M(t,s)$ satisfies either condition $\left(P_{g_1}\right)$ if $n-k$ is even or condition $\left(N_{g_1}\right)$ if $n-k$ is odd.

Let us consider the case where $n-k$ is even (the case where $n-k$ is odd is analogous).

From Theorem 1.2.10, if there exists $M^* \notin (\bar{M}-\lambda_1, \bar{M}-\lambda 2]$ such that $g_{M^*}(t,s)$ satisfies $\left(P_{g_1}\right)$, then $M^* > \bar{M}-\lambda_2$.

Without lost of generality, let us assume that $\lambda_2 = \lambda_2'$.

Suppose that there exists $\widehat{M} > \bar{M}-\lambda_2'$ such that $T_n[\widehat{M}]$ is inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, thus from Theorems 1.2.8 and 1.2.9, we can affirm that for every $M \in [\bar{M}-\lambda_2', \widehat{M}]$ operator $T_n[M]$ is inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and, moreover, $w_{\bar{M}-\lambda_2'} \geqslant w_M \geqslant w_{\widehat{M}}$.

In particular, $0 = w_{\bar{M}-\lambda_2'}^{(\beta)}(b) \geqslant w_M^{(\beta)}(b) \geqslant w_{\widehat{M}}^{(\beta)}(b)$ if β is even and, on the other hand, $0 = w_{\bar{M}-\lambda_2'}^{(\beta)}(b) \leqslant w_M^{(\beta)}(b) \leqslant w_{\widehat{M}}^{(\beta)}(b)$ if β is odd.

If $w_{\widehat{M}}^{(\beta)}(b) \neq 0$, then there exists $\rho > 0$ such that $w_{\widehat{M}}(t) < 0$ for all $t \in (b-\rho, b)$, which contradicts our assumption. So:

$$0 = w_{\bar{M}-\lambda_2'}^{(\beta)}(b) = w_M^{(\beta)}(b) = w_{\widehat{M}}^{(\beta)}(b), \quad \forall M \in [\bar{M}-\lambda_2', \widehat{M}],$$

and this contradicts the discrete character of the spectrum of the operator $T_n[\bar{M}]$ in the set $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.

Hence the result is proved. □

Example 3.7.2. *In Example 3.6.7, we have obtained the eigenvalues related to operator $T_4^0[0]$ in the different sets, $X_{\{0,2\}}^{\{1,2\}}$, $X_{\{0,2\}}^{\{0,1\}}$ and $X_{\{0,1\}}^{\{1,2\}}$. They are denoted by λ_1, λ_2' and λ_2'', respectively. We have that $\lambda_1 = m_1^4$ and:*

$$\lambda_2 = \max\left\{\lambda_2', \lambda_2''\right\} = \max\left\{-m_2^4, -4\,\pi^4\right\} = -4\,\pi^4,$$

where $m_1 \approxeq 2.36502$ and $m_2 \approxeq 5.550305$ have been introduced in Example 3.6.7 as the least positive solutions of (3.6.2) *and* (3.6.3)*, respectively.*

So, we can affirm that the related Green's function satisfies condition $\left(P_{g_1}\right)$ if, and only if, $M \in \left(-m_1^4, 4\,\pi^4\right] \approxeq \left(-2.365^4, 4\,\pi^4\right]$.

3.7.2 Study of Green's function in a different parameter set

In the previous section we have obtained a characterisation of the parameter set where the related Green's function to problem (3.0.1)–(3.0.3), $g_M(t,s)$, is either positive or negative on $I \times I$ if $n-k$ is even or odd, respectively.

Moreover, in some cases such as the $(k, n-k)$ boundary conditions, from Theorem 3.5.5, we can ensure that if $n-k$ is even, then there is not any $M \in \mathbb{R}$ such that $g_M(t,s)$ is non-positive on $I \times I$ and if $n-k$ is odd, then there is not any $M \in \mathbb{R}$ such that $g_M(t,s)$ is non-negative on $I \times I$. However, on the cases which do not fulfil the hypotheses of Theorem 3.5.5, we have not said anything about the negative constant sign of $g_M(t,s)$ if $n-k$ is even or, on the contrary, about the positive constant sign of $g_M(t,s)$ if $n-k$ is odd. We know that there is some cases when this parameter set are not empty, for instance see [22, 87] which refer to a fourth order problem coupled with the simply supported beam boundary conditions.

From Theorems 1.2.10, 1.2.12 and 1.2.7, if $n-k$ is even and there exists $\bar{M} \in \mathbb{R}$ such that $g_{\bar{M}}(t,s)$ is non-positive on $I \times I$, then the parameter set, for which such a property is fulfilled, is given by an interval whose maximum is given by $\bar{M} - \lambda_1$.

Moreover, from Theorems 1.2.11, 1.2.13 and 1.2.7, if $n-k$ is odd and there exists $\bar{M} \in \mathbb{R}$ such that $g_{\bar{M}}(t,s)$ is non-negative on $I \times I$, then the parameter set, for which such a property is fulfilled, is given by an interval whose minimum is given by $\bar{M} - \lambda_1$.

This section, is devoted to obtain a bound of the other extreme of the parameter set. Furthermore, we will see that, in such an interval, the Green's function satisfies a suitable property which allows us to prove that the obtained parameter set is optimal provided that the Green's function cannot have any zero on $(a,b) \times (a,b)$ for all M which belongs to that set.

Theorem 3.7.3. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ fulfil (N_a).*

Let us denote $g_M(t,s)$ as the related Green's function of the problem (3.0.1)–(3.0.3). *Then, the following properties are satisfied.*

- *If $n-k$ is even and $g_M(t,s)$ is non-positive on $I \times I$, then $M \in [\bar{M} - \lambda_3, \bar{M} - \lambda_1)$, where*
  - * $\lambda_1 > 0$ *is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * $\lambda_3 > 0$ *is the minimum between:*
    - · $\lambda_3' > 0$, *the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - · $\lambda_3'' > 0$, *the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*
- *If $n-k$ is odd and $g_M(t,s)$ is non-negative on $I \times I$, then $M \in (\bar{M} - \lambda_1, \bar{M} - \lambda_3]$, where*
  - * $\lambda_1 < 0$ *is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * $\lambda_3 < 0$ *is the maximum between:*
    - · $\lambda_3' < 0$, *the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - · $\lambda_3'' < 0$, *the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*

Proof. From Theorem 3.5.5, we can affirm that $\sigma_k \neq k-1$ and $\varepsilon_{n-k} \neq n-k-1$. Hence, by Corollary 3.6.6 the existence of λ_3' and λ_3'' is ensured.

First, let us focus on the case where $n-k$ is even.

Let us assume that there exists $M^* \notin [\bar{M} - \lambda_3, \bar{M} - \lambda_1)$, such that $g_M(t,s)$ is non-positive on $I \times I$. From Theorem 1.2.10, we know that $M^* < \bar{M} - \lambda_3$.

Moreover, by considering Theorems 1.2.7 and 1.2.12, we can affirm that for every M in $[M^*, \bar{M} - \lambda_1)$ $g_M(t,s)$ is non-positive on $I \times I$ and, by Theorem 1.2.9, we have that:

$$0 \geqslant g_{M^*}(t,s) \geqslant g_M(t,s) \geqslant g_{\bar{M}-\lambda_3}(t,s).$$

So, in particular:

$$0 \geqslant w_{M^*}(t) \geqslant w_M(t) \geqslant w_{\bar{M}-\lambda_3}(t),$$

and,

$$\begin{cases} 0 \geqslant y_{M^*}(t) \geqslant y_M(t) \geqslant y_{\bar{M}-\lambda_3}(t), & \text{if } \gamma \text{ is even,} \\ 0 \leqslant y_{M^*}(t) \leqslant y_M(t) \leqslant y_{\bar{M}-\lambda_3}(t), & \text{if } \gamma \text{ is odd.} \end{cases}$$

If $\lambda_3 = \lambda_3'$, then $w^{(\alpha)}_{\bar{M}-\lambda_3}(a) = 0$. So, we deduce that for all $M \in [M^*, \bar{M} - \lambda_3)$, $w^{(\alpha)}_M(a) = 0$ holds, which contradicts the discrete character of the spectrum of $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.

If $\lambda_3 = \lambda_3''$, then $y^{(\beta)}_{\bar{M}-\lambda_3}(b) = 0$. So, if $M \in [M^*, \bar{M} - \lambda_3)$, then $y^{(\beta)}_M(b) = 0$ in contradiction with the discrete character of the spectrum of $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Analogously, if $n-k$ is odd and we assume that there exists $M^* \notin (\bar{M} - \lambda_1, \bar{M} - \lambda_3]$, such that $g_M(t,s)$ is non-negative on $I \times I$. From Theorems 1.2.11 and 1.2.7, we know that $M^* > \bar{M} - \lambda_3$.

Moreover, using Theorem 1.2.13, we can affirm that for all $M \in (\bar{M} - \lambda_1, M^*]$, the function $g_M(t,s)$ is non-negative on $I \times I$ and, by Theorem 1.2.9, we have that:

$$g_{\bar{M}-\lambda_3}(t,s) \geqslant g_M(t,s) \geqslant g_{M^*}(t,s) \geqslant 0.$$

So, in particular:

$$w_{\bar{M}-\lambda_3}(t) \geqslant w_M(t) \geqslant w_{M^*}(t) \geqslant 0,$$

and,

$$\begin{cases} y_{\bar{M}-\lambda 3}(t) \geqslant y_M(t) \geqslant y_{M^*}(t) \geqslant 0, & \text{if } \gamma \text{ is even,} \\ y_{\bar{M}-\lambda 3}(t) \leqslant y_M(t) \leqslant y_{M^*}(t) \leqslant 0, & \text{if } \gamma \text{ is odd.} \end{cases}$$

If $\lambda_3 = \lambda_3'$, then $w^{(\alpha)}_{\bar{M}-\lambda_3}(a) = 0$ and we deduce that for all $M \in (\bar{M} - \lambda_3, M^*]$, $w^{(\alpha)}_M(a) = 0$, which contradicts the discrete character of the spectrum of operator $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}$.

If $\lambda_3 = \lambda_3''$, then $y^{(\beta)}_{\bar{M}-\lambda_3}(b) = 0$. Thus, we conclude that for all $M \in (\bar{M} - \lambda_3, M^*]$, $y^{(\beta)}_M(b) = 0$ holds, a contradiction with the discrete character of the spectrum of operator $T_n[\bar{M}]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

In all cases we arrive to a contradiction, thus the result is proved. □

Example 3.7.4. *Now, let us focus again in our recurrent example*

$$T_4^0[M]\,u(t) = u^{(4)}(t) + M\,u(t).$$

In Example 3.6.7, we have calculated the different eigenvalues of $T_4^0[0]$ on the sets $X^{\{1,2\}}_{\{0,2\}}$, $X^{\{1,2\}}_{\{0,1\}}$ and $X^{\{1,2\}}_{\{0,2\}}$. In particular, $\lambda_1 = m_1^4$ and

$$\lambda_3 = \min\{m_3^4, \pi^4\} = \pi^4,$$

where $m_1 \approxeq 2.36502$ *and* $m_3 \approxeq 3.9266$ *have been introduced in Example 3.6.7 as the least positive solutions of* (3.6.2) *and* (3.6.4)*, respectively.*

So, by using Theorem 3.7.3, we can affirm that if $g_M(t,s) \leqslant 0$ *on* $I \times I$*, then* M *belongs to the interval* $\big[-\pi^4, -m_1^4\big) \approxeq \big[-\pi^4, -2.365^4\big)$.

In Theorem 3.7.3, we have established a necessary condition on $g_M(t,s)$ to be either non-negative or non-positive on $I \times I$. Next result shows that this condition also ensures that the related Green's function satisfies a suitable condition on the boundary of $I \times I$.

Theorem 3.7.5. *Let* $\bar M \in \mathbb{R}$ *be such that* $T_n[\bar M]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *fulfil* (N_a)*. Moreover,* $\sigma_k \neq k-1$ *and* $\varepsilon_{n-k} \neq n-k-1$*. Then, the following properties are satisfied.*

- *If* $n-k$ *is even and* $M \in [\bar M - \lambda_3, \bar M - \lambda_1)$*, where* λ_1 *and* λ_3 *are given in Theorem 3.7.3. Then:*

  * $w_M(t) < 0$ *for all* $t \in (a,b)$*, where* w_M *has been introduced in Lemma 3.5.1.*
  * $(-1)^\gamma y_M(t) < 0$ *for all* $t \in (a,b)$*, where* y_M *has been introduced in Lemma 3.5.2 and* γ*, in* (3.4.8).
  * $\widehat{w}_M(s) < 0$ *for all* $s \in (a,b)$*, where* $\widehat{w}_M$ *has been introduced in Lemma 3.5.3.*
  * $(-1)^\beta \widehat{y}_M(s) < 0$ *for all* $s \in (a,b)$*, where* $\widehat{y}_M$ *has been introduced in Lemma 3.5.4 and* β*, in* (3.2.2).

- *If* $n-k$ *is odd and* $M \in (\bar M - \lambda_1, \bar M - \lambda_3]$*, where* λ_1 *and* λ_3 *are given in Theorem 3.7.3. Then:*

  * $w_M(t) > 0$ *for all* $t \in (a,b)$*, where* w_M *has been introduced in Lemma 3.5.1.*
  * $(-1)^\gamma y_M(t) > 0$ *for all* $t \in (a,b)$*, where* y_M *has been introduced in Lemma 3.5.2 and* γ*, in* (3.4.8).
  * $\widehat{w}_M(s) > 0$ *for all* $s \in (a,b)$*, where* $\widehat{w}_M$ *has been introduced in Lemma 3.5.3.*
  * $(-1)^\beta \widehat{y}_M(s) > 0$ *for all* $s \in (a,b)$*, where* $\widehat{y}_M$ *has been introduced in Lemma 3.5.4 and* β*, in* (3.2.2)..

Proof. From Lemmas 3.5.1, 3.5.2, 3.5.3, 3.5.4 and Propositions 3.6.8, 3.6.10, 3.6.20 and 3.6.21, we know that w_M, y_M, $\widehat{w}_M$ and $\widehat{y}_M$ do not have any zero in (a,b) for $M \in [\bar M - \lambda_3, \bar M - \lambda_1)$ if $n-k$ is even and for $M \in (\bar M - \lambda_1, \bar M - \lambda_3]$ whenever $n-k$ is odd.

Moreover, by Proposition 3.6.13, since we do not reach any eigenvalue of $T_n[\bar M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, we have that the related Green's function is well-defined for every M in those intervals. So, since we are moving continuously on M, we conclude that its sign is the same in all the interval.

Now, let us study the sign of these functions at a given M.

We consider w_M and $\widehat{w}_M$ at $M = \bar M - \lambda_3'$ and y_M and $\widehat{y}_M$ at $M = \bar M - \lambda_3''$. As we have proved before, at this values of the real parameter the functions are of constant sign and satisfy the maximal oscillation conditions, which means that, with the related boundary conditions, they satisfy the conditions at $t = a$ and $t = b$ to give the maximum number of zeros. Otherwise the function would be identically zero and this is not true. Moreover, we know that they satisfy for all $M \in \mathbb{R}$ the boundary conditions (3.5.2), (3.5.4), (3.5.5) and (3.5.7), respectively.

- Study of $w_{\bar{M}-\lambda_3'}$.

Let us consider $\alpha_1 \in \{0, \dots, n-1\}$, previously introduced in Notation 3.6.12.

Let us study the sign of $w^{(\alpha_1)}_{\bar{M}-\lambda_3'}(a)$ in order to obtain the sign of $w_{\bar{M}-\lambda_3'}$.

From Lemma 3.3.6, coupled with (3.5.2), we conclude that:

$$\begin{cases} T_{\sigma_k}\, w_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } n-\sigma_k \text{ is odd,} \\ T_{\sigma_k}\, w_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } n-\sigma_k \text{ is even.} \end{cases}$$

As we have said before, to allow the maximal oscillation, $T_{n-\ell} w_{\bar{M}-\lambda_3'}(a)$ must change its sign each time that it is different from zero, see Figure 3.3.2. Thus, since from α_1 to σ_k we have $k-\alpha_1$ zeros, with maximal oscillation the following inequalities are fulfilled:

If $n-\sigma_k$ is odd:

$$\begin{cases} T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } \sigma_k-\alpha_1-(k-\alpha_1) = \sigma_k - k \text{ is even,} \\ T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } \sigma_k - k \text{ is odd.} \end{cases}$$

If $n-\sigma_k$ is even:

$$\begin{cases} T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } \sigma_k - k \text{ is even,} \\ T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } \sigma_k - k \text{ is odd.} \end{cases}$$

From (3.3.3), we conclude that:

$$T_{\alpha_1}\, w_{\bar{M}-\lambda_3'}(a) = \frac{1}{v_1(a)\dots v_{\alpha_1}(a)} w^{(\alpha_1)}_{\bar{M}-\lambda_3'}(a),$$

hence, we can affirm that:

$$\begin{cases} w^{(\alpha_1)}_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } n-k \text{ is odd,} \\ w^{(\alpha_1)}_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } n-k \text{ is even.} \end{cases}$$

Thus, we have proved that:

* If $n-k$ is even and $M \in [\bar{M}-\lambda_3', \bar{M}-\lambda_1)$, then $w_M(t) < 0$ for all $t \in (a,b)$.
* If $n-k$ is odd and $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3']$, then $w_M(t) > 0$ for all $t \in (a,b)$.

- Study of $y_{\bar{M}-\lambda_3''}$

Now, let us consider $\beta_1 \in \{0,\dots,n-1\}$, introduced in Notation 3.6.12. In order to obtain the sign of $y_{\bar{M}-\lambda_3''}$, let us study the sign of $y_{\bar{M}-\lambda_3''}^{(\beta_1)}(b)$.

From Lemma 3.3.7 coupled with (3.5.4), we conclude that:

$$\begin{cases} T_{\varepsilon_{n-k}}\, y_{\bar{M}-\lambda_3''}(b) > 0, & \text{if } n-\varepsilon_{n-k} \text{ is even,} \\[2ex] T_{\varepsilon_{n-k}}\, y_{\bar{M}-\lambda_3''}(b) < 0, & \text{if } n-\varepsilon_{n-k} \text{ is odd.} \end{cases}$$

In this case, as we have said on the proof of Theorem 3.3.10, to allow the maximal oscillation, $T_{n-\ell}w_{\bar{M}-\lambda_3'}(b)$ must change its sign each time that it vanishes. Thus, since from β_1 to ε_{n-k} we have, with maximal oscillation, $n-k-\beta_1$ zeros, we deduce de following properties.

If $n-\varepsilon_{n-k}$ is even:

$$\begin{cases} T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) > 0, & \text{if } n-k-\beta_1 \text{ is even,} \\[2ex] T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) < 0, & \text{if } n-k-\beta_1 \text{ is odd.} \end{cases}$$

If $n-\varepsilon_{n-k}$ is odd:

$$\begin{cases} T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) < 0, & \text{if } n-k-\beta_1 \text{ is even,} \\[2ex] T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) > 0, & \text{if } n-k-\beta_1 \text{ is odd.} \end{cases}$$

From (3.3.3), we conclude that:

$$T_{\beta_1}\, y_{\bar{M}-\lambda_3''}(b) = \frac{1}{v_1(b)\dots v_{\beta_1}(b)}\, y_{\bar{M}-\lambda_3''}^{(\beta_1)}(b),$$

so, we can affirm that:

$$\begin{cases} y_{\bar{M}-\lambda_3''}^{(\beta_1)}(b) > 0, & \text{if } \varepsilon_{n-k}-k-\beta_1 \text{ is even,} \\[2ex] y_{\bar{M}-\lambda_3''}^{(\beta_1)}(b) < 0, & \text{if } \varepsilon_{n-k}-k-\beta_1 \text{ is odd.} \end{cases}$$

Thus, we have proved that:

$$\begin{cases} y_{\bar{M}-\lambda_3''}(t) \geqslant 0, \quad t \in I, & \text{if } \varepsilon_{n-k}-k \text{ is even,} \\[2ex] y_{\bar{M}-\lambda_3''}(t) \leqslant 0, \quad t \in I, & \text{if } \varepsilon_{n-k}-k \text{ is odd.} \end{cases}$$

Hence, since $\gamma = n-\varepsilon_{n-k}-1$, taking into account the fact that $y_{\bar{M}-\lambda_3''}$ cannot have any zero on (a,b), we conclude:

* If $n-k$ is even and $M \in [\bar{M}-\lambda_3'', \bar{M}-\lambda_1)$, then $(-1)^\gamma y_M(t) < 0$ for al $t \in (a,b)$.

* If $n-k$ is odd and $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3'']$, then $(-1)^\gamma y_M(t) > 0$ for all $t \in (a,b)$.

- Study of $\widehat{w}_{\bar{M}-\lambda_3'}$

Notation 3.7.6. *Let us define* $\eta_1 \in \{0,\dots,n-1\}$ *such that* $\eta_1 \notin \{\tau_1,\dots,\tau_{n-k-1},\eta\}$ *and* $\{0,\dots,\eta_1-1\} \subset \{\tau_1,\dots,\tau_{n-k-1},\eta\}$.

In order to obtain the sign of $\widehat{w}_{\bar{M}-\lambda_3'}$, let us study the sign of $\widehat{w}^{(\eta_1)}_{\bar{M}-\lambda_3'}(a)$.

From Lemma 3.4.9 coupled with (3.5.5), we conclude that:

$$\begin{cases} \widehat{T}_{\tau_{n-k}}\, \widehat{w}_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } \tau_{n-k} \text{ is odd,} \\ \widehat{T}_{\tau_{n-k}}\, \widehat{w}_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } \tau_{n-k} \text{ is even.} \end{cases}$$

As we have argued with T_k, to allow the maximal oscillation $\widehat{T}_{n-\ell}\widehat{w}_{\bar{M}-\lambda_3'}(a)$ must change its sign each time that it is non-null, see Figure 3.3.2. Thus, since from η_1 to τ_{n-k} we have $n-k-\eta_1$ zeros, with maximal oscillation, the following inequalities are fulfilled.

If τ_{n-k} is odd:

$$\begin{cases} \widehat{T}_{\eta_1}\, \widehat{w}_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } \tau_{n-k}-\eta_1-(n-k-\eta_1) = \tau_{n-k}-n+k \text{ is even,} \\ \widehat{T}_{\eta_1}\, \widehat{w}_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } \tau_{n-k}-n+k \text{ is odd.} \end{cases}$$

If τ_{n-k} is even:

$$\begin{cases} \widehat{T}_{\eta_1}\, \widehat{w}_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } \tau_{n-k}-n+k \text{ is even,} \\ \widehat{T}_{\eta_1}\, \widehat{w}_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } \tau_{n-k}-n+k \text{ is odd.} \end{cases}$$

From (3.4.25), we conclude that:

$$\widehat{T}_{\eta_1}\, \widehat{w}_{\bar{M}-\lambda_3'}(a) = v_1(a)\dots v_{n-\eta_1}(a) w^{(\eta_1)}_{\bar{M}-\lambda_3'}(a),$$

hence, we can affirm that:

$$\begin{cases} \widehat{w}^{(\eta_1)}_{\bar{M}-\lambda_3'}(a) > 0, & \text{if } n-k \text{ is odd,} \\ \widehat{w}^{(\eta_1)}_{\bar{M}-\lambda_3'}(a) < 0, & \text{if } n-k \text{ is even.} \end{cases}$$

Thus, we have proved that:

$$\begin{cases} \widehat{w}_{\bar{M}-\lambda_3'} \geqslant 0, & \text{on } I \text{ if } n-k \text{ is odd,} \\ \widehat{w}_{\bar{M}-\lambda_3'} \leqslant 0, & \text{on } I \text{ if } n-k \text{ is even.} \end{cases}$$

So, we conclude:

  * If $n-k$ is even and $M \in [\bar{M}-\lambda_3', \bar{M}-\lambda_1)$, then $\widehat{w}_M(s) < 0$ for all $s \in (a,b)$.
  * If $n-k$ is odd and $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3']$, then $\widehat{w}_M(s) > 0$ for all $s \in (a,b)$.

- Study of $\widehat{y}_{\bar{M}-\lambda_3''}$

Notation 3.7.7. *Let us denote* $\gamma_1 \in \{0,\ldots,n-1\}$*, such that* $\gamma_1 \notin \{\delta_1,\ldots,\delta_{k-1},\gamma\}$ *and* $\{0,\ldots,\gamma_1-1\} \subset \{\delta_1,\ldots,\delta_{k-1},\gamma\}$.

In order to obtain the sign of $\widehat{y}_{\bar{M}-\lambda_3''}$, let us study the sign of $\widehat{y}^{(\beta_1)}_{\bar{M}-\lambda_3''}(b)$.

From Lemma 3.4.10 coupled with (3.5.7), we conclude that:

$$\begin{cases} \widehat{T}_{\delta_k}\, \widehat{y}_{\bar{M}-\lambda_3''}(b) > 0, & \text{if } \delta_k \text{ is even,} \\ \\ \widehat{T}_{\delta_k}\, \widehat{y}_{\bar{M}-\lambda_3''}(b) < 0, & \text{if } \delta_k \text{ is odd.} \end{cases}$$

In this case, as we have argued with T_k, to allow the maximal oscillation, $\widehat{T}_{n-\ell} w_{\bar{M}-\lambda_3'}(b)$ changes its sign each time that it vanishes and it remains of constant sign if it does not vanish, see Figure 3.3.2. Thus, with maximal oscillation, since from γ_1 to δ_k we have $k-\gamma_1$ zeros

If δ_k is even:

$$\begin{cases} \widehat{T}_{\gamma_1}\, \widehat{y}_{\bar{M}-\lambda_3''}(b) > 0, & \text{if } k-\gamma_1 \text{ is even,} \\ \\ \widehat{T}_{\gamma_1}\, \widehat{y}_{\bar{M}-\lambda_3''}(b) < 0, & \text{if } k-\gamma_1 \text{ is odd.} \end{cases}$$

If δ_k is odd:

$$\begin{cases} \widehat{T}_{\gamma_1}\, \widehat{y}_{\bar{M}-\lambda_3''}(b) < 0, & \text{if } k-\gamma_1 \text{ is even,} \\ \\ \widehat{T}_{\gamma_1}\, \widehat{y}_{\bar{M}-\lambda_3''}(b) > 0, & \text{if } k-\gamma_1 \text{ is odd.} \end{cases}$$

From (3.4.25), we conclude that:

$$\widehat{T}_{\gamma_1}\, \widehat{y}_{\bar{M}-\lambda_3''}(b) = v_1(b)\ldots v_{n-\gamma_1}(b)\widehat{y}^{(\gamma_1)}_{\bar{M}-\lambda_3''}(b),$$

hence, we can affirm that:

$$\begin{cases} \widehat{y}^{(\gamma_1)}_{\bar{M}-\lambda_3''}(b) > 0, & \text{if } k-\delta_k-\gamma_1 \text{ is even,} \\ \\ \widehat{y}^{(\gamma_1)}_{\bar{M}-\lambda_3'}(b) < 0, & \text{if } k-\delta_k-\gamma_1 \text{ is odd.} \end{cases}$$

Thus, we have proved that:

$$\begin{cases} \widehat{y}_{\bar{M}-\lambda_3''}(t) \geqslant 0, & \text{if } k-\delta_k \text{ is even,} \\ \\ \widehat{y}_{\bar{M}-\lambda_3'}(t) \leqslant 0, & \text{if } k-\delta_k \text{ is odd.} \end{cases}$$

Hence, since $\beta = n-\delta_k-1$, we conclude:

* If $n-k$ is even and $M \in [\bar{M}-\lambda_3'', \bar{M}-\lambda_1)$, then $(-1)^\beta \widehat{y}_M(s) < 0$ for all $s \in (a,b)$.

* If $n-k$ is odd and $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_3'']$, then $(-1)^\beta \widehat{y}_M(s) > 0$ for all $s \in (a,b)$.

So, the proof is complete. $\square$

Remark 3.7.8. *Realise that from Theorems 3.7.3 and 3.7.5, if we are able to prove that the Green's function cannot have any zero on $(a,b)\times(a,b)$ and the properties imposed in Theorem 3.7.5 are fulfilled, then the intervals obtained in Theorem 3.7.3 are optimal. Moreover, we obtain that in those intervals the related Green's functions satisfies either property $\left(N_{g_1}\right)$ or $\left(P_{g_1}\right)$.*

Example 3.7.9. *Now, let us apply Remark 3.7.8 to our recurrent example, the operator $T_4^0[M]$.*

Let us assume that there exists $M^* \in [-\pi^4, m_1^4)$ such that $g_{M^*}(t,s)$ changes its sign. Then, from Theorem 3.7.5 it must exist $s^* \in (0,1)$ such that $u^*(t) = g_{M^*}(t,s^*)$ has at least two zeros, $0 < c_1 < c_2 < 1$.

By the definition of the Green's function $u^* \in C^2\left([0,1]\right)$. So, there exists $c^* \in (c_1,c_2)$ such that $u^{*\prime}(c^*) = 0$. There are two possibilities:

- $c^* \leqslant s^*$.

  In such a case, u^* is a solution of $T_4^0[M^*]u^*(t) = 0$ on $[0,c^*]$ satisfying the boundary conditions $u^*(0) = u^{*\prime\prime}(0) = u^{*\prime}(c^*) = 0$. Moreover, it satisfies $u^*(c_1) = 0$.

  Function $y^*(t) = u^*(c^* t)$ satisfies $y^*(0) = y^{*\prime\prime}(0) = y^{*\prime}(1) = 0$ and $y^*\left(\dfrac{c_1}{c^*}\right) = 0$. Moreover, it is a solution of $T_4^0[{c^*}^4 M^*]y^*(t) = 0$ on $[0,1]$, with

  $$0 > {c^*}^4 M^* > M^* > -\pi^4 > -m_3^4,$$

  where m_3^4 has been introduced in Example 3.6.7. But, this is is a contradiction with Proposition 3.6.10.

- $c^* > s^*$.

  In this case, u^* is a solution of $T_4^0[M^*]u^*(t) = 0$ on $[c^*,1]$ satisfying the boundary conditions $u^{*\prime}(c^*) = u^{*\prime}(1) = u^{*\prime\prime}(1) = 0$. Moreover, it satisfies $u^*(c_2) = 0$.

  The function $y^*(t) = u^*\big((1-c^*)t + c^*\big)$ satisfies

  $$y^{*\prime}(0) = y^{*\prime}(1) = y^{*\prime\prime}(1) = 0. \tag{3.7.6}$$

  Moreover, it is a solution of $T_4^0\big[(1-c^*)^4 M^*\big]y^*(t) = 0$ on $[0,1]$ and it satisfies $y^*\left(\dfrac{c_2-c^*}{1-c^*}\right) = 0$, with $0 > (1-c^*)^4 M^* > M^* > -\pi^4$.

  We can see that π^4 is the least positive eigenvalue of $T_4^0[0]$ in $X_{\{0,1\}}^{\{1,2\}}$.

  Now, let us see that every solution of $u^{(4)}(t) + M\,u(t) = 0$, satisfying the given boundary conditions (3.7.6), cannot have any zero on $(0,1)$ whenever $M \in (-\pi^4, 0)$. This is a contradiction of supposing that there is a sign change on the Green's function.

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First, let us choose $M=-\left(\dfrac{\pi}{2}\right)^4$*, the solution is given as a multiple of:*

$$u(t)=f_1(1-t)+f_2(1-t),$$

where, for $t\in[0,1]$*,*

$$f_1(t)=\left(1-\sinh\left(\frac{\pi}{2}\right)\right)\left(\sinh\left(\frac{\pi t}{2}\right)-\sin\left(\frac{\pi t}{2}\right)\right)\geqslant f_1(1)=-\left(\sinh\left(\frac{\pi}{2}\right)-1\right)^2,$$

$$f_2(t)=\cosh\left(\frac{\pi}{2}\right)\left(\cos\left(\frac{\pi t}{2}\right)+\cosh\left(\frac{\pi t}{2}\right)\right)\geqslant f_2(0)=2\cosh\left(\frac{\pi}{2}\right).$$

So, $u(t)\geqslant-\left(\sinh\left(\dfrac{\pi}{2}\right)-1\right)^2+2\cosh\left(\dfrac{\pi}{2}\right)>0$ *for all* $t\in[0,1]$.

Now, let us move continuously on M *to obtain the different solutions of*

$$T_4^0[M]\,u(t)=0,\quad t\in[0,1]$$

coupled with the boundary conditions (3.7.6).

Let us see that it is not possible that these solutions start to change sign on $(0,1)$*. If this was the case, we would have that there exist* $\widehat{M}\in(-\pi^4,0)$ *and* $\hat{t}\in(0,1)$ *such that* $\hat{u}$ *is a solution of* $T_4^0\left[\widehat{M}\right]\hat{u}(t)=0$ *on* $\left[\hat{t},1\right]$*, satisfying*

$$\hat{u}\left(\hat{t}\right)=\hat{u}'\left(\hat{t}\right)=\hat{u}(1)=\hat{u}''(1)=0.$$

Then, the function $\hat{y}(t)=\hat{u}\left(\left(1-\hat{t}\right)t+\hat{t}\right)$ *is an eigenfunction related to the eigenvalue* $-\left(1-\hat{t}\right)^4\widehat{M}\in(0,\pi^4)$ *of the operator* $T_4^0[0]$ *in* $X_{\{0,1\}}^{\{1,2\}}$*, which is a contradiction.*

Analogously, if there exists $\widehat{M}\in(-\pi^4,0)$*, for which there is a non-trivial solution of* $T_4^0\left[\widehat{M}\right]u(t)=0$ *on* $[0,1]$*, satisfying* $u(0)=0$*, coupled with the boundary conditions* (3.7.6)*, then there is an eigenvalue* $-\widehat{M}\in(0,\pi^4)$ *of the operator* $T_4^0[0]$ *in* $X_{\{0,1\}}^{\{1,2\}}$*, which is again a contradiction.*

Finally, since there is not any positive eigenvalue of $T_4^0[0]$ *in* $X_{\{1\}}^{\{0,1,2\}}$*, we can affirm that it is not possible that the sign change begins at* $t=1$.

So, we have proved that every solution of $u^{(4)}(t)+M\,u(t)=0$ *coupled with the boundary conditions* (3.7.6) *does not not have any zero on* $(0,1)$ *for* $M\in(-\pi^4,0)$*, and as a consequence, we have also arrived to a contradiction when* $c^*>s$.

So, from Remark 3.7.8, coupled with Theorems 3.7.3 and 3.7.5, we can affirm that the related Green's function satisfies property $\left(N_{g_1}\right)$ *if, and only if,* $M\in[-\pi^4,-m_1^4)$*, where* m_1 *has been introduced in Example 3.6.7.*

Open Problem 3. *In Example 3.7.9, we have proved, by means of Remark 3.7.8, the parameter set given in Theorem 3.7.3 is the optimal one. Moreover, by means of Theorem 3.5.5, we know that this parameter set is empty for some boundary conditions.*

However, it remains as an open problem to obtain a characterisation to know when this set is or not empty. Furthermore, it is also a pending task to find an example, with different boundary conditions from those which Theorem 3.5.5 covers, where the parameter set given in Theorem 3.7.3 is not the optimal one; or, on another hand, to prove that this is always an optimal interval.

If we opt for the second option, as we have mentioned in Remark 3.7.8, we need to make sure that the related Green's function does not have any zero on $(a, b) \times (a, b)$ for all M in the considered intervals. With our techniques, this is not possible, even if we have proved that this parameter set is not empty. This is due to the fact that in Theorem 3.7.1, we avoid the possibility of having a double zero while the Green's function is of constant sign. Nevertheless, here we cannot skip such a possibility. We have the same problem if we try to generalise Theorem 3.7.1 for different boundary conditions such as periodic boundary conditions.

On the other hand, finding an example where this property is not fulfilled does not seem easy. Probably, if it exists, it will be given by a differential equation of higher order which is difficult to deal with.

3.8 Hypothesis (T_d) cannot be weakened

In this last section, we show that, in general, we cannot avoid the hypothesis (T_d) on operator $T_n[\bar{M}]$.

To this end, we consider the operator:

$$T_4^6[M]\, u(t) = u^{(4)}(t) - 1000\, u'(t) + M\, u(t), \quad t \in [0, 1], \tag{3.8.1}$$

previously introduced in Chapter 2, coupled with two-point boundary value conditions:

$$u(0) = u'(0) = u''(0) = u(1) = 0. \tag{3.8.2}$$

The equation (3.8.1) is not disconjugate for $M = 0$, indeed:

$$u(t) = -e^{10(t-1)} - 2e^{5-5t} \cos\left(5\sqrt{3}(t-1)\right) + 3,$$

is a solution of $T_4^6[0]\, u(t) = 0$ with 5 zeros on $[0, 1]$. Thus, there is not a decomposition (T_d) for operator $T_4^6[0]$.

In fact, for the boundary conditions (3.8.2), the disconjugacy of (3.8.1) is equivalent to the existence of the decomposition (T_d), see [29].

In a first moment we will verify that Green's function related to problem (3.8.1)-(3.8.2) satisfies condition (N_g) for $\bar{M} = 0$. So, by means of Theorem 1.2.11, we will know that $N_T = [-\mu, -\lambda_1)$ for some $\mu \geqslant 0$.

In a second part, we will prove that $\mu \neq \lambda_2$, with λ_2 the first eigenvalue related to operator $T_4^6[0]$ on the space $X_2 = \left\{u \in C^4(I) \;\mid\; u(0) = u'(0) = u(1) = u'(1) = 0\right\}$.

As a consequence, we deduce that the applicability of Theorem 3.7.1 is not ensured when the disconjugacy assumption fails, which is, in this case, equivalent to property (T_d).

We point out that, since the existence of at least one $\bar{M}$ for which operator $T_4^6[\bar{M}]$ is disconjugate on $[0,1]$ implies the applicability of Theorem 3.7.1, operator $T_4^6[M]$ cannot be disconjugate on $[0,1]$ for any real parameter M and not only for $\bar{M}=0$, see Appendix C, where this property is proved without taking into account this argument.

First, we obtain the Green's function expression related to the operator $T_4^6[0]\,u(t)$ in X_3, $g_0(t,s)$. By means of the *Mathematica* package developed in [21], we have that if follows the expression:

$$\begin{cases} \dfrac{-1}{3000}\left(\dfrac{e^{-5(2s+t)}\left(-3e^{10s+5}+2e^{15s}\cos\left(5\sqrt{3}(s-1)\right)+e^{15}\right)\left(-3e^{5t}+e^{15t}+2\cos\left(5\sqrt{3}t\right)\right)}{-3e^5+e^{15}+2\cos\left(5\sqrt{3}\right)}\right. & 0\leqslant s\leqslant t\leqslant 1,\\ \qquad\qquad \left. -e^{10(t-s)}-2e^{5s-5t}\cos\left(5\sqrt{3}(t-s)\right)+3\right), & \\ -\dfrac{e^{-5(2s+t)}\left(-3e^{10s+5}+2e^{15s}\cos\left(5\sqrt{3}(s-1)\right)+e^{15}\right)\left(-3e^{5t}+e^{15t}+2\cos\left(5\sqrt{3}t\right)\right)}{3000\left(-3e^5+e^{15}+2\cos\left(5\sqrt{3}\right)\right)}, & 0<t<s\leqslant 1. \end{cases}$$

Let us see now that $g_0(t,s)\leqslant 0$ on $[0,1]\times[0,1]$ and that it satisfies condition (N_g), i.e., the following inequality is satisfied:

$$\frac{g_0(t,s)}{t^3\,(t-1)}>0, \quad \text{for all } (t,s)\in[0,1]\times(0,1).$$

To study the behaviour on a neighbourhood of $t=0$ and $t=1$, we define the following functions:

$$k_1(s)=\lim_{t\to 0^+}\frac{g_0(t,s)}{t^3\,(t-1)}=\frac{e^{-10s}\left(-3e^{10s+5}+2e^{15s}\cos\left(5\sqrt{3}(s-1)\right)+e^{15}\right)}{6\left(-3e^5+e^{15}+2\cos\left(5\sqrt{3}\right)\right)},$$

$$k_2(s)=\lim_{t\to 1^-}\frac{g_0(t,s)}{t^3\,(t-1)}=\frac{1}{300}e^{-10s-5}\left(e^{15s}\left(\sqrt{3}\sin\left(5\sqrt{3}(s-1)\right)-\cos\left(5\sqrt{3}(s-1)\right)\right)+e^{15}\right.$$

$$\left.+\frac{\left(-3e^{10s+5}+2e^{15s}\cos\left(5\sqrt{3}(s-1)\right)+e^{15}\right)\left(-e^{15}+\sqrt{3}\sin\left(5\sqrt{3}\right)+\cos\left(5\sqrt{3}\right)\right)}{-3e^5+e^{15}+2\cos\left(5\sqrt{3}\right)}\right).$$

In the sequel we will prove that both functions are strictly positive on $(0,1)$.

It is not difficult to verify that $k_1(1)=k_1'(1)=k_1''(1)=0$ and that:

$$k_1^{(3)}(1)=-\frac{500e^5}{-3e^5+e^{15}+2\cos\left(5\sqrt{3}\right)}<0.$$

If we prove that $k_1^{(3)}(s)$ is strictly negative on $[0,1]$, since, in such a case, $k_1''(s)$ would be positive and $k_1'(s)$ negative, we will deduce that $k_1(s)>0$ for $s\in(0,1)$.

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Due to the fact that:

$$k_1^{(3)}(s) = -\frac{500e^{-10s}\left(2e^{15s}\cos\left(5\sqrt{3}(s-1)\right) + e^{15}\right)}{3\left(-3e^5 + e^{15} + 2\cos\left(5\sqrt{3}\right)\right)},$$

we only must check that:

$$k_{10}(s) := 2e^{15s}\cos\left(5\sqrt{3}(s-1)\right) + e^{15} > 0, \quad s \in [0,1].$$

But previous inequality holds immediately from the fact that:

$$\min_{s\in[0,1]} k_{10}(s) = e^{15}\left(1 - e^{-\frac{2\pi}{\sqrt{3}}}\right) > 0.$$

Consider now function k_2. We have that $k_2(0) = 0$ and:

$$k_2'(0) = \frac{1 + e^5\left(e^{15} - \sqrt{3}\left(e^{10} - 1\right)\sin\left(5\sqrt{3}\right) - \left(1 + e^{10}\right)\cos\left(5\sqrt{3}\right)\right)}{10e^5\left(-3e^5 + e^{15} + 2\cos\left(5\sqrt{3}\right)\right)} > 0.$$

So, we study the sign of its first derivative:

$$k_2'(s) = \frac{e^{-10s-5}}{30\left(-3e^5 + e^{15} + 2\cos\left(5\sqrt{3}\right)\right)} k_{20}(s),$$

with:

$$\begin{aligned} k_{20}(s) =& e^{15s}\left(\sqrt{3}\left(e^5\left(2e^{10} - 3\right)\sin\left(5\sqrt{3}(s-1)\right) + \sin\left(5\sqrt{3}s\right)\right) - 3e^5\cos\left(5\sqrt{3}(s-1)\right)\right. \\ &\left. + 3\cos\left(5\sqrt{3}s\right)\right) + e^{15}\left(3e^5 - \sqrt{3}\sin\left(5\sqrt{3}\right) - 3\cos\left(5\sqrt{3}\right)\right). \end{aligned}$$

It is clear that such function satisfies:

$$k_{20}(s) > \left(-3 - 3e^5 + \sqrt{3}\left(-1 + 3e^5 - 2e^{15}\right)\right)e^{15s} + e^{15}\left(3e^5 - \sqrt{3}\sin\left(5\sqrt{3}\right) - 3\cos\left(5\sqrt{3}\right)\right),$$

which is positive for:

$$s < \frac{1}{15}\log\left(\frac{3e^{20} - \sqrt{3}e^{15}\sin\left(5\sqrt{3}\right) - 3e^{15}\cos\left(5\sqrt{3}\right)}{3 + \sqrt{3} + 3e^5 - 3\sqrt{3}e^5 + 2\sqrt{3}e^{15}}\right) \approxeq 0.32389.$$

Moreover, for $s \in \left[1 - \dfrac{2\pi}{5\sqrt{3}}, 1 - \dfrac{\pi}{5\sqrt{3}}\right] \approxeq [0.27448, 0.63724]$ we have that:

$$k_{20}(s) > \left(-3 - \sqrt{3} - 3e^5\right)e^{15s} + e^{15}\left(3e^5 - \sqrt{3}\sin\left(5\sqrt{3}\right) - 3\cos\left(5\sqrt{3}\right)\right),$$

and right part of previous equality is positive for:

$$s < \frac{1}{15} \log \left(\frac{3e^{20} - \sqrt{3}e^{15} \sin\left(5\sqrt{3}\right) - 3e^{15} \cos\left(5\sqrt{3}\right)}{3 + \sqrt{3} + 3e^{5}} \right) \approxeq 0.99944.$$

Then, we have that $k_2'(s) > 0$ for $s \in \left[0, 1 - \dfrac{\pi}{5\sqrt{3}}\right]$, and, as a consequence, the same holds for $k_2(s)$.

On the other hand, we have that $k_2(1) = k_2'(1) = 0$ and $k_2''(1) = 1$, moreover:

$$k_2''(s) = \frac{e^{-10s-5}}{3\Big(-3e^{5} + e^{15} + 2\cos\left(5\sqrt{3}\right)\Big)} k_{21}(s),$$

where:

$$\begin{aligned} k_{21}(s) =& e^{15s} \Big(\sqrt{3}\Big(e^{15} \sin\left(5\sqrt{3}(s-1)\right) - \sin\left(5\sqrt{3}s\right)\Big) + 3e^{5}\left(e^{10} - 2\right) \cos\left(5\sqrt{3}(s-1)\right) + 3\cos\left(5\sqrt{3}s\right)\Big) \\ &+ e^{15}\Big(-3e^{5} + \sqrt{3}\sin\left(5\sqrt{3}\right) + 3\cos\left(5\sqrt{3}\right)\Big). \end{aligned}$$

Now, we must verify that $k_{21}(s) > 0$.

If $s > 0.9$ we can bound it from below by the following function:

$$e^{15s}\left[-3 - \sqrt{3}\left(1 + e^{15}\right) + 3e^{5}\left(e^{10} - 2\right) \cos\left(\frac{\sqrt{3}}{2}\right)\right] + e^{15}\Big(-3e^{5} + \sqrt{3}\sin\left(5\sqrt{3}\right) + 3\cos\left(5\sqrt{3}\right)\Big).$$

It is clear that it is positive for $s \in (s_1, 1]$, where:

$$s_1 = \frac{1}{15} \log \left(\frac{-3e^{20} + \sqrt{3}e^{15} \sin\left(5\sqrt{3}\right) + 3e^{15} \cos\left(5\sqrt{3}\right)}{3 + \sqrt{3} + \sqrt{3}e^{15} + 6e^{5} \cos\left(\frac{\sqrt{3}}{2}\right) - 3e^{15} \cos\left(\frac{\sqrt{3}}{2}\right)} \right) \approxeq 0.510335,$$

which ensures that $k_2(s) > 0$ on $(0.9, 1)$.

On the other hand, for every $s \in [0, 1]$, function $300\, e^{10s+5}\, k_2(s)$ is bounded from below by:

$$k_{22} = \frac{1}{100}\left(-476e^{15s} + 303e^{10s+5} - e^{15}\right),$$

which is positive for $s \in (s_2, s_3)$, where:

$$\begin{aligned} s_2 = 1 + \frac{1}{5} \log &\left[\frac{101}{476} + \frac{101}{476}\sqrt{3} \sin\left(\frac{1}{3}\tan^{-1}\left(\frac{476\sqrt{973657}}{917013}\right)\right) - \frac{101}{476}\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{476\sqrt{973657}}{917013}\right)\right)\right] \\ &\approxeq 0.438593, \end{aligned}$$

and,

$$s_3 = 1 + \frac{1}{5} \log \left[\frac{101}{476} + \frac{101}{238} \cos\left(\frac{1}{3}\tan^{-1}\left(\frac{476\sqrt{973657}}{917013}\right)\right)\right] \approxeq 0.908.$$

So, we conclude that $k_2(s) > 0$ for every $s \in (0,1)$.

Now, in order to deduce condition (N_g), we only have to verify that $g_0(t,s) < 0$ for every (t,s) in $(0,1)\times(0,1)$.

If $t < s$ we can express:

$$g_0(t,s) = -\frac{e^{-5(2s+t)}\,\ell_1(s)\,\ell_2(t)}{3000\Big(-3e^5 + e^{15} + 2\cos\big(5\sqrt{3}\big)\Big)},$$

where:

$$\ell_1(s) = \Big(-3e^{10s+5} + 2e^{15s}\cos\Big(5\sqrt{3}(s-1)\Big) + e^{15}\Big),$$

$$\ell_2(t) = \Big(-3e^{5t} + e^{15t} + 2\cos\Big(5\sqrt{3}t\Big)\Big).$$

So, we must prove that both functions are positive on $(0,1)$.

$\ell_1(s)$ is a positive multiple of $k_1(s)$, so, as we have proved before, it is positive for $s \in (0,1)$.

To study the sign of ℓ_2, since it satisfies that $\ell_2(0) = \ell_2'(0) = \ell_2''(0) = 0$, from the following expressions, applicable for all $t \in [0,1]$,

$$\ell_2^{(3)}(t) = 375\Big(-e^{5t} + 9e^{15t} + 2\sqrt{3}\sin\Big(5\sqrt{3}t\Big)\Big) \geqslant 375\left(-e^{5t} + 9e^{15t} - 2\sqrt{3}\right) > 0,$$

we deduce that $\ell_2(t) > 0$ for every $t \in (0,1)$.

Let us see now what happens for $0 < s \leqslant t < 1$.

We can express $g_0(t,s)$ as follows:

$$g_0(t,s) = \frac{1}{3000}\big(p_2(t-s) - p_1(t,s)\big), \quad 0 < s \leqslant t < 1,$$

where:

$$p_1(t,s) = \frac{e^{-5(2s+t)}\,\ell_1(s)\,\ell_2(t)}{-3\,e^5 + e^{15} + 2\,\cos\big(5\sqrt{3}\big)},$$

and,

$$p_2(r) = e^{10\,r} + 2\,e^{-5\,r}\,\cos\Big(5\,\sqrt{3}\,r\Big) - 3.$$

From the previously proved positiveness of ℓ_1 and ℓ_2, we know that $p_1(t,s) > 0$.

On the other hand, since $p_2(0) = p_2'(0) = p_2''(0) = 0$, if we verify that $p_2^{(3)}(r) > 0$ for every $r \in [0,1]$, then we conclude that the same holds for p_2 on $(0,1]$. In this case:

$$p_2^{(3)}(r) = 1000e^{10r} + 2000e^{-5r}\cos\Big(5\sqrt{3}r\Big).$$

This function is trivially positive whenever $0 \leqslant r \leqslant \dfrac{\pi}{10\sqrt{3}} \approxeq 0.18138$.

Moreover, for every $r \in [0,1]$, we have that:

$$p_2^{(3)}(r) > 1000e^{10r} - 2000e^{-5r},$$

which is positive if, and only if, $r > \dfrac{\log(2)}{15} \approxeq 0.0462$.

As a consequence we deduce that $p_2(r) > 0$ for every $r \in (0,1]$.

Then if we prove that $p_2(t-s) < p_1(t,s)$ for $0 < s \leqslant t < 1$, we can conclude that $g_0(t,s) < 0$.

Notice that, if we have two strictly convex functions on a suitable interval, we may affirm that they have at most two common points. In the sequel, to prove our result, we use this property.

Since by definition $g_0(1,s) = 0$, we know that $p_1(1,s) = p_2(1-s)$, for every fixed $s \in (0,1)$.

From the fact, proved before, that $k_2 > 0$ on $(0,1)$, we know that $g_0(t,s) < 0$ on a neighbourhood of $t = 1$ for every $s \in (0,1)$. Then $p_1(t,s) > p_2(t-s)$ on a neighbourhood of $t = 1$ for every $s \in (0,1)$.

Let us see now that, for every $s \in (0,1)$, $p_1(t,s)$ and $p_2(t-s)$ are convex functions of t. By direct calculations, we have that:

$$\frac{\partial^2}{\partial t^2}p_1(t,s) = \frac{100e^{-5(2s+t)}\left(e^{15t} + \sqrt{3}\sin\left(5\sqrt{3}t\right) - \cos\left(5\sqrt{3}t\right)\right)\ell_1(s)}{-3\,e^5 + e^{15} + 2\,\cos\left(5\sqrt{3}\right)},$$

so we only need to verify that:

$$p_{11}(t) = \left(e^{15t} + \sqrt{3}\sin\left(5\sqrt{3}t\right) - \cos\left(5\sqrt{3}t\right)\right) > 0, \quad t \in (0,1).$$

The following inequality is trivially fulfilled:

$$p_{11}(t) > e^{15t} + \sqrt{3}\sin\left(5\sqrt{3}t\right) - 1 = q_1(t), \quad t \in [0,1].$$

We have that:

$$q_1'(t) = 15\,e^{15\,t} + 15\,\cos\left(5\,\sqrt{3}\,t\right) > 15\big(e^{15\,t} - 1\big) > 0, \quad t > 0,$$

since $q_1(0) = 0$, we conclude that $q_1 > 0$ and, as a consequence, $p_{11}(t) > 0$ on $(0,1]$ and also $\dfrac{\partial^2}{\partial t^2}p_1(t,s) > 0$.

We have already proved that $p_2^{(3)}(r) > 0$, for $r \in [0,1]$, and $p_2''(0) = 0$, so for every fixed $s \in (0,1)$ we have that $p_2''(t-s) > 0$ for every $t \in (s,1]$.

As a consequence, for any fixed $s \in (0,1)$, both $p_1(t,s)$ and $p_2(t-s)$ are convex functions of t.

From the fact that $p_1(t,s) > p_2(t-s)$ on a neighbourhood of $t=1$, $p_1(1,s) = p_2(1-s)$ and, also, $p_1(s,s) > 0 = p_2(0)$, we can affirm that $p_1(t,s) > p_2(t-s)$ for $t \in [s,1)$, and then $g_0(t,s) < 0$ for $0 < s \leqslant t < 1$. So, condition (N_g) is fulfilled.

Now, as a consequence of Theorem 1.2.11, we know that $g_M(t,s) \leqslant 0$ for $M \in [0, -\lambda_1)$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_4^6[0]\, u(t)$ in X_3.

To verify that Theorem 3.7.1 does not hold in this case we will prove that for $M < 0$ the sign change does not come on the least positive eigenvalue of $T_4^6[0]\, u(t)$ in X_2.

As in the previous section, we can obtain numerically the first eigenvalues of $T_4^6[0]$, which can be given by the following approximated values:

The biggest negative eigenvalue in X_1 is $\lambda_3 \approxeq -(12.529)^4$.

The least positive eigenvalue in X_2 is $\lambda_2 \approxeq (10.895)^4$.

The biggest negative eigenvalue in X_3 in $\lambda_1 \approxeq -(9.458)^4$.

Remark 3.8.1. *Realise that, since $T_4^6[0]\, u(t) = 0$ is not disconjugate on $[0,1]$, we have no a priori information about the sign of the eigenvalues λ_3 and λ_2. However, since g_0 satisfies (N_g), we can ensure, without calculating it, that $\lambda_1 < 0$.*

Finally, let us see that there exists $M^* > -\lambda_2$ for which g_{M^*} has not constant sign on $I \times I$.

We study the function w_M introduced in Lemma 3.5.1 with $\eta = 1$. As we have seen in the proof of Theorem 3.5.5, if this function has not constant sign on I then the Green's function must necessarily change sing in a neighbourhood of $s=0$.

For $M^* = -\dfrac{59584}{9} \approxeq -(9.02032)^4$, we have that $v(t)$ follows the next expression

$$\begin{aligned}
&\frac{3e^{-\frac{1}{3}\left(9+\sqrt{669}\right)t}}{8441871944}\Bigg[277e^{6t}\left(\left(213\sqrt{669}-27875\right)e^{2\sqrt{\frac{223}{3}}t}-27875-213\sqrt{669}\right)+446e^{\sqrt{\frac{223}{3}}t}\left(537\sqrt{831}\sin\left(\sqrt{\frac{277}{3}}t\right)\right.\\
&\left.+34625\cos\left(\sqrt{\frac{277}{3}}t\right)\right)\Bigg]-223\left(537\sqrt{831}\sin\left(\sqrt{\frac{277}{3}}\right)+34625\cos\left(\sqrt{\frac{277}{3}}\right)\right)+277e^{6}\left(213\sqrt{669}\sinh\left(\sqrt{\frac{223}{3}}\right)\right.\\
&\left.-27875\cosh\left(\sqrt{\frac{223}{3}}\right)\right)\Bigg/\Bigg[8441871944\left(-277\left(2007+152\sqrt{669}\right)e^{6}+277\left(152\sqrt{669}-2007\right)e^{6+2\sqrt{\frac{223}{3}}}\right.\\
&\left.+446e^{\sqrt{\frac{223}{3}}}\left[2493\cos\left(\sqrt{\frac{277}{3}}\right)-98\sqrt{831}\sin\left(\sqrt{\frac{277}{3}}\right)\right]\right)\Bigg]+6e^{\sqrt{\frac{223}{3}}-\frac{1}{3}\left(9+\sqrt{669}\right)t}\Bigg[446e^{\sqrt{\frac{223}{3}}t}\left(2493\cos\left(\sqrt{\frac{277}{3}}t\right)\right.\\
&\left.-98\sqrt{831}\sin\left(\sqrt{\frac{277}{3}}t\right)\right)+277e^{6t}\left(\left(152\sqrt{669}-2007\right)e^{2\sqrt{\frac{223}{3}}t}-2007-152\sqrt{669}\right)\Bigg],
\end{aligned}$$

which, see Figure 3.8.1, changes sign on I.

As a consequence the Green's function has not constant sign for a value of M bigger than $-\lambda_2$.

Even more, we can verify numerically which is the interval for M, where $g_M(t,s)$ is non-positive on $I \times I$. We observe that change sign come first on the interior of $I \times I$. It comes at the point $(t,s) \approxeq (0.7186, 0.0307) \in (0,1) \times (0,1)$ for $M \approxeq -(7.87022)^4$. So we deduce that the constant sign interval is given by $[-(7.87022)^4, -\lambda_1)$.

As a consequence we conclude the example that show us that if we suppress the existence of the decomposition (T_d), Theorem 3.7.1 is not true in general.

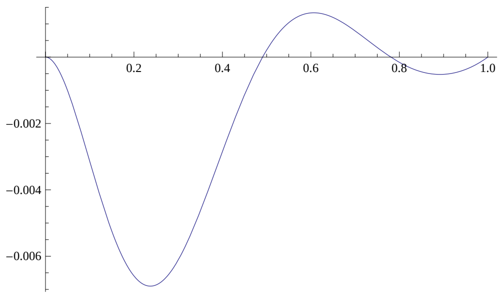

Figure 3.8.1: Graph of v

Open Problem 4. *In this section, we have proved that, in general, we cannot avoid or weaken property* (T_d).

However, we have not considered to eliminate property (N_a) *in the boundary conditions. This is due to the fact that if* $\lambda = 0$ *is an eigenvalue of the operator, so we cannot obtain the related Green's function because the solutions of the problem are not unique. Even though, it could be possible to obtain a characterisation similar to the one given in Theorem 3.7.1 for problems with different boundary conditions by using other techniques.*

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Chapter 4

Strongly inverse positive (negative) operators

Contents

In previous chapter (Section 3.2, Theorems 3.2.5 and 3.2.6), we have established a relationship between the Green's function sign and the strongly inverse positive (negative) character of the related operator. In this chapter, we use this relationship to obtain a characterisation of the parameter set for which the operator $T_n[M]$ is either strongly inverse positive or negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. The results and examples which appear in this chapter together with the ones proved in Chapter 3 are published in [29, 30].

As in Chapter 3 (Section 3.7), we start by studying the behaviour at a neighbourhood of $\bar{M}$, where $\bar{M}$ is such that $T_n[\bar{M}]$ satisfies the hypotheses of Theorem 3.7.1. By using the results obtained in that chapter, we obtain a characterisation of either the strongly inverse positive or negative character of operator $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

After that, we study the behaviour for $M < \bar{M} - \lambda_1$ or $M > \bar{M} - \lambda_1$, respectively.

Thus, in the first part of the chapter we obtain different results related with the strongly inverse positive or negative character of the operator.

In the second part, we focus on the study of problems with non-homogeneous boundary conditions. In this case, in general, we cannot characterise the strongly inverse positive or negative character, if we consider this properties as the given in Definitions 3.2.3 and 3.2.4 for the homogeneous case. Nevertheless, in this case we will introduce the concept of strongly inverse positive character or negative for a problem with non-homogeneous boundary conditions.

4.1 Strongly inverse positive (negative) character on a neighbourhood of $\bar{M}$

As a direct consequence of Theorem 3.7.1, together with Theorems 3.2.5 and 3.2.6, we obtain the following result.

Theorem 4.1.1. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy (N_a). The following properties are fulfilled.*

- *If $n-k$ is even and $2 \leqslant k \leqslant n-1$, then $T_n[M]$ is strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, where:*
    - * $\lambda_1 > 0$ *is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - * $\lambda_2 < 0$ *is the maximum between:*
        - · $\lambda_2' < 0$*, the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
        - · $\lambda_2'' < 0$*, the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*

- *If $k=1$ and n is odd, then $T_n[M]$ is strongly inverse positive in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$ if, and only if, $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, where:*
    - * $\lambda_1 > 0$ *is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
    - * $\lambda_2 < 0$ *is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.*

- *If $n-k$ is odd and $2 \leqslant k \leqslant n-2$, then $T_n[M]$ is strongly inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, where:*
    - * $\lambda_1 < 0$ *is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - * $\lambda_2 > 0$ *is the minimum between:*
        - · $\lambda_2' > 0$*, the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
        - · $\lambda_2'' > 0$*, the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*

- *If $k=1$ and $n>2$ is even, then $T_n[M]$ is strongly inverse negative in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$ if, and only if, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, where:*
    - * $\lambda_1 < 0$ *is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
    - * $\lambda_2 > 0$ *is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.*

- *If $k=n-1$ and $n>2$, then $T_n[M]$ is strongly inverse negative in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$ if, and only if, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, where:*

* $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.
* $\lambda_2 > 0$ *is the least positive eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-2}\}}^{\{\varepsilon_1|\beta\}}$.

- *If* $n = 2$, *then* $T_n[M]$ *is a strongly inverse negative operator in* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$ *if, and only if,* $M \in (-\infty, \bar{M} - \lambda_1)$, *where:*

  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[\bar{M}]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.

Proof. It is clear that condition $\left(P_{g_1}\right)$ implies that $g_M(t,s) > 0$ and condition $\left(N_{g_1}\right)$ implies that $g_M(t,s) < 0$ for all $(t,s) \in (a,b) \times (a,b)$.

Moreover, in the proof of Theorem 3.7.1, we have seen that the functions $\tilde{w}_M$ and $\tilde{y}_M$, previously defined in Lemmas 3.5.3 and 3.5.4, satisfy the following properties:

- $(-1)^{n-k}\widehat{w}_M > 0$ on (a,b) for the given intervals.
- $(-1)^{n-k+\beta}\widehat{y}_M > 0$ on (a,b) for the given intervals.

Thus, from Theorems 3.2.5 and 3.2.6, we have ensured that if M is in the given intervals, then the strongly inverse positive (negative) character holds.

Moreover, from Step 4 in the proof of Theorem 3.7.1, these intervals cannot be increased and the result is proved. $\square$

Example 4.1.2. *For the operator* $T_4^0[M]$ *introduced in Chapter 4, by using Example 3.6.7, we conclude that the operator* $T_4^0[M]$ *is strongly inverse negative in* $X_{\{0,2\}}^{\{1,2\}}$ *if, and only if,* $M \in \left(-m_1^4, 4\,\pi^4\right]$, *where* $m_1 \approx 2.365$ *has been introduced in Example 3.6.7.*

As a direct corollary of Theorem 4.1.1, by using the method of upper and lower solutions (see [22] for details), we obtain the characterisation of either the strongly inverse positive character or the strongly inverse negative character of the following operator:

$$T_n[M,c]\,u(t) = T_n[M]\,u(t) + c(t)\,u(t), \quad t \in I, \tag{4.1.1}$$

in the set $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, where $c \in C(I)$.

Corollary 4.1.3. *Consider the operator* $T_n[M,c]$ *defined in* (4.1.1).

Let $\bar{M} \in \mathbb{R}$ *be such that* $T_n[\bar{M}]$ *satisfies property* (T_d) *on* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and the set of indices* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfies* (N_a).

Let λ_1 *and* λ_2 *be as in Theorem 4.1.1. Then the following properties are fulfilled.*

- *If* $n-k$ *is even and* $-\lambda_1 < c(t) \leqslant -\lambda_2$ *for all* $t \in I$, *then* $T_n[\bar{M},c]$ *is strongly inverse positive in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
- *If* $n-k$ *is odd,* $n > 2$ *and* $-\lambda_2 \leqslant c(t) < -\lambda_1$ *for all* $t \in I$, *then* $T_n[\bar{M},c]$ *is strongly inverse negative in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
- *If* $n = 2$, *and* $c(t) < -\lambda_1$ *for all* $t \in I$, *then* $T_n[\bar{M},c]$ *is strongly inverse negative in* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.

4.2 Strongly inverse positive (negative) character in a different parameter set

The aim of this section is to give a necessary condition for the operator $T_n[M]$ to be either strongly inverse negative if $n-k$ is even or strongly inverse positive if $n-k$ is odd. Obviously, from Theorem 4.1.1, $\bar{M}$ does not belongs to such intervals.

Such a result is obtained directly from Theorems 3.7.3 and 1.2.7.

Theorem 4.2.1. *Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies property (T_d) on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ fulfil (N_a). Then, the following properties are satisfied.*

- *If $n-k$ is even and $T_n[M]$ is inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then M belongs to the interval $[\bar{M}-\lambda_3, \bar{M}-\lambda_1)$, where:*
  - * *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * *$\lambda_3 > 0$ is the minimum between:*
    - · *$\lambda_3' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - · *$\lambda_3'' > 0$, the least positive eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*

- *If $n-k$ is odd and $T_n[M]$ is inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then M belongs to the interval $(\bar{M}-\lambda_1, \bar{M}-\lambda_3]$, where:*
  - * *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * *$\lambda_3 < 0$ is the maximum between:*
    - · *$\lambda_3' < 0$, the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - · *$\lambda_3'' < 0$, the biggest negative eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}}$.*

As in Remark 3.7.8, if we can ensure that the parameter set where the operator is either strongly inverse negative or positive is non-empty and that the related Green's function must begin to change sign on the boundary of $I \times I$, then the intervals given on Theorem 4.2.1 are the optimal ones.

Indeed, from Example 3.7.9, we arrive to the following conclusion.

Example 4.2.2. *In addition to Example 4.1.2, from the calculus done in Example 3.7.9, we conclude that the operator $T_4^0[M]$ is strongly inverse negative in $X_{\{0,2\}}^{\{1,2\}}$ if, and only if, M is in the interval $\big[-\pi^4, -m_1^4\big)$, where $m_1 \approxeq 2.365$ has been introduced in Example 3.6.7.*

Moreover, in Chapter 5, we will study a fourth order operator coupled with the simply supported beam boundary conditions. In such a case, we will also obtain the optimal interval.

4.3 Particular cases

In addition to examples which we will give in Chapter 5 for the simply supported beam boundary conditions, this section is devoted to show several examples where we consider different operators and spaces.

Along the Chapters 3 and 4, we have been considering the particular case where $n = 4$, $\{\sigma_1, \sigma_2\} = \{0, 2\}$, $\{\varepsilon_1, \varepsilon_2\} = \{1, 2\}$ and

$$T_4^0[M]\, u(t) = u^{(4)}(t) + M\, u(t),$$

which satisfies the hypotheses of Theorem 4.1.1 for $\bar{M} = 0$.

Indeed, in Example 4.1.2, we have obtained the parameter set where the operator is strongly inverse positive in $X_{\{0,2\}}^{\{1,2\}}$.

If we were interested in studying the strongly inverse positive character of $T_4^0[M]$ in $X_{\{0,2\}}^{\{1,2\}}$ without taking into account Theorem 4.1.1, we would have to study the related Green's function, which is given by the following expression for $M = m^4 > 0$, obtained by means of the *Mathematica* program developed in [21]:

$$\begin{cases} \begin{array}{r} e^{-\sqrt{2}m(s+t-2)} \left[\left(e^{\frac{m(3s+t-4)}{\sqrt{2}}} - e^{\frac{m(s+3t-4)}{\sqrt{2}}} \right) \sin\left(\frac{m(s-t+2)}{\sqrt{2}} \right) \right. \\ + \left(e^{\frac{m(s+t-4)}{\sqrt{2}}} - e^{\frac{m(3s+3t-4)}{\sqrt{2}}} \right) \sin\left(\frac{m(s+t-2)}{\sqrt{2}} \right) \\ + \left(2e^{\frac{m(s+t-4)}{\sqrt{2}}} - e^{\frac{m(3s+t-2)}{\sqrt{2}}} + e^{\frac{m(s+3t-6)}{\sqrt{2}}} - 2e^{\frac{m(3s+3t-4)}{\sqrt{2}}} \right) \sin\left(\frac{m(s-t)}{\sqrt{2}} \right) \\ + \left(-e^{\frac{m(s+t-2)}{\sqrt{2}}} + e^{\frac{3m(s+t-2)}{\sqrt{2}}} + 2e^{\frac{m(3s+t-4)}{\sqrt{2}}} - 2e^{\frac{m(s+3t-4)}{\sqrt{2}}} \right) \sin\left(\frac{m(s+t)}{\sqrt{2}} \right) \\ + \left(e^{\frac{m(3s+t-2)}{\sqrt{2}}} + e^{\frac{m(s+3t-6)}{\sqrt{2}}} \right) \cos\left(\frac{m(s-t)}{\sqrt{2}} \right) - e^{\frac{m(3s+3t-4)}{\sqrt{2}}} \cos\left(\frac{m(s+t-2)}{\sqrt{2}} \right) \\ + \left(e^{\frac{m(3s+t-4)}{\sqrt{2}}} + e^{\frac{m(s+3t-4)}{\sqrt{2}}} \right) \cos\left(\frac{m(s-t+2)}{\sqrt{2}} \right) - e^{\frac{m(s+t-4)}{\sqrt{2}}} \cos\left(\frac{m(s+t-2)}{\sqrt{2}} \right) \\ \left. - e^{\frac{m(s+t-2)}{\sqrt{2}}} \left(e^{\sqrt{2}m(s+t-2)} + 1 \right) \cos\left(\frac{m(s+t)}{\sqrt{2}} \right) \right] / \left(4\sqrt{2}m^3 \left(\sin(\sqrt{2}m) + \sinh(\sqrt{2}m) \right) \right) \end{array} & 0 \leqslant s \leqslant t \leqslant 1, \\ \\ \begin{array}{r} e^{-\frac{m(3s+t-6)}{\sqrt{2}}} \left[e^{2\sqrt{2}m(s-1)} \sin\left(\frac{m(s-t-2)}{\sqrt{2}} \right) - e^{\sqrt{2}m(s+t-2)} \sin\left(\frac{m(s-t-2)}{\sqrt{2}} \right) \right. \\ +2e^{\sqrt{2}m(s-2)} \sin\left(\frac{m(s-t)}{\sqrt{2}} \right) - e^{\sqrt{2}m(2s-3)} \sin\left(\frac{m(s-t)}{\sqrt{2}} \right) + e^{\sqrt{2}m(s+t-1)} \sin\left(\frac{m(s-t)}{\sqrt{2}} \right) \\ -2e^{\sqrt{2}m(2s+t-2)} \sin\left(\frac{m(s-t)}{\sqrt{2}} \right) + e^{\sqrt{2}m(s-2)} \sin\left(\frac{m(s+t-2)}{\sqrt{2}} \right) \\ -e^{\sqrt{2}m(2s+t-2)} \sin\left(\frac{m(s+t-2)}{\sqrt{2}} \right) + \left(e^{\sqrt{2}m(s+t-2)} + e^{2\sqrt{2}m(s-1)} \right) \cos\left(\frac{m(s-t-2)}{\sqrt{2}} \right) \\ + \left(-2e^{\sqrt{2}m(s+t-2)} + e^{\sqrt{2}m(2s+t-3)} - e^{\sqrt{2}m(s-1)} + 2e^{2\sqrt{2}m(s-1)} \right) \sin\left(\frac{m(s+t)}{\sqrt{2}} \right) \\ + \left(e^{\sqrt{2}m(s+t-1)} + e^{\sqrt{2}m(2s-3)} \right) \cos\left(\frac{m(s-t)}{\sqrt{2}} \right) \\ -e^{\sqrt{2}m(2s+t-2)} \cos\left(\frac{m(s+t-2)}{\sqrt{2}} \right) - \left(e^{\sqrt{2}m(2s+t-3)} + e^{\sqrt{2}m(s-1)} \right) \cos\left(\frac{m(s+t)}{\sqrt{2}} \right) \\ \left. -e^{\sqrt{2}m(s-2)} \cos\left(\frac{m(s+t-2)}{\sqrt{2}} \right) \right] / \left(2\sqrt{2}m^3 \left(e^{2\sqrt{2}m} + 2e^{\sqrt{2}m} \sin\left(\sqrt{2}m \right) - 1 \right) \right) \end{array} & 0 < t < s \leqslant 1, \end{cases}$$

which gives an example where the direct applicability of Theorem 4.1.1 is posted.

Characterising the Green's function constant sign attains an even higher difficulty in more complicated problems. Its expression may be unapproachable. Moreover, in some cases, for instance in problems with non-constant coefficients, we cannot even obtain its expression. So, Theorem 4.1.1 is very useful because it allows us to see which is the sign of the related Green's function without knowing its expression. We point out that to calculate the corresponding eigenvalues is very simple in the constant coefficient case and can be numerically approached in the non-constant case.

$(k, n-k)$ boundary conditions

First, consider the examples previously introduced in Chapter 2, for the particular boundary conditions $(k, n-k)$, that is, they correspond to the choice:

$$X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}} = X_{\{0,1,\dots,k-1\}}^{\{0,1,\dots,n-k-1\}} \equiv X_k.$$

In these particular cases, the existence of a $\bar{M} \in \mathbb{R}$ such that the equation $T_n[\bar{M}]u(t) = 0$ is disconjugate on I is a necessary and sufficient condition to ensure that the hypotheses of Theorem 4.1.1 are fulfilled, see [29]. In the sequel, we show the different results obtained by using the calculus done in Section 2.1.

Operator $T_n^0[M] \equiv \dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n} + M$

Firstly, we consider problems where $T_n^0[M]\,u(t) \equiv u^{(n)}(t) + M\,u(t)$, with $[a,b] = [0,1]$.

Remark 4.3.1. *Remember that $u^{(n)}(t) = 0$ is always a disconjugate equation on every real interval. Thus, we can apply Theorem 4.1.1.*

- 2^{nd}-order:
  - $\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ is a strongly inverse negative operator in X_1 if, and only if,

    $$M \in \big(-\infty, \pi^2\big).$$

- 3^{rd}-order:
  - $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is a strongly inverse positive operator in X_1 if, and only if,

    $$M \in \Big(-\big(\lambda_3^1\big)^3, \big(\lambda_3^1\big)^3\Big] \approx \big(-4.23^3, 4.23^3\big],$$

    where λ_3^1 is the least positive solution of (2.1.4).
  - $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is a strongly inverse negative operator in X_2 if, and only if,

    $$M \in \Big[-\big(\lambda_3^1\big)^3, \big(\lambda_3^1\big)^3\Big) \approx \big[-4.23^3, 4.23^3\big).$$

- 4^{th}-order:
  - $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}+M$ is a strongly inverse negative operator in X_1 or X_3 if, and only if,

$$M\in\Big[-\big(\lambda_4^2\big)^4,\big(\lambda_4^1\big)^4\Big)\approxeq\big[-4.73^4,5.55^4\big),$$

where λ_4^1 and λ_4^2 are the least positive solutions of (2.1.5) and (2.1.6), respectively.

  - $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}+M$ is a strongly inverse positive operator in X_2 if, and only if,

$$M\in\Big(-\big(\lambda_4^2\big)^4,\big(\lambda_4^1\big)^4\Big]\approxeq\big(-4.73^4,5.55^4\big].$$

- 5^{th}-order:
  - $\dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}+M$ is a strongly inverse positive operator in X_1 if, and only if,

$$M\in\Big(-\big(\lambda_5^1\big)^5,\big(\lambda_5^2\big)^5\Big]\approxeq\big(-6.95^5,5.64^5\big],$$

where λ_5^1 and λ_5^2 have been introduced in Section 2.1.

  - $\dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}+M$ is a strongly inverse negative operator in X_2 if, and only if,

$$M\in\Big[-\big(\lambda_5^2\big)^5,\big(\lambda_5^2\big)^5\Big)\approxeq\big[-5.64^5,5.64^5\big).$$

  - $\dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}+M$ is a strongly inverse positive operator in X_3 if, and only if,

$$M\in\Big(-\big(\lambda_5^2\big)^5,\big(\lambda_5^2\big)^5\Big]\approxeq\big(-5.64^5,5.64^5\big].$$

  - $\dfrac{\mathrm{d}^5}{\mathrm{d}\,t^5}+M$ is a strongly inverse negative operator in X_4 if, and only if,

$$M\in\Big[-\big(\lambda_5^2\big)^5,\big(\lambda_5^1\big)^5\Big)\approxeq\big[-5.64^5,5.64^5\big).$$

- 6^{th}-order:
  - $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}+M$ is a strongly inverse negative operator in X_1 or X_5 if, and only if,

$$M\in\Big[-\big(\lambda_6^2\big)^6,\big(\lambda_6^1\big)^6\Big)\approxeq\big(-6.71^6,8.38^6\big],$$

where λ_6^1 and λ_6^2 are the least positive solutions of (2.1.7) and (2.1.8), respectively.

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- $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} + M$ is a strongly inverse positive operator in X_2 or X_4 if, and only if,

$$M \in \left(-\left(\lambda_6^2\right)^6, \left(\lambda_6^3\right)^6\right] \approx \left(-6.71^6, 6.28^6\right],$$

  where λ_6^3 is the least positive solution (2.1.9).

- $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} + M$ is a strongly inverse negative operator in X_3 if, and only if,

$$M \in \left[-\left(\lambda_6^2\right)^6, \left(\lambda_6^3\right)^6\right) \approx \left[-6.71^6, 6.28^6\right).$$

◦ 7^{th}-order:

- $\dfrac{\mathrm{d}^7}{\mathrm{d}\,t^7} + M$ is a strongly inverse positive operator in X_1 if, and only if,

$$M \in \left(-\left(\lambda_7^1\right)^7, \left(\lambda_7^2\right)^7\right] \approx \left(-9.83^7, 7.86^7\right],$$

  where λ_7^1 and λ_7^2 have been introduced in Section 2.1.

- $\dfrac{\mathrm{d}^7}{\mathrm{d}\,t^7} + M$ is a strongly inverse negative operator in X_2 if, and only if,

$$M \in \left[-\left(\lambda_7^3\right)^7, \left(\lambda_7^2\right)^7\right) \approx \left[-7.13^7, 7.86^7\right),$$

  where λ_7^3 has been introduced in Section 2.1.

- $\dfrac{\mathrm{d}^7}{\mathrm{d}\,t^7} + M$ is a strongly inverse positive operator in X_3 if, and only if,

$$M \in \left(-\left(\lambda_7^3\right)^7, \left(\lambda_7^3\right)^7\right] \approx \left(-7.13^7, 7.13^7\right].$$

- $\dfrac{\mathrm{d}^7}{\mathrm{d}\,t^7} + M$ is a strongly inverse negative operator in X_4 if, and only if,

$$M \in \left[-\left(\lambda_7^3\right)^7, \left(\lambda_7^2\right)^7\right) \approx \left[-7.86^7, 7.13^7\right).$$

- $\dfrac{\mathrm{d}^7}{\mathrm{d}\,t^7} + M$ is a strongly inverse positive operator in X_5 if, and only if,

$$M \in \left(-\left(\lambda_7^2\right)^7, \left(\lambda_7^3\right)^7\right] \approx \left(-7.86^7, 9.83^7\right].$$

- $\dfrac{\mathrm{d}^7}{\mathrm{d}\,t^7} + M$ is a strongly inverse negative operator in X_6 if, and only if,

$$M \in \left[-\left(\lambda_7^2\right)^7, \left(\lambda_7^1\right)^7\right) \approx \left[-7.86^7, 9.83^7\right).$$

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- 8^{th}-order:
  - $\dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}+M$ is a strongly inverse negative operator in X_1 or X_7 if, and only if,

$$M\in\Big[-\left(\lambda_8^2\right)^8,\left(\lambda_8^1\right)^8\Big)\approxeq\big[-9.06^8,11.28^8\big),$$

where λ_8^1 and λ_8^2 have been introduced in Section 2.1.

  - $\dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}+M$ is a strongly inverse positive operator in X_2 or X_6 if, and only if,

$$M\in\Big(-\left(\lambda_8^2\right)^8,\left(\lambda_8^3\right)^8\Big]\approxeq\big(-9.06^8,8.1^8\big],$$

where λ_8^3 is has been introduced in Section 2.1.

  - $\dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}+M$ is a strongly inverse negative operator in X_3 or X_5 if, and only if,

$$M\in\Big[-\left(\lambda_8^4\right)^8,\left(\lambda_8^3\right)^8\Big)\approxeq\big[-7.82^8,8.1^8\big),$$

where λ_8^4 has been introduced in Section 2.1.

  - $\dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8}+M$ is a strongly inverse positive operator in X_4 if, and only if,

$$M\in\Big(-\left(\lambda_8^4\right)^8,\left(\lambda_8^3\right)^8\Big]\approxeq\big(-7.82^8,8.1^8\big].$$

Operators with constant coefficients

Now, let us study more complex operators with constant coefficients, the ones considered in Section 2.1.

- $T_4^1[M]\equiv\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}+10\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}+10\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}+10\dfrac{\mathrm{d}}{\mathrm{d}\,t}+M$

Let us consider the fourth order operator $T_4^1[M]$, previously defined in (2.1.10). By using the related eigenvalues obtained in Section 2.1, we have:

  - $T_4[M]$ is strongly inverse negative in X_1 if, and only if,

$$M\in\left[-5.27208^4,7.02782^4\right).$$

  - $T_4[M]$ is strongly inverse positive in X_2 if, and only if,

$$M\in\left(-5.27208^4,5.97041^4\right].$$

  - $T_4[M]$ is strongly inverse negative in X_3 if, and only if,

$$M\in\left[-5.27208^4,5.97041^4\right).$$

∘ $T_4^2[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 10\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 550\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

Now, let us choose the operator $T_4^2[M]$ which has been defined in (2.1.11).

- $T_4^2[M]$ is strongly inverse negative in X_1 if, and only if,
$$M \in [-611.5685, 8956.55).$$
- $T_4^2[M]$ is strongly inverse positive in X_2 if, and only if,
$$M \in (-611.5685, -571.0247].$$
- $T_4^2[M]$ is strongly inverse negative in X_3 if, and only if,
$$M \in [-611.5685, -571.0247).$$

∘ $T_6^1[M] \equiv \dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} - 8\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$

Consider, now, the sixth order example introduced in (2.1.12).

- $T_6^1[M]$ is a strongly inverse negative operator in X_1 if, and only if,
$$M \in \left[-6.698^6, 8.355^6\right).$$
- $T_6^1[M]$ is a strongly inverse positive operator in X_2 if, and only if,
$$M \in \left(-6.698^6, 6.2835^6\right].$$
- $T_6^1[M]$ is a strongly inverse negative operator in X_3 if, and only if,
$$M \in \left[-6.698^6, 6.2835^6\right).$$
- $T_6^1[M]$ is a strongly inverse positive operator in X_4 if, and only if,
$$M \in \left(-6.717^6, 6.2835^6\right].$$
- $T_6^1[M]$ is a strongly inverse negative operator in X_5 if, and only if,
$$M \in \left[-6.717^6, 8.40247^6\right).$$

∘ $T_4^3[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 50\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$

Finally, let us consider the operator fourth order operator $T_4^3[M]$, previously defined in (2.1.13).

- $T_4^3[M]$ is a strongly inverse negative operator in X_1 or X_3 if, and only if,
$$M \in [-140.324, 389.73).$$
- $T_4^3[M]$ is a strongly inverse positive operator in X_2 if, and only if,
$$M \in (-140.324, 389.73].$$

Operators with non-constant coefficients

Finally, let us study the operators with non-constant coefficients which have been introduced in Section 2.1.

∘ $T_3^1[M] \equiv \dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + \cos(10\,t)\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$

First, consider the third order linear differential operator $T_3^1[M]$ defined in (2.1.16).

- $T_3^1[M]$ is a strongly inverse positive operator in X_1 if, and only if,
$$M \in \left(-4.29055^3, 4.33149^3\right].$$
- $T_3^1[M]$ is a strongly inverse negative operator in X_2 if, and only if,
$$M \in \left[-4.29055^3, 4.33149^3\right).$$

∘ $T_3^2[M] \equiv \dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + t\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

Now, consider the operator $T_3^2[M]$, previously introduced in (2.1.17).

- $T_3^2[M]$ is a strongly inverse positive operator in X_1 if, and only if,
$$M \in \left(-4.19369^3, 4.21255^3\right].$$
- $T_3^2[M]$ is a strongly inverse negative operator in X_2 if, and only if,
$$M \in \left[-4.19369^3, 4.21255^3\right).$$

∘ $T_4^4[M] \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + e^{2\,t}\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$

Finally, let us consider the fourth order operator $T_4^4[M]$ defined in (2.1.18).

- $T_4^4[M]$ is a strongly inverse negative operator in X_1 if, and only if,
$$M \in \left[-4.7235^4, 5.5325^4\right).$$
- $T_4^4[M]$ is a strongly inverse positive operator in X_2 if, and only if,
$$M \in \left(-4.7235^4, 5.5325^4\right].$$
- $T_4^4[M]$ is a strongly inverse negative operator in X_3 if, and only if,
$$M \in \left[-4.7235^4, 5.5815^4\right).$$

Remark 4.3.2. *Realise that in all of theses cases we are under the hypotheses of Theorem 3.5.5, since both $\sigma_k = k-1$ and $\varepsilon_{n-k} = n-k-1$. Thus, combining this result with Theorem 1.2.7, we obtain that the following assertions are fulfilled.*

- *If $n-k$ is even, then there is not any $M \in \mathbb{R}$ such that $T_n[M]$ is inverse negative in X_k.*
- *If $n-k$ is odd, then there is not any $M \in \mathbb{R}$ such that $T_n[M]$ is inverse positive in X_k.*

More general boundary conditions

Once we have shown the applicability of the result for the $(k, n-k)$ boundary conditions, let us show different examples with more general boundary conditions where the hypotheses of Theorem 4.1.1 are fulfilled.

Operator $T_n^0[M] \equiv \dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n} + M$

Before giving some results for this kind of operator, we take into account the following remarks:

Remark 4.3.3. *Choosing* $v_1(t) = \cdots = v_n(t) = 1$ *for all* $t \in I$*, we have that operator* $T_n^0[0]$ *fulfils the hypotheses of Theorem 4.1.1 if we choose* $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ *satisfying* (N_a).

Remark 4.3.4. *Realise that for this kind of operators we can characterise the behaviour in* $X_{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}^{\{\sigma_1,\ldots,\sigma_k\}}$ *by studying the behaviour in* $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$*. This is due to the fact that the eigenvalues are the same if* n *is even or the opposed if* n *is odd.*

Indeed, if u *is a non-trivial solution of* $u^{(n)}(t) + M\,u(t) = 0$ *on* $X_{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}^{\{\sigma_1,\ldots,\sigma_k\}}$*, then* $y(t) = u(1-t)$ *is a solution of* $y^{(n)}(t) + (-1)^n\,M\,y(t) = 0$ *on* $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.

So, we do not need to study all the cases to obtain conclusions about the strongly inverse positive (negative) character.

- 2^{nd}-order

In second order, the only possibility is to consider $k = 1$. There are three options for the choice of $\{\sigma_1\} - \{\varepsilon_1\}$. The first of them is $\sigma_1 = \varepsilon_1 = 0$ which correspond to the Dirichlet case, that is the boundary conditions $(1,1)$. This case has been considered before.

The other two choices correspond to the mixed boundary conditions and are equivalent, $\sigma_1 = 0$ and $\varepsilon_1 = 1$ or $\sigma_1 = 1$ and $\varepsilon_1 = 0$.

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in $X_{\{0\}}^{\{1\}}$ is $\lambda_1 = -\dfrac{\pi^2}{4}$. So, using Theorem 4.1.1, we can affirm that:

- $\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ is a strongly inverse negative operator in $X_{\{0\}}^{\{1\}}$ or in $X_{\{1\}}^{\{0\}}$ if, and only if, $M \in \left(-\infty, \dfrac{\pi^2}{4}\right)$.

Moreover, from Theorems 3.5.5 and 1.2.7, we conclude that there is not any $M \in \mathbb{R}$ such that $\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ is inverse positive either in $X_{\{0\}}^{\{1\}}$ or $X_{\{1\}}^{\{0\}}$.

- 3^{rd}-order

In this case the number of possible cases increases up to twelve, which we can reduce to six. The cases $\{\sigma_1, \sigma_2\} = \{0,1\}$, $\{\varepsilon_1\} = \{0\}$ and $\{\sigma_1\} = \{0\}$, $\{\varepsilon_1, \varepsilon_2\} = \{0,1\}$ have been considered previously because they belong to the $(k, n-k)$ boundary conditions. Let us see some of the rest.

First, let us consider $\{\sigma_1, \sigma_2\} = \{1, 2\}$ and $\{\varepsilon_1\} = \{0\}$.

The biggest negative eigenvalue of operator $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{1,2\}}^{\{0\}}$ is given by $\lambda_1 = -m_4^3$, where $m_4 \approxeq 1.85$ is the least positive solution of the equation:

$$e^{-m} + 2\, e^{m/2} \cos\left(\frac{\sqrt{3}}{2} m\right) = 0. \tag{4.3.1}$$

In order to apply Theorem 4.1.1, we need to obtain the least positive eigenvalue of $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{1\}}^{\{0,1\}}$, which is given by $\lambda_2 = m_5^3$, where $m_5 \approxeq 3.017$ is the least positive solution of:

$$e^{-m} - e^{m/2}\left(\cos\left(\frac{\sqrt{3}}{2} m\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2} m\right)\right) = 0. \tag{4.3.2}$$

Since, $k = 2 = n - 1$, we can apply Theorem 4.1.1 to affirm that:

- $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is strongly inverse negative in $X_{\{1,2\}}^{\{0\}}$ if, and only if, M is in the interval $\left[-m_5^3, m_4^3\right) \approxeq \left[-3.017^3, 1.85^3\right)$.

Moreover, from Theorems 3.5.5 and 1.2.7, we can conclude that there is not any $M \in \mathbb{R}$ such that $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is strongly inverse positive in $X_{\{1,2\}}^{\{0\}}$.

Now, from Remark 4.3.4, we can affirm that:

- $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is strongly inverse positive in $X_{\{0\}}^{\{1,2\}}$ if, and only if, $M \in \left(-m_4^3, m_5^3\right]$.

And, we also conclude that there is not any $M \in \mathbb{R}$ such that $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is strongly inverse negative in $X_{\{0\}}^{\{1,2\}}$.

Now, consider $\{\sigma_1, \sigma_2\} = \{0, 2\}$ and $\{\varepsilon_1\} = \{1\}$.

The biggest negative eigenvalue of $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{0,2\}}^{\{1\}}$ is $\lambda_1 = -m_4^3$, where $m_4 \approxeq 1.85$ has been defined as the least positive solution of (4.3.1).

Moreover, the least positive eigenvalue of $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{0\}}^{\{0,1\}}$ is given by $\lambda_2 = \left(\lambda_3^1\right)^3$, where $\lambda_3^1 \approxeq 4.223$ is the least positive solution of equation (2.1.4). Thus, from Theorem 4.1.1, we conclude that:

- $\frac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is strongly inverse negative in $X_{\{0,2\}}^{\{1\}}$ if, and only if, M belongs to the interval $\left[-\left(\lambda_3^1\right)^3, m_4^3\right) \approxeq \left[-4.223^3, 1.85^3\right)$.

We note that in this case $\sigma_2 = 2 > 1$ and $\varepsilon_1 = 1 > 0$, thus we cannot apply Theorem 3.5.5 to obtain conclusions about the inverse positive character of $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ in $X_{\{0,2\}}^{\{1\}}$.

However, we can apply Theorem 4.2.1. In order to do that, we need to obtain the biggest negative eigenvalues of $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{0,1\}}^{\{1\}}$ and $X_{\{0,2\}}^{\{0\}}$.

From Remark 4.3.4, the biggest negative eigenvalue of operator $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{0,1\}}^{\{1\}}$ is given by $\lambda_3' = -m_5^3$, where $m_5 \approxeq 3.017$ is the least positive solution of (4.3.2).

Moreover, the biggest negative eigenvalue of $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{0,2\}}^{\{0\}}$ is also given by $\lambda_3'' = -m_5^3$. Thus, from Theorem 4.2.1:

- If there exists $M \in \mathbb{R}$ such that $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is a strongly inverse positive operator, then $M \in \left(m_4^3, m_5^3\right] \approxeq \left(1.85^5, 3.017^3\right]$.

∘ 4^{th}-order

There are forty possibilities, which we can decrease, by using Remark 4.3.4, to twenty one. There are three possibilities which we have studied with the $(k, n-k)$ boundary conditions, they correspond to the sets $X_{\{0,1,2\}}^{\{0\}}$, $X_{\{0\}}^{\{0,1,2\}}$ and $X_{\{0,1\}}^{\{0,1\}}$. We will develop the characterisation in $X_{\{0,2\}}^{\{0,2\}}$ in Chapter 5. Moreover, along the chapter we have studied the case $X_{\{0,2\}}^{\{1,2\}}$. From Remark 4.3.4, the obtained characterisation remains suitable for the set $X_{\{1,2\}}^{\{0,2\}}$.

In the sequel, let us see a pair of different cases. For instance, $X_{\{1,2,3\}}^{\{0\}}$ $\Big($which also gives the characterisation in $X_{\{0\}}^{\{1,2,3\}}\Big)$ and $X_{\{0,2\}}^{\{1,2\}}$ $\left(X_{\{1,3\}}^{\{0,2\}}\right)$.

Let us work on the space $X_{\{1,2,3\}}^{\{0\}}$.

First, we obtain the necessary eigenvalues in order to apply Theorem 4.1.1 to this case.

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{1,2,3\}}^{\{0\}}$ is $\lambda_1 = -\dfrac{\pi^4}{4}$.

The least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{1,2\}}^{\{0,1\}}$ is $\lambda_2 = \pi^4$.

- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ is strongly inverse negative in $X_{\{1,2,3\}}^{\{0\}}$ if, and only if, $M \in \left[-\pi^4, \dfrac{\pi^4}{4}\right)$.

Since $\varepsilon_1 = 0$, we can apply Theorems 3.5.5 and 1.2.7 to conclude that there is not any $M \in \mathbb{R}$ such that $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ is inverse positive in $X_{\{1,2,3\}}^{\{0\}}$.

Concerning to the space $X_{\{0,2\}}^{\{1,3\}}$, we have the following eigenvalues.

The least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,2\}}^{\{1,3\}}$ is $\lambda_1 = \dfrac{\pi^4}{16}$.

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_2' = -4\,\pi^4$.
The biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0\}}^{\{0,1,3\}}$ is $\lambda_2'' = -4\,\pi^4$.
So, $\lambda_2 = \max\big\{-4\,\pi^4, -4\,\pi^4\big\} = -4\,\pi^4$.
Hence, we conclude, from Theorem 4.1.1, that:

- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}+M$ is strongly inverse positive in $X_{\{0,2\}}^{\{1,3\}}$ if, and only if, $M \in \left(-\dfrac{\pi^4}{16}, 4\,\pi^4\right]$.

Since $\sigma_2 = 2 > 1$ and $\varepsilon_2 = 3 > 1$, we cannot apply Theorem 3.5.5 to affirm that it cannot be strongly inverse negative for any $M \in \mathbb{R}$.

But, we can apply Theorem 4.2.1. We need to obtain the least positive eigenvalues of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,1\}}^{\{1,3\}}$ and $X_{\{0,2\}}^{\{0,1\}}$.

The least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,1\}}^{\{1,3\}}$ is $\lambda_1 = m_1^4$, where $m_1 \approxeq 2.36502$ has been introduced in Example 3.6.7.

The least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,2\}}^{\{0,1\}}$ is $\lambda_1 = m_3^4$, where $m_3 \approxeq 3.9266$ has been introduced in Example 3.6.7.

Thus, from Theorem 4.2.1, we have that:

- If there exists $M \in \mathbb{R}$ such that $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ is a inverse negative operator, then M remains in $\left[-m_1^3, -\dfrac{\pi^4}{16}\right)$.

∘ Higher order

If we increase the order of the problem, due to the fact that the related Green's function gets more complexity, the usefulness of Theorem 4.1.1 also increases. Even if we cannot obtain the eigenvalues analytically, we can obtain them numerically, by using different methods.

Now, let us show an example of sixth order, where we can obtain the eigenvalues analytically.

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,2,4\}}^{\{0,2,4\}}$ is $\lambda_1 = -\pi^6$.

The least positive eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,1,2,4\}}^{\{0,2\}}$ and $X_{\{0,2\}}^{\{0,1,2,4\}}$ is $\lambda_2 = \lambda_2' = \lambda_2'' = m_6^6$, where $m_6 \approxeq 5.47916$ is the least positive solution of:

$$\cos\left(\sqrt{3}m\right) - \cosh(m) + 8\cos\left(\frac{\sqrt{3}m}{2}\right)\sinh^2\left(\frac{m}{2}\right)\cosh\left(\frac{m}{2}\right) = 0.$$

Hence, from Theorem 4.1.1, we conclude that:

- $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} + M$ is a strongly inverse negative operator in $X_{\{0,2,4\}}^{\{0,2,4\}}$ if, and only if, M is in $\big[-m_6^6, \pi^6\big) \approxeq \big[-5.47916^6, \pi^6\big)$.

Since $\sigma_3 = 4 > 2$ and $\varepsilon_3 = 4 > 2$, we cannot apply Theorem 3.5.5 to obtain any conclusion about the inverse positive character.

However, we can apply Theorem 4.2.1. In order to do that, let us obtain the biggest negative eigenvalues of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,1,2\}}^{\{0,2,4\}}$ and $X_{\{0,2,4\}}^{\{0,1,2\}}$. From Remark 4.3.4 they both coincide. They are given by $\lambda_3 = \lambda_3' = \lambda_3'' = -m_7^6$, where $m_7 \approxeq 4.7135$ is the least positive solution of:

$$\cos(2m) + 2\cos(m)\cosh\left(\sqrt{3}m\right) - 3 = 0.$$

- If there exists $M \in \mathbb{R}$ such that $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} + M$ is a strongly inverse positive operator in $X_{\{0,2,4\}}^{\{0,2,4\}}$, then $M \in \left(\pi^6, m_7^6\right] \approxeq \left(\pi^6, 4.7135^6\right]$.

Operator $T_4^N[M]u(t) \equiv \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + N\dfrac{\mathrm{d}}{\mathrm{d}\,t} + M$ in $X_{\{0,2\}}^{\{1,2\}}$

Let us denote $N = n^3$. Now, let us consider the fourth order operator:

$$T_4^{n^3}[M]\,u(t) = u^{(4)}(t) + n^3\,u'(t) + M\,u(t),$$

in $X_{\{0,2\}}^{\{1,2\}}$. Realise that for $n = 0$ this operator coincides with the example that we have been considering in the different examples along the chapter.

Let us see that for $n \in \left(-\dfrac{4\,\pi}{3\,\sqrt{3}}, \dfrac{2\,\pi}{3\,\sqrt{3}}\right)$, $T_4^{n^3}[0]$ satisfies property (T_d) in $X_{\{0,2\}}^{\{1,2\}}$.

In order to do that, let us consider the following fundamental system of solutions:

$$\begin{aligned}
y_1^n(t) &= 1,\\
y_2^n(t) &= \sqrt{3}\,e^{\frac{n\,t}{2}}\,\cos\left(\frac{\sqrt{3}}{2}n\,t\right) + e^{\frac{n\,t}{2}}\,\sin\left(\frac{\sqrt{3}}{2}n\,t\right),\\
y_3^n(t) &= e^{\frac{n\,t}{2}}\,\sin\left(\frac{\sqrt{3}}{2}n\,t\right),\\
y_4^n(t) &= e^{-n\,t},
\end{aligned}$$

and the correspondent Wronskians:

$$\begin{aligned}
W_1^n(t) &= 1,\\
W_2^n(t) &= e^{\frac{n\,t}{2}}\,n\left(\cos\left(\frac{\sqrt{3}}{2}n\,t\right) - \sin\left(\frac{\sqrt{3}}{2}n\,t\right)\right),\\
W_3^n(t) &= \frac{3}{2}\,n^3\,e^{n\,t},\\
W_4^n(t) &= -\frac{9\,n^6}{2}.
\end{aligned}$$

If, $n \neq 0$, then W_1^n, W_3^n and W_4^n are non-null in $[0,1]$.

Moreover, we can see that if $n \in \left(-\dfrac{4\,\pi}{3\,\sqrt{3}}, \dfrac{2\,\pi}{3\,\sqrt{3}}\right)$, then $W_2^n(t) \neq 0$ for all $t \in [0,1]$. So, we obtain the representation given in (3.3.1)-(3.3.2).

We construct $v_1, \ldots, v_4$ following the recurrence formula (1.1.3):

$$\begin{aligned}
v_1^n(t) &= 1, \\
v_2^n(t) &= W_2^n(t) = e^{\frac{n\,t}{2}}\, n \left(\cos\left(\frac{\sqrt{3}}{2} n\,t\right) - \sin\left(\frac{\sqrt{3}}{2} n\,t\right)\right), \\
v_3^n(t) &= \frac{W_3^n(t)}{W_2^{n\,2}(t)}, \\
v_4^n(t) &= \frac{W_2^n(t)\, W_4^n(t)}{W_3^{n\,2}(t)}.
\end{aligned}$$

In Example 3.4.7, we have proved that a fourth order operator satisfies property (T_d) in $X_{\{0,2\}}^{\{1,2\}}$ if, and only if, there exists the decomposition (3.3.1)-(3.3.2) and (3.4.21)-(3.4.22) are fulfilled.

Let us check it. Obviously, (3.4.22) is satisfied. Now, since $v_1^{n\prime}(0) = 0$ and $v_2^n(0) \neq 0$, we have to verify that $v_2^{n\prime}(0) = 0$. But, from the fact that:

$$v_2^{n\prime}(t) = -2\, e^{\frac{n\,t}{2}}\, n^2 \sin\left(\frac{\sqrt{3}}{2} n\,t\right),$$

we deduce that it is trivially satisfied that $v_2^{n\prime}(0) = 0$. So, as a consequence, (3.4.21) is fulfilled and we conclude that if $n \in \left(-\dfrac{4\,\pi}{3\,\sqrt{3}}, \dfrac{2\,\pi}{3\,\sqrt{3}}\right)$, then $T_4^{n^3}[0]$ verifies property (T_d) in $X_{\{0,2\}}^{\{1,2\}}$.

Remark 4.3.5. *Realise that the interval* $\left(-\dfrac{4\,\pi}{3\,\sqrt{3}}, \dfrac{2\,\pi}{3\,\sqrt{3}}\right)$ *is not necessarily optimal.*

If we study the disconjugacy set of $T_4^{n^3}[0]\,u(t) = 0$ *on* $[0,1]$, *we obtain that such an equation is disconjugate if, and only if,* $n \in (-n_1, n_1)$, *where* $n_1 \approxeq 5.55$ *is the least positive solution of:*

$$-3 + e^{-n} + 2^{n/2} \cos\left(\frac{\sqrt{3}\,n}{2}\right) = 0.$$

Then, it is possible that we may find different values of $n \in (-n_1, n_1)$ *such that* $T_4^{n^3}[0]$ *satisfies property* (T_d) *in* $X_{\{0,2\}}^{\{1,2\}}$ *with a suitable choice of the fundamental system of solutions.*

For instance, repeating the previous arguments for the following fundamental system of *solutions:*

$$\begin{aligned}
y_1^n(t) &= 1,\\
y_2^n(t) &= \frac{2}{\sqrt{3}}\, e^{\frac{n\,t}{2}}\, \sin\left(\frac{\sqrt{3}}{2} n\,t\right) - e^{-n\,t},\\
y_3^n(t) &= e^{-n\,t},\\
y_4^n(t) &= e^{\frac{n\,t}{2}}\, \cos\left(\frac{\sqrt{3}}{2} n\,t\right),
\end{aligned}$$

we obtain a decomposition to ensure that $T_4^{n^3}[0]$ *satisfy property* (T_d) *for* $n \in \left(-\frac{\pi}{\sqrt{3}}, \frac{\pi}{\sqrt{3}}\right)$.

Thus, we can say that if:

$$n \in \left(-\frac{4\,\pi}{3\,\sqrt{3}}, \frac{2\,\pi}{3\,\sqrt{3}}\right) \cup \left(-\frac{\pi}{\sqrt{3}}, \frac{\pi}{\sqrt{3}}\right) = \left(-\frac{4\,\pi}{3\,\sqrt{3}}, \frac{\pi}{\sqrt{3}}\right) \subset (-n_1, n_1),$$

then $T_4^{n^3}[0]$ *satisfies property* (T_d).

However, we cannot even affirm that such an interval is the optimal one.

Let us choose, for instance, $n = -\frac{\pi}{\sqrt{3}} \in \left(-\frac{4\,\pi}{3\,\sqrt{3}}, \frac{\pi}{\sqrt{3}}\right) \subset (-n_1, n_1)$ and we obtain the different eigenvalues numerically, by using Mathematica.

The least positive eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ in $X_{\{0,2\}}^{\{1,2\}}$ is $\lambda_1 \approxeq 2.21152^4$.

The biggest negative eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ in $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_2' \approxeq -4.53073^4$.

The biggest negative eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ in $X_{\{0\}}^{\{0,1,2\}}$ is $\lambda_2'' \approxeq -5.5014^4$.

So, $\lambda_2 = \max\left\{\lambda_2', \lambda_2''\right\} = \lambda_2' \approxeq -4.53073^4$.

From Theorem 4.1.1, we conclude:

- $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[M]$ is strongly inverse positive in $X_{\{0,2\}}^{\{1,2\}}$ if, and only if, M belongs to the interval $(-\lambda_1, -\lambda_2] \approxeq \left(-2.21152^4, 4.53073^4\right]$.

Since $\sigma_2 = \varepsilon_2 = 2 > 1$, we cannot obtain any conclusion about the strongly inverse positive character from Theorem 3.5.5.

However, we can obtain numerically the least positive eigenvalues in $X_{\{0,1\}}^{\{1,2\}}$ and $X_{\{0,2\}}^{\{0,1\}}$ in order to apply Theorem 4.2.1.

The least positive eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ in $X_{\{0,1\}}^{\{1,2\}}$ is $\lambda_3' \approxeq 3.06562^4$.

The least positive eigenvalue of $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[0]$ in $X_{\{0,2\}}^{\{0,1\}}$ is $\lambda_3'' \approxeq 3.92719^4$.

So, $\lambda_3 = \min\{\lambda_3', \lambda_3''\} = \lambda_3' \approxeq 3.06562^4$.

Thus, from Theorem 4.2.1:

- if there exists $M \in \mathbb{R}$ such that $T_4^{-\frac{\pi^3}{3\sqrt{3}}}[M]$ is strongly inverse negative in $X_{\{0,2\}}^{\{1,2\}}$, then $M \in [-\lambda_3, -\lambda_1) \approxeq \left[-3.06562^4, -2.21152^4\right)$.

4.4 Non-homogeneous boundary conditions

This section is devoted to study operator $T_n[M]$, coupled with different non-homogeneous boundary conditions.

First, let us consider the following set:

$$\begin{aligned}\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}} = \Big\{ u \in C^n(I) \ \mid \ & u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k-1)}(a) = 0, \ (-1)^{n-\sigma_k-1} u^{(\sigma_k)}(a) \geqslant 0, \\ & u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k-1})}(b) = 0, \ u^{(\varepsilon_{n-k})}(b) \leqslant 0 \Big\}.\end{aligned} \tag{4.4.1}$$

That is, we consider a set where some of the boundary conditions do not have to be necessarily homogeneous. This information is very useful in order to apply the lower and upper solutions method and monotone iterative techniques for non-linear boundary value problems, see for instance [22].

Since we are including non-homogeneous boundary conditions we lose the control on the derivatives on the boundary in some cases, thus we need to define what we understand by strongly inverse positive or negative operator in a set with non-homogeneous boundary conditions, $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Definition 4.4.1. *Operator $T_n[M]$ is said to be strongly inverse positive in $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if every function $u \in \tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\,u \gneqq 0$ on I, must satisfy $u > 0$ on (a,b) and, moreover,*

- *If $\sigma_k > \alpha$ or $u^{(\sigma_k)}(a) = 0$, then $u^{(\alpha)}(a) > 0$.*
- *If $\varepsilon_{n-k} > \beta$ or $u^{(\varepsilon_{n-k})}(b) = 0$, then $\begin{cases} u^{(\beta)}(b) > 0, & \text{if } \beta \text{ is even,} \\ u^{(\beta)}(b) < 0, & \text{if } \beta \text{ is odd,} \end{cases}$*

where α and β are defined in (3.2.1) *and* (3.2.2)*, respectively.*

Definition 4.4.2. *Operator $T_n[M]$ is said to be strongly inverse negative in $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if every function $u \in \tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ such that $T_n[M]\,u \gneqq 0$ on I, must satisfy $u < 0$ on (a,b) and, moreover,*

- *If $\sigma_k > \alpha$ or $u^{(\sigma_k)}(a) = 0$, then $u^{(\alpha)}(a) < 0$.*
- *If $\varepsilon_{n-k} > \beta$ or $u^{(\varepsilon_{n-k})}(b) = 0$, then $\begin{cases} u^{(\beta)}(b) < 0, & \text{if } \beta \text{ is even,} \\ u^{(\beta)}(b) > 0, & \text{if } \beta \text{ is odd,} \end{cases}$*

where α and β are defined in (3.2.1) *and* (3.2.2)*, respectively.*

We introduce the boundary conditions which a function $u \in \tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ must satisfy:

$$u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k-1)}(a) = 0, \quad u^{(\sigma_k)}(a) = c_1, \tag{4.4.2}$$

$$u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k-1})}(b) = 0, \quad u^{(\varepsilon_{n-k})}(b) = c_2, \tag{4.4.3}$$

where $(-1)^{n-\sigma_k-1}c_1 \geqslant 0$ and $c_2 \leqslant 0$ are arbitrary.

We can connect the non-homogeneous problem (3.0.1), (4.4.2)-(4.4.3) with the homogeneous problem (3.0.1)–(3.0.3) by means of the following result.

Lemma 4.4.3. *If problem* (3.0.1)–(3.0.3) *has only the trivial solution for* $\sigma(t) = 0$ *for all* $t \in I$. *Then problem* (3.0.1), *coupled with the boundary conditions* (4.4.2)-(4.4.3) *has a unique solution given by:*

$$u(t) = \int_a^b g_M(t,s)\,\sigma(s)\,\mathrm{d}\,s + c_1\,x_M(t) + d_1\,z_M(t), \tag{4.4.4}$$

where $g_M(t,s)$ *is the related Green's function of* $T_n[M]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *and:*

- x_M *is defined as the unique solution of:*

$$\begin{cases} T_n[M]\,u(t) = 0, \quad t \in I, \\ u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{k-1})}(a) = 0, \\ u^{(\sigma_k)}(a) = 1, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0. \end{cases} \tag{4.4.5}$$

- z_M *is defined as the unique solution of:*

$$\begin{cases} T_n[M]\,u(t) = 0, \quad t \in I \\ u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k-1})}(b) = 0, \\ u^{(\varepsilon_{n-k})}(b) = 1. \end{cases} \tag{4.4.6}$$

Using this lemma we can obtain the following result which characterises the strongly inverse positive (negative) character of $T_n[M]$ in $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Theorem 4.4.4. $T_n[M]$ *is strongly inverse positive (negative) in* $\tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ *if, and only if, it is strongly inverse positive (negative) in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. Since $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}} \subset \tilde{X}_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ the necessary condition is obvious.

Now, let us see the sufficient one. From the strongly inverse positive (negative) character of $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, using Theorems 3.2.5 and 3.2.6, coupled with Lemma 4.4.3, we only need to study the sign of x_M and z_M.

In order to do that, we establish a relationship between these functions and some derivatives of $g_M(t,s)$.

Taking into account the boundary conditions, it is clear that:

$$x_M(t) = (-1)^{n-1-\sigma_k}\,w_M(t) \text{ and } z_M(t) = (-1)^{n-\varepsilon_{n-k}}\,y_M(t),$$

where w_M and y_M have been defined in Lemmas 3.5.1 and 3.5.2, respectively.

Now, if $T_n[M]$ is strongly inverse positive (negative) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, then $w_M(t) \geqslant 0$ ($\leqslant 0$). Moreover, clearly $(-1)^\beta w_M^{(\beta)}(b) \geqslant 0$ ($\leqslant 0$) and if $\sigma_k > \alpha$, then $w_M^{(\alpha)}(a) \geqslant 0$ ($\leqslant 0$).

On another hand, clearly, $(-1)^\gamma\, y_M(t) \geqslant 0$ ($\leqslant 0$ for the negative case). So, taking into account that $\gamma = n-1-\varepsilon_{n-k}$, we have that $(-1)^{n-\varepsilon_{n-k}} y_M(t) \leqslant 0$ ($\geqslant 0$). Furthermore, from the boundary conditions, $(-1)^\gamma y_M^{(\alpha)}(a) \geqslant 0$ ($\leqslant 0$) and if, in addition, $\varepsilon_{n-k} > \beta$, then $(-1)^{\gamma+\beta} y_M^{(\beta)}(b) \geqslant 0$ ($\leqslant 0$).

Thus, the result is proved. $\square$

Remark 4.4.5. *It is easy to see that if we consider the strongly inverse positive or negative character as in Definitions 3.2.3 and 3.2.4, then Theorem 4.1.1 is not true.*

Consider, for instance the second order problem:

$$u''(t) = t, \quad t \in [0,1], \quad u(0) = 0,\ u(1) = -1,$$

In this case, $\sigma_1 = \varepsilon_1 = 0$ *and* $\alpha = \beta = 1$. *The unique solution is given by:*

$$u(t) = \frac{1}{6}\left(t^3 - 7t\right),$$

and, clearly, $u'(1) = -\dfrac{2}{3} < 0$. *Thus, it does not satisfy the properties given by the strongly inverse negative character in Definition 3.2.4.*

This is due to the fact that imposing $u(1) = -1 < 0$, *we lose the control in* $u'(1)$. *However, the solution is strictly negative at* $t = 1$ *and it clearly fulfils Definition 4.4.2.*

4.4.1 A particular kind of operators

In this section we consider a particular kind of operators which satisfy property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, thus we can apply previous results to these operators.

After that, we obtain some results which characterise either the strongly inverse positive character or the strongly inverse negative character of $T_n[M]$ if $n-k$ is even or odd, respectively, in different sets where more general non-homogeneous boundary conditions are considered.

First, we introduce the following notation.

Notation 4.4.6. *Let us denote* $\alpha_2 \in \{-1,0,1,\dots,n-2\}$, *such that* $\alpha_2 \notin \{\sigma_1,\dots,\sigma_k\}$ *and* $\{\alpha_2+1, \alpha_2+2,\dots,\sigma_k\} \subset \{\sigma_1,\dots,\sigma_k\}$; *and* $\beta_2 \in \{-1,0,1,\dots,n-2\}$, *such that* $\beta_2 \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *and* $\{\beta_2+1,\beta_2+2,\dots,\varepsilon_{n-k}\} \subset \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$.

We denote $\mu = \max\{\alpha_2,\beta_2\}$.

Remark 4.4.7. *Realise that if* $\sigma_k = k-1$ *then* $\alpha_2 = -1$. *Otherwise,* $\alpha_2 \geqslant \alpha \geqslant 0$.

Moreover, if $\varepsilon_{n-k} = n-k-1$ *then* $\beta_2 = -1$. *Otherwise,* $\beta_2 \geqslant \beta \geqslant 0$.

Now, we introduce the following sufficient condition for an operator to satisfy property (T_d).

Proposition 4.4.8. *If the linear differential equation of* $(n-\mu-1)^{\mathrm{th}}$*-order:*

$$L_{n-\mu-1}\, u(t) \equiv u^{(n-\mu-1)}(t) + p_1(t)\, u^{(n-\mu-2)}(t) + \cdots + p_{n-\mu-1}(t)\, u(t) = 0,\ t \in I \quad (4.4.7)$$

with $p_j \in C^{n-j}(I)$, is disconjugate on I, then the operator:

$$\tilde{T}_n[0]\, u(t) = u^{(n)}(t) + p_1(t)\, u^{(n-1)}(t) + \cdots + p_{n-\mu-1}(t)\, u^{(\mu+1)}(t),$$

satisfies property (T_d) in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.

Proof. From Theorems 1.1.4 and 1.1.6, since the linear differential equation (4.4.7) is disconjugate on I, there exist positive functions $v_1, \ldots, v_{n-\mu-1}$ such that $v_k \in C^{n-\mu-k}(I)$ for $k = 1, \ldots, n-\mu-1$, and:

$$L_{n-\mu-1}\, u \equiv v_1 \cdots v_{n-\mu-1} \frac{\mathrm{d}}{\mathrm{d}\,t}\left[\frac{1}{v_{n-\mu-1}} \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\cdots \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{u}{v_1}\right)\right)\right].$$

To clarify the proof, we divide it in two steps.

Step 1. Let us see that, in fact, $v_k \in C^n(I)$ for $k = 1, \ldots, n-\mu-1$.

Since, $p_j \in C^{n-j}(I)$ for $j \in \{\mu+1, \ldots, n-1\}$, every solution of (4.4.7) belongs to $C^n(I)$.

If we look at the proof of Theorem 1.1.4, we observe that v_k is given by the recurrence formula (1.1.3), where W_k are the correspondent Wronskians related to a Markov fundamental system of solutions of (4.4.7), $\{y_1, \ldots, y_{n-\mu-1}\}$.

Thus, taking into account that $y_1, \ldots, y_{n-\mu-1} \in C^n(I)$, we conclude that $v_1 \in C^n(I)$, $v_2 \in C^{n-1}(I), \ldots, v_{n-\mu-1} \in C^{\mu+2}(I)$.

Now, let us consider the expression (3.3.3), with $\ell = n-\mu-1$ and $p_{\ell j} = p_j \in C^{n-j}(I)$, $j \in \{\mu+1, \ldots, n-1\}$ given by the expressions (3.3.4)–(3.3.7).

First, let us see that $v_1 \in C^n(I)$, $v_2 \in C^n(I)$, $v_3 \in C^{n-1}(I), \ldots, v_{n-\mu-1} \in C^{\mu+3}(I)$.

If $\mu = n-2$, then $n-\mu-1 = 1$ and the result is proved since $v_1 \in C^n(I)$.

Otherwise, $p_1 \in C^{n-1}(I) \subset C^{\mu+2}(I)$, since $v_1, \ldots, v_{n-\mu-2} \in C^{\mu+3}(I)$ and, moreover, $v_{n-\mu-1} \in C^{\mu+2}(I)$, from (3.3.4) we obtain that $v'_{n-\mu-1} \in C^{\mu+2}(I)$, then we have that $v_{n-\mu-1} \in C^{\mu+3}(I)$.

Let us assume that $v_{k+1} \in C^{n-k}(I)$, $v_{k+2} \in C^{n-k-1}(I), \ldots, v_{n-\mu-1} \in C^{\mu+3}(I)$, then since $v_k \in C^{n-k}(I)$ considering the expression of $p_{n-\mu-k}$, given in equation (3.3.7) for $\ell_\ell = n-\mu-k$, we obtain that $v_k^{(n-\mu-k)} \in C^{\mu+1}(I)$, hence $v_k \in C^{n-k+1}(I)$.

Thus, we have proved by induction that $v_1 \in C^n(I)$, $v_2 \in C^n(I)$, $v_3 \in C^{n-1}(I), \ldots$, $v_{n-\mu-1} \in C^{\mu+3}$.

If $\mu = n-3$, then the result is proved, since $v_1, v_2 \in C^n(I)$.

Now, let us assume that $\mu < n-3$. Considering the expression of $p_{n-\mu-3} \in C^{\mu+3}(I)$, given in (3.3.7) for $\ell_\ell = n-\mu-k$. Since $v_2 \in C^n(I)$, we conclude that $v_3^{(n-\mu-3)} \in C^{\mu+3}(I)$; so, $v_3 \in C^n(I)$.

If we suppose that $v_1, \ldots, v_{k-1} \in C^n(I)$, then by considering the expression of the coefficient $p_{n-\mu-k} \in C^{\mu+k}(I)$, we conclude that $v_k^{(n-\mu-k)} \in C^{\mu+k}(I)$, thus $v_k \in C^n(I)$.

Then, we have proved that $v_1, \ldots, v_{n-\mu-1} \in C^n(I)$.

Step 2. Construction of the decomposition satisfying property (T_d).

Now, we consider the decomposition of $\tilde{T}[0]$ as follows:

$$\tilde{T}[0]\, u \equiv v_1 \dots v_{n-\mu-1} \frac{\mathrm{d}}{\mathrm{d}\, t}\left[\frac{1}{v_{n-\mu-1}} \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\cdots \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{u^{(\mu+1)}}{v_1}\right)\right)\right]. \tag{4.4.8}$$

Hence, if we denote $\tilde{v}_1 = \cdots = \tilde{v}_{\mu+1} = 1$ and $\tilde{v}_{\mu+2} = v_1, \dots, \tilde{v}_n = v_{n-\mu-1}$, we can decompose $\tilde{T}_n[0]$ in the following sense:

$$\tilde{T}_0\, u = u, \quad \tilde{T}_k\, u = \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{\tilde{T}_{k-1}\, u}{\tilde{v}_k}\right),\ k = 1, \dots, n.$$

Trivially $\tilde{T}_n[0]\, u = \tilde{v}_1 \dots \tilde{v}_n \tilde{T}_n\, u$. Now, let us see that this decomposition satisfies property (T_d).

We have that $\tilde{T}_0 u = u$, $\tilde{T}_1 u = u'$, $\dots$, $\tilde{T}_{\mu+1} u = u^{(\mu+1)}$. Hence, if $\sigma_i < \alpha_2 \leqslant \mu$ then $\tilde{T}_{\sigma_i} u(a) = u^{(\sigma_i)}(a) = 0$. Analogously, if $\varepsilon_i < \beta_2 \leqslant \mu$, then $\tilde{T}_{\varepsilon_i} u(b) = u^{(\varepsilon_i)}(b) = 0$.

If $h > \mu + 1$, then:

$$\tilde{T}_h u = \frac{u^{(h)}}{v_1 \dots v_h} + p_{h_1}\, u^{(h-1)} + \cdots + p_{h_{h-\mu-1}}\, u^{(\mu+1)},$$

where p_{h_i} is given by equations (3.3.4)–(3.3.7).

If $\sigma_i > \mu$, then by definition of μ, $u^{(\mu+1)}(a) = u^{(\mu+2)}(a) = \cdots = u^{(\sigma_i)}(a) = 0$. Hence $\tilde{T}_{\sigma_i} u(a) = 0$.

Analogously, if $\varepsilon_i > \mu$, then $u^{(\mu+1)}(b) = u^{(\mu+2)}(b) = \cdots = u^{(\varepsilon_i)}(b) = 0$. Hence $\tilde{T}_{\varepsilon_i} u(b) = 0$.

Thus, the result is proved. □

As a direct consequence of this result, we can apply Theorems 4.1.1, 4.2.1 and 4.4.4 to operator $\tilde{T}_n[M]$. Moreover, we can apply Theorems 3.7.1, 3.7.3 and 3.7.5 to the related Green's function.

Furthermore, for this particular case, we will be able to obtain a characterisation of strongly inverse positive (negative) character in different spaces with non-homogeneous boundary conditions.

Definition 4.4.9. *Let us consider* $\{\sigma_{\epsilon_1}, \dots, \sigma_{\epsilon_\ell}\} \subset \{\sigma_1, \dots, \sigma_k\}$ *such that:*

$$\sigma_{\epsilon_1} < \sigma_{\epsilon_2} < \cdots < \sigma_{\epsilon_\ell} = \sigma_k, \text{ with } \sigma_{\epsilon_{\ell-1}} < \mu.$$

And $\{\varepsilon_{\kappa_1}, \dots, \varepsilon_{\kappa_h}\} \subset \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ *such that*

$$\varepsilon_{\kappa_1} < \varepsilon_{\kappa_2} < \cdots < \varepsilon_{\kappa_h} = \varepsilon_{n-k}, \text{ with } \varepsilon_{\kappa_{h-1}} < \mu.$$

Let us define the set of functions $X_{\{\sigma_1,\dots,\sigma_k\}\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}$ as follows:

$$\left\{ u \in C^n(I) \ \mid \ u^{(\sigma_j)}(a) = \begin{cases} 0, & j \notin \{\epsilon_1, \dots, \epsilon_\ell\}, \\ (-1)^{n-\sigma_j-(k-j)+1} \varphi_j, & j \in \{\epsilon_1, \dots, \epsilon_\ell\}, \end{cases} \quad \text{for some } \varphi_j \geqslant 0,\ j = 1, \dots, k, \right.$$

$$u^{(\varepsilon_i)}(b) = \left\{ \begin{array}{ll} 0, & i \notin \{\kappa_1, \ldots, \kappa_h\}, \\ (-1)^{n-k+i-1}\psi_i, & i \in \{\kappa_1, \ldots, \kappa_h\}, \end{array} \text{ for some } \psi_i \geqslant 0,\ i = 1, \ldots, n-k \right\}. \tag{4.4.9}$$

Our aim is to characterise the strongly inverse positive or negative character of operators with non-homogeneous boundary conditions. In this case, we also have to point out what we understand by these properties in this case.

Definition 4.4.10. *Operator $T_n[M]$ is strongly inverse positive in $X_{\{\sigma_1,\ldots,\sigma_k\}_{\{\sigma_{\epsilon_1},\ldots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\ldots,\varepsilon_{\kappa_h}\}}}$ if every function $u \in X_{\{\sigma_1,\ldots,\sigma_k\}_{\{\sigma_{\epsilon_1},\ldots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\ldots,\varepsilon_{\kappa_h}\}}}$ such that $T_n[M]\,u \gneqq 0$ on I, must satisfy $u > 0$ on (a,b) and, moreover,*

- *If $u^{(\sigma_j)}(a) = 0$ for all $j \in \{\epsilon_1, \ldots, \epsilon_\ell\}$ such that $\sigma_j < \alpha$, then $u^{(\alpha)}(a) > 0$.*
- *If $u^{(\varepsilon_i)}(a) = 0$ for all $i \in \{\kappa_1, \ldots, \kappa_h\}$ such that $\varepsilon_i < \beta$, then $(-1)^\beta u^{(\beta)}(b) > 0$.*

Where α and β are defined in (3.2.1) *and* (3.2.2)*, respectively.*

Definition 4.4.11. *Operator $T_n[M]$ is strongly inverse negative in $X_{\{\sigma_1,\ldots,\sigma_k\}_{\{\sigma_{\epsilon_1},\ldots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\ldots,\varepsilon_{\kappa_h}\}}}$ if every function $u \in X_{\{\sigma_1,\ldots,\sigma_k\}_{\{\sigma_{\epsilon_1},\ldots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\ldots,\varepsilon_{\kappa_h}\}}}$ such that $T_n[M]\,u \gneqq 0$ on I, must satisfy $u < 0$ on (a,b) and, moreover,*

- *If $u^{(\sigma_j)}(a) = 0$ for all $j \in \{\epsilon_1, \ldots, \epsilon_\ell\}$ such that $\sigma_j < \alpha$, then $u^{(\alpha)}(a) < 0$.*
- *If $u^{(\varepsilon_i)}(a) = 0$ for all $i \in \{\kappa_1, \ldots, \kappa_h\}$ such that $\varepsilon_i < \beta$, then $(-1)^\beta u^{(\beta)}(b) < 0$.*

Where α and β are defined in (3.2.1) *and* (3.2.2)*, respectively.*

Remark 4.4.12. *The strongly inverse positive (negative) character, defined previously and in Definitions* 4.4.1 *and* 4.4.2*, implies that the possible solutions belong to the interior of a suitable cone related to the set $X_{\{\sigma_1,\ldots,\sigma_k\}_{\{\sigma_{\epsilon_1},\ldots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\ldots,\varepsilon_{\kappa_h}\}}}$. This is a good property to apply different iterative techniques to attain the existence of solutions for related non-linear problems, see [22].*

Now, we enunciate a similar result to Lemma 4.4.3 for this more general case.

Lemma 4.4.13. *If problem* (3.0.1)–(3.0.3) *has only the trivial solution for $\sigma(t) = 0$ for all $t \in I$. Then problem* (3.0.1)*, coupled with the boundary conditions:*

$$u^{(\sigma_j)}(a) = \begin{cases} 0, & j \notin \{\epsilon_1, \ldots, \epsilon_\ell\}, \\ c_j, & j \in \{\epsilon_1, \ldots, \epsilon_\ell\}, \end{cases} \qquad j = 1, \ldots, k, \tag{4.4.10}$$

and,

$$u^{(\varepsilon_i)}(b) = \begin{cases} 0, & i \notin \{\kappa_1, \ldots, \kappa_h\}, \\ d_i, & i \in \{\kappa_1, \ldots, \kappa_h\}, \end{cases} \qquad i = 1, \ldots, n-k, \tag{4.4.11}$$

has a unique solution given by:

$$u(t)=\int_a^b g_M(t,s)\,\sigma(s)\,\mathrm{d}\,s+\sum_{j=1}^{\ell} c_{\epsilon_j}\,x_M^{\sigma_{\epsilon_j}}(t)+\sum_{i=1}^{h} d_{\kappa_i}\,z_M^{\varepsilon_{\kappa_i}}(t), \tag{4.4.12}$$

where $g_M(t,s)$ is the related Green's function of $T_n[M]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and,

- $x_M^{\sigma_{\epsilon_j}}$ *is defined as the unique solution of:*

$$\left\{\begin{array}{c} T_n[M]\,u(t)=0,\quad t\in I\\ \\ u^{(\sigma_{\epsilon_j})}(a)=1,\\ \\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_{\epsilon_j-1})}(a)=u^{(\sigma_{\epsilon_j+1})}(a)=\cdots=u^{(\sigma_k)}(a)=0,\\ \\ u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k})}(b)=0. \end{array}\right. \tag{4.4.13}$$

- $z_M^{\varepsilon_{\kappa_i}}$ *is defined as the unique solution of:*

$$\left\{\begin{array}{c} T_n[M]\,u(t)=0,\quad t\in I,\\ \\ u^{(\sigma_k)}(a)=\cdots=u^{(\sigma_k)}(a)=0,\\ \\ u^{(\varepsilon_{\kappa_j})}(b)=1,\\ \\ u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{\kappa_i-1})}(b)=u^{(\varepsilon_{\kappa_i+1})}(b)=\cdots=u^{(\varepsilon_{n-k})}(b)=0. \end{array}\right. \tag{4.4.14}$$

We have the following results, which ensure the existence of the different eigenvalues.

Lemma 4.4.14.

- *If $\sigma_{\epsilon_j}>\alpha$, then $\tilde{T}_n[0]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
- *Operator $\tilde{T}_n[0]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
- *If $\varepsilon_{\kappa_i}>\beta$, then $\tilde{T}_n[0]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}$.*
- *Operator $\tilde{T}_n[0]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}$.*

Proof. Let us study the different possibilities:

- If $\sigma_j<\mu$, then $T_{\sigma_j}u(a)=u^{(\sigma_j)}(a)=0$.
- If $\sigma_j=\sigma_k$, then $T_{\sigma_k}u(a)=0$, by the definition of μ.
- If $\varepsilon_i<\mu$, then $T_{\varepsilon_i}u(b)=u^{(\varepsilon_i)}(b)=0$.

- If $\varepsilon_i = \varepsilon_{n-k}$, then $T_{\varepsilon_{n-k}} u(b) = 0$, by the definition of μ.
- If $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$ or $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$ and $\sigma_{\epsilon_j} > \alpha$, then $T_\alpha u(a) = \dfrac{1}{v_1(a)\, \dots\, v_\alpha(a)} u^{(\alpha)}(a) = 0$.
- Analogously, if $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$ or $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\varepsilon_{\kappa_i} > \beta$, then $T_\beta u(b) = \dfrac{1}{v_1(b)\, \dots\, v_\alpha(b)} u^{(\beta)}(b) = 0$.

$\square$

Remark 4.4.15. *Realise that if we ensure that $T_{\sigma_j} u(a) = u^{(\sigma_j)}(a)$ or $T_{\varepsilon_i} u(b) = u^{(\varepsilon_i)}(b)$, we do not need the assumption that $\sigma_j < \mu$ or $\varepsilon_i < \mu$ given by the choice of $\{\epsilon_1, \dots, \epsilon_\ell\}$ and $\{\kappa_1, \dots, \kappa_h\}$ on Definition 4.4.9.*

This is true, in particular, if we can choose on decomposition (3.3.1)-(3.3.2)*,*

$$v_1 \equiv \cdots \equiv v_{\sigma_j} \equiv 1 \text{ or } v_1 \equiv \cdots \equiv v_{\varepsilon_i} \equiv 1.$$

We note that such a choice is applicable for the operator $T_n^0[M] = \dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n} + M$, where we can choose $v_1 \equiv \cdots \equiv v_n \equiv 1$.

The following results are also true under the hypotheses of this remark.

Lemma 4.4.16.

- *Let $n-k$ be even, then the following assertions are satisfied.*
  - *If $\sigma_{\epsilon_j} > \alpha$, then there exists $\lambda^1_{\sigma_{\epsilon_j}} > 0$, the least positive eigenvalue of operator $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$.*
  - *There exists $\lambda^2_{\sigma_{\epsilon_j}} < 0$,the biggest negative eigenvalue of operator $\tilde{T}_n[0]$ in the space $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$.*
  - *If $\varepsilon_{\kappa_i} > \beta$, then there exists $\lambda^1_{\varepsilon_{\kappa_i}} > 0$, the least positive eigenvalue of operator $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.*
  - *There exists $\lambda^2_{\varepsilon_{\kappa_i}} < 0$ the biggest negative eigenvalue of operator $\tilde{T}_n[0]$ in the space $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.*
- *Let $n-k$ be odd, then the following assertions are satisfied.*
  - *If $\sigma_{\epsilon_j} > \alpha$, then there exists $\lambda^1_{\sigma_{\epsilon_j}} < 0$, the biggest negative eigenvalue of operator $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$.*
  - *There exists $\lambda^2_{\sigma_{\epsilon_j}} > 0$, the least positive eigenvalue of operator $\tilde{T}_n[0]$ in the space $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$.*

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

– *If $\varepsilon_{\kappa_i} > \beta$, then there exists $\lambda^1_{\varepsilon_{\kappa_i}} < 0$, the biggest negative eigenvalue of operator $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}$.*

– *There exists $\lambda^2_{\varepsilon_{\kappa_i}} > 0$, the least positive eigenvalue of operator $\tilde{T}[0]$ in the space $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}\}}$.*

Proof. Since $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy property (N_a), this property is satisfied in all the spaces involved in the result.

Moreover, from Lemma 4.4.14, property (T_d) is also satisfied. Then, by applying Theorems 3.3.10, 1.2.10 and 1.2.11, the result is proved. □

Now, let us see two results which allow us to ensure that functions $x_M^{\sigma_{\epsilon_j}}$ and $z_M^{\varepsilon_{\kappa_i}}$ are of constant sign for suitable values of M.

Proposition 4.4.17. *Let $u \in C^n(I)$ be a solution of $\tilde{T}[M]\,u(t) = 0$ for $t \in (a,b)$, which satisfies the boundary conditions:*

$$\begin{cases} u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{\epsilon_j}-1)}(a) = u^{(\sigma_{\epsilon_j}+1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0. \end{cases} \tag{4.4.15}$$

Then, the function u does not have any zero on (a,b) provided that one of the following assertions is fulfilled.

- *If $n-k$ is even, $k>1$, $\sigma_{\epsilon_j} > \alpha$ and $M \in \left[-\lambda^1_{\sigma_{\epsilon_j}}, -\lambda^2_{\sigma_{\epsilon_j}}\right]$, where:*
  * *$\lambda^1_{\sigma_{\epsilon_j}} > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$,*
  * *$\lambda^2_{\sigma_{\epsilon_j}} < 0$ is the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
- *If $n-k$ is even, $k>1$, $\sigma_{\epsilon_j} < \alpha$ and $M \in \left[-\lambda_1, -\lambda^2_{\sigma_{\epsilon_j}}\right]$, where:*
  * *$\lambda_1 > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$,*
  * *$\lambda^2_{\sigma_{\epsilon_j}} < 0$ is the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
- *If $k=1$, n odd, $\sigma_{\epsilon_j} > \alpha$ and $M \in \left[-\lambda^1_{\sigma_{\epsilon_j}}, +\infty\right)$, where:*
  * *$\lambda^1_{\sigma_{\epsilon_j}} > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
- *If $k=1$, n odd, $\sigma_{\epsilon_j} < \alpha$ and $M \in [-\lambda_1, +\infty)$, where:*
  * *$\lambda_1 > 0$ is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
- *If $n-k$ is odd, $k>1$, $\sigma_{\epsilon_j} > \alpha$ and $M \in \left[-\lambda^2_{\sigma_{\epsilon_j}}, -\lambda^1_{\sigma_{\epsilon_j}}\right]$, where:*

  * $\lambda^2_{\sigma_{\epsilon_j}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$,
  * $\lambda^1_{\sigma_{\epsilon_j}} < 0$*, the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$.

- *If* $n-k$ *is odd,* $k>1$*,* $\sigma_{\epsilon_j} < \alpha$ *and* $M \in \left[-\lambda^2_{\sigma_{\epsilon_j}}, -\lambda_1\right]$*, where:*

  * $\lambda^2_{\sigma_{\epsilon_j}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}$,
  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

- *If* $k=1$*,* n *odd,* $\sigma_{\epsilon_j} > \alpha$ *and* $M \in \left(-\infty, -\lambda^1_{\sigma_{\epsilon_j}}\right]$*, where:*

  * $\lambda^1_{\sigma_{\epsilon_j}} < 0$*, the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$.

- *If* $k=1$*,* n *odd,* $\sigma_{\epsilon_j} < \alpha$ *and* $M \in (-\infty, -\lambda_1]$*, where:*

  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Proof. Firstly, let us see what happens for $M=0$. As we have seen in the previous results, without taking into account the boundary conditions, if u is a solution of $\tilde{T}_n[0]\,u(t)=0$ on (a,b), then u has at most $n-1$ zeros.

However, from the boundary conditions (4.4.15) and property (T_d), we have that either $\tilde{T}_\ell u(a)=0$ or $\tilde{T}_\ell u(b)=0$ at least $n-1$ times from $\ell=0$ to $n-1$. Thus, we lose the $n-1$ possible zeros and u does not have any zero on (a,b).

Now, let us consider $u_M \in C^n(I)$ a solution of $\tilde{T}_n[M]\,u_M(t)=0$ on (a,b).

Assume that $u_0>0$ on (a,b) (if $u_0<0$ on (a,b) the arguments are applicable by multiplying by -1) and we move continuously on M to obtain u_M.

We will see that while $u_M \geqslant 0$, it cannot have any double zero, which implies that it is positive on (a,b).

We know that $\tilde{T}[0]\,u_M(t) = -M\,u_M(t)$, on (a,b), thence $\tilde{T}_{n-1}u_M$ is a monotone function on I, with at most one zero. Then, arguing as before, we conclude, without taking into account the boundary conditions, that u_M can have at most n zeros. But, if we consider the boundary conditions (4.4.15), we lose $n-1$ possible oscillation and u_M is only allowed to have a simple zero on (a,b), which is not possible if it is of constant sign. Hence, we can affirm that $u_M>0$ on (a,b) up to one of the following assertions is satisfied:

- $\sigma_{\epsilon_j} > \alpha$ and $u_M^{(\alpha)}(a)=0$.
- $\sigma_{\epsilon_j} < \alpha$ and $u_M^{(\sigma_{\epsilon_j})}(a)=0$.
- $u_M^{(\beta)}(b)=0$.

Now, let us study separately the cases where $M>0$ or $M<0$ to see with which of the previous assertions the sign change begins in each case.

USC UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

If $M \geqslant 0$, then $\tilde{T}[0]\, u_M(t) = -M\, u_M(t) \leqslant 0$ for $t \in (a,b)$. Thus, $\tilde{T}_n u_M(a) \leqslant 0$ and $\tilde{T}_n u_M(b) \leqslant 0$.

With maximal oscillation $\tilde{T}_{n-\ell}\, u_M(a)$ changes its sign each time that it is not null and $\tilde{T}_{n-\ell}\, u_M(b)$ changes its sign as many times as it vanishes, see Figure 3.3.2.

- If $\sigma_{\epsilon_j} > \alpha$, from $\ell = 0$ to $n-\alpha$, $\tilde{T}_{n-\ell} u_M(a)$ vanishes $k-1-\alpha$ times.

  If $\tilde{T}_{n-\ell}\, u_M(a) = 0$ for $\ell < n-\alpha$ and $n-\ell \notin \{\sigma_1, \dots, \sigma_{\epsilon_j -1}, \sigma_{\epsilon_j+1}, \dots, \sigma_k | \alpha\}$, then $\tilde{T}_\alpha u_M(a) \neq 0$ and u_M remains positive on (a,b). So, we can assume that this situation cannot be fulfilled.

  Hence, with maximal oscillation, we have:

$$\begin{cases} \tilde{T}_\alpha\, u_M(a) \geqslant 0, & \text{if } n-\alpha-(k-1-\alpha) = n-k+1 \text{ is odd,} \\[2ex] \tilde{T}_\alpha\, u_M(a) \leqslant 0, & \text{if } n-k+1 \text{ is even.} \end{cases}$$

  Since $\sigma_{\epsilon_j} > \alpha$, from (3.3.3), we have that:

$$\tilde{T}_\alpha u_M(a) = \frac{u^{(\alpha)}(a)}{v_1(a)\dots v_\alpha(a)},$$

  so, with maximal oscillation:

$$\begin{cases} u_M^{(\alpha)}(a) \geqslant 0, & \text{if } n-k \text{ is even,} \\[2ex] u_M^{(\alpha)}(a) \leqslant 0, & \text{if } n-k \text{ is odd.} \end{cases}$$

- If $\sigma_{\epsilon_j} < \alpha$, from $\ell = 0$ to $n - \sigma_{\epsilon_j}$, $\tilde{T}_{n-\ell} u_M(a)$ vanishes $k-1-\sigma_{\epsilon_j}$ times. Again, let us assume that $\tilde{T}_{n-\ell} u_M(a) \neq 0$ for $\ell < n-\sigma_{\epsilon_j}$ if $n-\ell \notin \{\sigma_1, \dots, \sigma_k\}$. Then, with maximal oscillation, we have:

$$\begin{cases} \tilde{T}_{\sigma_{\epsilon_j}}\, u_M(a) \geqslant 0 & \text{if } n-\sigma_{\epsilon_j}-(k-1-\sigma_{\epsilon_j}) = n-k+1 \text{ is odd,} \\[2ex] \tilde{T}_{\sigma_{\epsilon_j}}\, u_M(a) \leqslant 0 & \text{if } n-k+1 \text{ is even.} \end{cases}$$

  Since $\sigma_{\epsilon_j} < \alpha$, from (3.3.3), we have that:

$$\tilde{T}_{\sigma_{\epsilon_j}} u_M(a) = \frac{u^{(\sigma_{\epsilon_j})}(a)}{v_1(a)\dots v_{\sigma_{\epsilon_j}}(a)}.$$

  In particular, if $\sigma_{\epsilon_j} < \mu$, then $v_1(t)\dots v_{\sigma_{\epsilon_j}}(t) = 1$ for all $t \in I$.

  Thus, with maximal oscillation:

$$\begin{cases} u_M^{(\sigma_{\epsilon_j})}(a) \geqslant 0, & \text{if } n-k \text{ is even,} \\[2ex] u_M^{(\sigma_{\epsilon_j})}(a) \leqslant 0, & \text{if } n-k \text{ is odd.} \end{cases}$$

- On another hand, from $\ell = 0$ to $n-\beta$, $\tilde{T}_{n-\ell}u_M(b)$ vanishes $n-k-\beta$ times. We can also assume that $\tilde{T}_{n-\ell}u_M(b) \neq 0$ if $n-\ell \notin \{\varepsilon_1, \dots, \varepsilon_{n-k}|\beta\}$. Then, with maximal oscillation:
$$\begin{cases} \tilde{T}_\beta\, u_M(b) \geqslant 0, & \text{if } n-k-\beta \text{ is odd,} \\ \\ \tilde{T}_\beta\, u_M(b) \leqslant 0, & \text{if } n-k-\beta \text{ is even.} \end{cases}$$
From (3.3.3), we have that:
$$\tilde{T}_\beta u_M(b) = \frac{u^{(\beta)}(b)}{v_1(b)\dots v_\beta(b)}.$$
Thus:

- If $n-k$ is even, to set maximal oscillation, we need:
$$\begin{cases} u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is even,} \\ \\ u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

- If $n-k$ is odd, to ensure maximal oscillation is necessary:
$$\begin{cases} u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is even,} \\ \\ u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

Since, we are considering $u_M \geqslant 0$, we know that:
$$\begin{cases} u_M^{(\alpha)}(a) \geqslant 0, & \text{if } \sigma_{\epsilon_j} > \alpha, \\ \\ u_M^{(\sigma_{\epsilon_j})}(a) \geqslant 0, & \text{if } \sigma_{\epsilon_j} < \alpha, \end{cases} \tag{4.4.16}$$
and,
$$\begin{cases} u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is even,} \\ \\ u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is odd,} \end{cases} \tag{4.4.17}$$

Taking into account that if $k = 1$, then $u_M^{(\beta)}(b) \neq 0$ for all $M \in \mathbb{R}$, we obtain the following conclusions for $M \geqslant 0$:

- If $n-k$ is odd and $\sigma_{\epsilon_j} > \alpha$, then $u_M \geqslant 0$ whenever $u_N^{(\alpha)}(a) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$ is found.

- If $n-k$ is odd and $\sigma_{\epsilon_j} < \alpha$, then $u_M \geqslant 0$ whenever $u_N^{(\sigma_k)}(a) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ is found.

USC UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

- If $n-k$ is even and $k>1$, then $u_M \geqslant 0$ whenever $u_N^{(\beta)}(b) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$ is found.
- If $k=1$ and n is odd, then $u_M \geqslant 0$ for all $M \geqslant 0$.

Now, let us see what happens for $M \leqslant 0$. In this case, we have that:

$$\tilde{T}_n[0]u_M(t) = -M\,u_M(t) \geqslant 0,\ t \in (a,b).$$

Then, $\tilde{T}_n u_M(a) \geqslant 0$ and $\tilde{T}_n u_M(b) \geqslant 0$. Hence, we conclude that with maximal oscillation, the inequalities are reversed from te case $M \geqslant 0$. So, we obtain that:

- If $\sigma_{\epsilon_j} > \alpha$, with maximal oscillation:

$$\begin{cases} u_M^{(\alpha)}(a) \leqslant 0, & \text{if } n-k \text{ is even,} \\ u_M^{(\alpha)}(a) \geqslant 0, & \text{if } n-k \text{ is odd.} \end{cases}$$

- If $\sigma_{\epsilon_j} < \alpha$, with maximal oscillation:

$$\begin{cases} u_M^{(\sigma_{\epsilon_j})}(a) \leqslant 0, & \text{if } n-k \text{ is even,} \\ u_M^{(\sigma_{\epsilon_j})}(a) \geqslant 0, & \text{if } n-k \text{ is odd,} \end{cases}$$

and,

- If $n-k$ is even, with maximal oscillation:

$$\begin{cases} u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is even,} \\ u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

- If $n-k$ is odd, with maximal oscillation:

$$\begin{cases} u_M^{(\beta)}(b) \leqslant 0, & \text{if } \beta \text{ is even,} \\ u_M^{(\beta)}(b) \geqslant 0, & \text{if } \beta \text{ is odd.} \end{cases}$$

Then, taking into account that $u_M \geqslant 0$, (4.4.16) and (4.4.17) are also satisfied.

Hence, using that if $k=1$, then $u_M^{(\beta)}(b) \neq 0$ for all $M \in \mathbb{R}$, we obtain the following conclusions for $M \leqslant 0$.

- If $n-k$ is even and $\sigma_{\epsilon_j} > \alpha$, then $u_M \geqslant 0$ whenever $u_N^{(\alpha)}(a) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ is found.

- If $n-k$ is even and $\sigma_{\epsilon_j} < \alpha$, then $u_M \geqslant 0$ whenever $u_N^{(\sigma_{\epsilon_j})}(a) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ is found.

- If $n-k$ is odd and $k>1$, then $u_M \geqslant 0$ whenever $u_N^{(\beta)}(b) \neq 0$ for all N between 0 and M; i.e., up to an eigenvalue of $\tilde{T}_n[0]$ on $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$ is found.

- If $k=1$ and n is even, then $u_M \geqslant 0$ for all $M \leqslant 0$.

The proof is complete. □

Proposition 4.4.18. *Let $u \in C^n(I)$ be a solution of $\tilde{T}[M]\,u(t) = 0$ for $t \in (a,b)$, which satisfies the boundary conditions:*

$$\begin{cases} u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{\kappa_i}-1)}(b) = u^{(\varepsilon_{\kappa_i}+1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0. \end{cases} \tag{4.4.18}$$

Then, u does not have any zero on (a,b) provided that one of the following assertions is fulfilled.

- *If $n-k$ is even, $\varepsilon_{\kappa_i} > \beta$ and $M \in \left[-\lambda^1_{\varepsilon_{\kappa_i}}, -\lambda^2_{\varepsilon_{\kappa_i}}\right]$, where:*
  * $\lambda^1_{\varepsilon_{\kappa_i}} > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}|\beta\}}$,*
  * $\lambda^2_{\varepsilon_{\kappa_i}} < 0$*, the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}$.*

- *If $n-k$ is even, $\varepsilon_{\kappa_i} < \alpha$ and $M \in \left[-\lambda_1, -\lambda^2_{\varepsilon_{\kappa_i}}\right]$, where:*
  * $\lambda_1 > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$,*
  * $\lambda^2_{\varepsilon_{\kappa_i}} < 0$*, the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}$.*

- *If $n-k$ is odd, $k<n-1$, $\varepsilon_{\kappa_i} > \beta$ and $M \in \left[-\lambda^2_{\varepsilon_{\kappa_i}}, -\lambda^1_{\varepsilon_{\kappa_i}}\right]$, where:*
  * $\lambda^2_{\varepsilon_{\kappa_i}} > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}$,*
  * $\lambda^1_{\varepsilon_{\kappa_i}} < 0$*, the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}|\beta\}}$.*

- *If $n-k$ is odd, $k<n-1$, $\varepsilon_{\kappa_i} < \alpha$ and $M \in \left[-\lambda^2_{\varepsilon_{\kappa_i}}, -\lambda_1\right]$, where:*
  * $\lambda^2_{\varepsilon_{\kappa_i}} > 0$ *is the least positive eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i}-1,\varepsilon_{\kappa_i}+1,\dots,\varepsilon_{n-k}\}}$,*
  * $\lambda_1 < 0$*, the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

- *If $k=n-1$, $\varepsilon_{\kappa_i} > \alpha$ and $M \in \left(-\infty, -\lambda^1_{\sigma_{\epsilon_j}}\right]$, where:*

* $\lambda^1_{\varepsilon_{\kappa_i}} < 0$, *the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

- *If* $k = n-1$, $\varepsilon_{\kappa_i} < \alpha$ *and* $M \in (-\infty, -\lambda_1]$, *where:*

  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Proof. The proof is analogous to the proof of Proposition 4.4.17. □

Now, we are in a position to prove a result which gives a relationship on the eigenvalues of the different spaces $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k|\alpha\}}$ with the closest to zero eigenvalue of $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. The result is the following.

Proposition 4.4.19. *Let* $j_1 \in \{\epsilon_1,\dots,\epsilon_\ell\}$ *be such that* $\alpha < \sigma_{j_1}$, *then the following assertions are true.*

- *If* $n-k$ *is even, then* $0 < \lambda_1 < \lambda^1_{\sigma_{j_1}}$, *where:*

  * $\lambda^1_{\sigma_{j_1}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k|\alpha\}}$.
  * $\lambda_1 > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

- *If* $n-k$ *is odd, then* $\lambda^1_{\sigma_{j_1}} < \lambda_1 < 0$, *where:*

  * $\lambda^1_{\sigma_{j_1}} < 0$, *the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k|\alpha\}}$.
  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

Proof. To prove this result, let us denote $v_M \in C^n(I)$ as a solution of $\tilde{T}[M]\, v_M(t) = 0$ on (a,b), coupled with the following boundary conditions:

$$\begin{cases} v_M^{(\sigma_j)}(a) = 0, \quad j = 0,\dots,k, \quad j \neq j_1, \\[2ex] v_M^{(\varepsilon_i)}(b) = 0, \qquad i = 0,\dots,n-k. \end{cases} \tag{4.4.19}$$

Let us study v_0. With the arguments used before, we know that, without taking into account the boundary conditions, v_0 has at most $n-1$ zeros. However, from the boundary conditions (4.4.19), we conclude that $n-1$ possible zeros are lost. Hence, since v_0 is a non-trivial function, the boundary conditions for the maximal oscillation are satisfied.

Let us choose $v_0 \geqslant 0$ (if $v_0 \leqslant 0$, the arguments are applicable by multiplying by -1), then $v_0^{(\alpha)}(a) \geqslant 0$. From (3.3.3) $T_\alpha v_0(a)$ also satisfies this inequality.

Let us study the sign of $v_M^{(\sigma_{j_1})}(a)$. Realize that, to achieve the maximal oscillation, $T_\ell v_M(a)$ must change its sign each time that it is non-null, see Figure 3.3.2.

From $\ell = \alpha$ to σ_{j_1}, $T_\ell v_0(a)$ vanishes $j_1 - 1 - \alpha$ times, then, with maximal oscillation:

$$\begin{cases} T_{\sigma_{j_1}} v_0(a) > 0, \quad \text{if } \sigma_{j_1} - \alpha - (j_1 - 1 - \alpha) = \sigma_{j_1} - j_1 + 1 \text{ is even}, \\[2ex] T_{\sigma_{j_1}} v_0(a) < 0, \qquad\qquad\qquad \text{if } \sigma_{j_1} - j_1 + 1 \text{ is odd}. \end{cases}$$

From the choice of $j_1 \in \{\epsilon_1, \dots, \epsilon_\ell\}$, see Definition 4.4.9 and Remark 4.4.15, we can affirm that:

$$\begin{cases} v_0^{(\sigma_{j_1})}(a) < 0, & \text{if } \sigma_{j_1} - j_1 \text{ is even}, \\ \\ v_0^{(\sigma_{j_1})}(a) > 0, & \text{if } \sigma_{j_1} - j_1 \text{ is odd}. \end{cases} \tag{4.4.20}$$

Now, let us move with continuity on M up to $-\lambda_{\sigma_{j_1}}$ and study the sign of $v_{-\lambda^1_{\sigma_{j_1}}}^{(\sigma_{j_1})}(a)$.

From Proposition 4.4.17, we know that $v_{-\lambda_{\sigma_{j_1}}} > 0$ on (a, b). Moreover, we have that $v_{-\lambda_{\sigma_{j_1}}}^{(\alpha)}(a) = 0$. Thus, with the calculations done before, we conclude that the maximal oscillation is satisfied too.

So, we can study in this case the sign of $v_{-\lambda^1_{\sigma_{j_1}}}^{(\sigma_{j_1})}(a)$.

Let us consider $\alpha_1 \in \{0, \dots, n-1\}$, previously introduced in Notation 3.6.12. Since $v_{-\lambda_{\sigma_{j_1}}} \geqslant 0$ on I, we can affirm that $v_{-\lambda_{\sigma_{j_1}}}^{(\alpha_1)}(a) > 0$.

From $\ell = \alpha_1$ to σ_{j_1}, there are $j_1 - \alpha_1$ zeros for $T_\ell v_{-\lambda^1_{\sigma_{j_1}}}(a)$, then, with maximal oscillation:

$$\begin{cases} T_{\sigma_{j_1}} v_{-\lambda^1_{\sigma_{j_1}}}(a) > 0, & \text{if } \sigma_{j_1} - \alpha_1 - (j_1 - \alpha_1) = \sigma_{j_1} - j_1 \text{ is even}, \\ \\ T_{\sigma_{j_1}} v_{-\lambda^1_{\sigma_{j_1}}}(a) < 0, & \text{if } \sigma_{j_1} - j_1 \text{ is odd}. \end{cases}$$

From the choice of $j_1 \in \{\epsilon_1, \dots, \epsilon_\ell\}$, we can affirm that:

$$\begin{cases} v_{-\lambda^1_{\sigma_{j_1}}}^{(\sigma_{j_1})}(a) > 0, & \text{if } \sigma_{j_1} - j_1 \text{ is even}, \\ \\ v_{-\lambda^1_{\sigma_{j_1}}}^{(\sigma_{j_1})}(a) < 0, & \text{if } \sigma_{j_1} - j_1 \text{ is odd}. \end{cases} \tag{4.4.21}$$

Hence, in this case, since we have been moving continuously on M, we can affirm that there exists $-\tilde{\lambda}_1$ between 0 and $-\lambda^1_{\sigma_{j_1}}$ such that $v_{-\tilde{\lambda}_1}^{(\sigma_{j_1})}(a) = 0$, i.e. we have proved the existence of an eigenvalue of $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ between 0 and $-\lambda^1_{\sigma_{j_1}}$, and the result is proved. □

In an analogous way, we can prove the following result for the eigenvalues of operator $\tilde{T}[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_i-1},\varepsilon_{\kappa_i+1},\dots,\varepsilon_{n-k}|\beta\}}$, comparing them with the closest to zero eigenvalue in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proposition 4.4.20. *Let $i_1 \in \{\kappa_1, \dots, \kappa_h\}$ be such that $\varepsilon_{i_1} > \beta$, then the following assertions are true.*

- *If $n-k$ is even, then $0 < \lambda_1 < \lambda^1_{\varepsilon_{i_1}}$, where:*
    - * $\lambda^1_{\varepsilon_{i_1}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{i_1-1},\varepsilon_{i_1+1},\dots,\varepsilon_{n-k}|\beta\}}$.

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* $\lambda_1 > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

- *If* $n-k$ *is odd, then* $0 > \lambda_1 > \lambda^1_{\varepsilon_{i_1}}$*, where:*

  * $\lambda^1_{\varepsilon_{i_1}} < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{i_1-1},\varepsilon_{i_1+1},\dots,\varepsilon_{n-k}|\beta\}}$.
  * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. The proof is analogous to the one of Proposition 4.4.19. □

Now, let us establish a comparison between the eigenvalues in $X_{\{\sigma_1,\dots,\sigma_{\epsilon_j-1},\sigma_{\epsilon_j+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.

Proposition 4.4.21. *Let* $j_1,\ j_2 \in \{\epsilon_1,\dots,\epsilon_\ell\}$ *be such that* $j_1 < j_2$*. Then the following assertions are fulfilled:*

- *If* $n-k$ *is even and* $k>1$*, then* $0 > \lambda^2_{\sigma_{j_1}} > \lambda^2_{\sigma_{j_2}}$*, where:*

  * $\lambda^2_{\sigma_{j_1}} < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  * $\lambda^2_{\sigma_{j_2}} < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_{j_2-1},\sigma_{j_2+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.

- *If* $n-k$ *is odd and* $k>1$*, then* $0 < \lambda^2_{\sigma_{j_1}} < \lambda^2_{\sigma_{j_2}}$*, where:*

  * $\lambda^2_{\sigma_{j_1}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_{j_1-1},\sigma_{j_1+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.
  * $\lambda^2_{\sigma_{j_2}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_{j_2-1},\sigma_{j_2+1},\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.

Proof. First, let us denote $v1_M \in C^n(I)$ as a solution of $\tilde{T}[M]\, v1_M(t) = 0$ on (a,b), coupled with the following boundary conditions:

$$\begin{cases} v1_M^{(\sigma_j)}(a) = 0, & j = 0,\dots,k, \quad j \neq j_1,\ j_2, \\ v1_M^{(\varepsilon_i)}(b) = 0, & i = 0,\dots,n-k, \\ v1_M^{(\beta)}(b) = 0. \end{cases} \tag{4.4.22}$$

Again, from the boundary conditions (4.4.22), to ensure that is is a non-trivial solution, $v1_0$ satisfies the conditions of maximal oscillation at $t=a$ and $t=b$.

There are three possible cases with different behaviours.

Case 1. $\sigma_{j_1} > \alpha$

Case 2. $\sigma_{j_1} < \alpha < \sigma_{j_2}$

Case 3. $\sigma_{j_2} < \alpha$

Now, let us study these three cases separately.

Case 1. $\sigma_{j_1} > \alpha$

Let us choose $v_{10} \geqslant 0$ (if $v_{10} \leqslant 0$, then the arguments are applicable by multiplying by -1), then $v_{10}^{(\alpha)}(a) \geqslant 0$. From (3.3.3) we have $T_\alpha v_{10}(a) \geqslant 0$.

To study the sign of $v_0^{(\sigma_{j_2})}(a)$, realise that, to achieve the maximal oscillation, $T_\ell v_M(a)$ changes its sign each time that it is non-null, see Figure 3.3.2.

From $\ell = \alpha$ to σ_{j_2}, there are $j_2 - 2 - \alpha$ zeros for $T_\ell v_{10}(a)$, then, with maximal oscillation:

$$\begin{cases} T_{\sigma_{j_2}} v_{10}(a) > 0, & \text{if } \sigma_{j_2} - \alpha - (j_2 - 2 - \alpha) = \sigma_{j_2} - j_2 + 2 \text{ is even,} \\ T_{\sigma_{j_2}} v_{10}(a) < 0, & \text{if } \sigma_{j_2} - j_2 + 2 \text{ is odd.} \end{cases}$$

From the choice of $j_2 \in \{\epsilon_1, \ldots, \epsilon_\ell\}$, we can affirm that:

$$\begin{cases} v_{10}^{(\sigma_{j_2})}(a) > 0, & \text{if } \sigma_{j_2} - j_2 \text{ is even,} \\ v_{10}^{(\sigma_{j_2})}(a) < 0, & \text{if } \sigma_{j_2} - j_2 \text{ is odd.} \end{cases} \tag{4.4.23}$$

Now, let us move with continuity on M up to $-\lambda_{\sigma_{j_2}}^2$ and analyse the sign of $v_{1\,-\lambda_{\sigma_{j_2}}^2}^{(\sigma_{j_2})}(a)$. Let us denote $\bar\lambda_2 = -\lambda_{\sigma_{j_2}}^2$, from Proposition 4.4.17, we know that $v_{1\bar\lambda_2} > 0$ on (a, b). Moreover, $v_{1\bar\lambda_2}^{(\sigma_{j_1})}(a) = 0$. Thus, since another possible zero on the boundary will imply that $v_{1\bar\lambda_2} \equiv 0$, we conclude that the maximal oscillation is satisfied too.

So, we can study, in this case, the sign of $v_{1\bar\lambda_2}^{(\sigma_{j_1})}(a)$.

Since $v_{1\bar\lambda_2} \geqslant 0$ on I, we can affirm that, as for $M = 0$, $v_{1\bar\lambda_2}^{(\alpha)}(a) > 0$.

From $\ell = \alpha$ to σ_{j_2}, there are $j_2 - 1 - \alpha$ zeros of $T_\ell v_{1\bar\lambda_2}(a)$. Then, with maximal oscillation:

$$\begin{cases} T_{\sigma_{j_2}} v_{1\bar\lambda_2}(a) > 0, & \text{if } \sigma_{j_2} - \alpha - (j_2 - 1 - \alpha) = \sigma_{j_2} - j_2 + 1 \text{ is even,} \\ T_{\sigma_{j_2}} v_{1\bar\lambda_2}(a) < 0, & \text{if } \sigma_{j_2} - j_2 + 1 \text{ is odd.} \end{cases}$$

From the choice of $j_2 \in \{\epsilon_1, \ldots, \epsilon_\ell\}$, we can affirm that:

$$\begin{cases} v_{1\bar\lambda_2}^{(\sigma_{j_2})}(a) < 0, & \text{if } \sigma_{j_2} - j_2 \text{ is even,} \\ v_{1\bar\lambda_2}^{(\sigma_{j_2})}(a) > 0, & \text{if } \sigma_{j_2} - j_2 \text{ is odd.} \end{cases} \tag{4.4.24}$$

Case 2. $\sigma_{j_1} < \alpha < \sigma_{j_2}$

In this case, $\sigma_{j_1} = j_1 - 1$.

For $M = 0$, since $v_{10} \geqslant 0$, we have that $v_{10}^{(\sigma_{j_1})}(a) \geqslant 0$. From (3.3.3) we have that $T_{\sigma_{j_1}} v_{10}(a) \geqslant 0$.

Let us study the sign of $v_{10}^{(\sigma_{j_2})}(a)$ in this case.

From $\ell = \sigma_{j_1}$ to σ_{j_2}, there are $j_2 - 2 - (j_1 - 1) = j_2 - j_1 - 1$ zeros of $T_\ell v_{10}(a)$. Then, with maximal oscillation:

$$\begin{cases} T_{\sigma_{j_2}} v_{10}(a) > 0, & \text{if } \sigma_{j_2} - j_1 - 1 - (j_2 - j_1 - 1) = \sigma_{j_2} - j_2 \text{ is even}, \\ T_{\sigma_{j_2}} v_{10}(a) < 0, & \text{if } \sigma_{j_2} - j_2 \text{ is odd}. \end{cases}$$

From the choice of $j_2 \in \{\epsilon_1, \ldots, \epsilon_\ell\}$, we can affirm that (4.4.23) holds.

Now, we study the sign of $v_{1_{\bar{\lambda}_2}}^{(\sigma_{j_2})}(a)$ if the conditions to allow the maximal oscillation hold.

Since $v_{1_{-\lambda^2_{\sigma_{j_2}}}} \geqslant 0$ on I, we can affirm that $v_{1_{-\lambda_{\sigma_{j_1}}}}^{(\alpha)}(a) > 0$.

From $\ell = \alpha$ to σ_{j_2}, there are $j_2 - 1 - \alpha$ zeros for $T_\ell v_{1_{-\lambda^2_{\sigma_{j_2}}}}(a)$, then, with maximal oscillation and repeating the previous arguments, we obtain that (4.4.24) is satisfied.

Case 3. $\sigma_{j_2} < \alpha$

In this situation, we have $\sigma_{j_1} = j_1 - 1$ and $\sigma_{j_2} = j_2 - 1$.

For $M = 0$, since $v_{10} \geqslant 0$, we have that $v_{1_0}^{(\sigma_{j_1})}(a) \geqslant 0$ and from (3.3.3) $T_\alpha v_{10}(a) \geqslant 0$.

Let us study the sign of $v_{1_0}^{(\sigma_{j_2})}(a)$ in this situation. Since $\alpha > \sigma_{j_2}$, for $\ell = \sigma_{j_1}, \ldots, \sigma_{j_2}$, we have that $\tilde{T}_\ell v_{10}(a) = 0$. So, to allow the maximal oscillation, it must be satisfied that $T_{\sigma_{j_2}} v_{10}(a) < 0$. And this inequality also holds for $v_{1_0}^{(\sigma_{j_2})}(a)$.

In this case, for $M = \bar{\lambda}_2$, since $v_{1_{\bar{\lambda}_2}} > 0$ on (a, b) and $v_{1_{\bar{\lambda}_2}}^{(\sigma_{j_1})}(a) = 0$, we, necessarily, have that $v_{1_{\bar{\lambda}_2}}^{(\sigma_{j_2})}(a) > 0$.

Hence, in all the cases, since we have been moving continuously on M, we can affirm that there exists $-\tilde{\lambda}_1$ lying between 0 and $-\lambda^2_{\sigma_{j_2}}$, such that $v_{1_{-\tilde{\lambda}_1}}^{(\sigma_{j_2})}(a) = 0$. As a consequence, we have proved the existence on an eigenvalue of $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1, \ldots, \varepsilon_{n-k}\}}_{\{\sigma_1, \ldots, \sigma_{j_1 - 1}, \sigma_{j_1 + 1}, \ldots, \sigma_k\}}$ between 0 and $-\lambda^2_{\sigma_{j_2}}$, and the result is proved. □

Before introducing the final result which characterizes the strongly inverse positive (negative) character in the different spaces $X^{\{\varepsilon_1, \ldots, \varepsilon_{n-k}\}\{\varepsilon_{\kappa_1}, \ldots, \varepsilon_{\kappa_h}\}}_{\{\sigma_1, \ldots, \sigma_k\}\{\sigma_{\epsilon_1}, \ldots, \sigma_{\epsilon_\ell}\}}$, we show a result which gives an order on the eigenvalues associated to different spaces $X^{\{\varepsilon_1, \ldots, \varepsilon_{\kappa_i - 1}, \varepsilon_{\kappa_i + 1}, \ldots, \varepsilon_{n-k}\}}_{\{\sigma_1, \ldots, \sigma_k | \alpha\}}$.

Proposition 4.4.22. *Let $i_1,\ i_2 \in \{\kappa_1, \ldots,\ \kappa_h\}$ be such that $i_1 < i_2$, then the following assertions are true.*

- *If $n - k$ is even, then $0 > \lambda^2_{\varepsilon_{i_1}} > \lambda^2_{\varepsilon_{i_2}}$, where:*
  - * *$\lambda^2_{\varepsilon_{i_1}} < 0$ is the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1, \ldots, \varepsilon_{i_1 - 1}, \varepsilon_{i_1 + 1}, \ldots, \varepsilon_{n-k}\}}_{\{\sigma_1, \ldots, \sigma_k | \alpha\}}$.*
  - * *$\lambda^2_{\varepsilon_{i_2}} < 0$ is the biggest negative eigenvalue of $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1, \ldots, \varepsilon_{i_2 - 1}, \varepsilon_{i_2 + 1}, \ldots, \varepsilon_{n-k}\}}_{\{\sigma_1, \ldots, \sigma_k | \alpha\}}$.*
- *If $n - k$ is odd and $k < n - 1$, then $0 < \lambda^2_{\varepsilon_{i_1}} < \lambda^2_{\varepsilon_{i_2}}$, where:*

   * $\lambda^2_{\varepsilon_{i_1}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{i_1-1},\varepsilon_{i_1+1},\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.
   * $\lambda^2_{\varepsilon_{i_2}} > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{i_2-1},\varepsilon_{i_2+1},\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.

Proof. The proof follows the same structure and arguments as Proposition 4.4.21. □

Once we have obtained the previous results, which allow us to characterise the constant sign of the functions $x_M^{\sigma_{\epsilon_j}}$ and $z_M^{\varepsilon_{\kappa_i}}$ for $j = 1,\dots,\ell$ and $i = 0,\dots,h$, respectively, we can obtain a characterisation of the strongly inverse positive (negative) character of operator $\tilde{T}_n[M]$ in the spaces $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}_{\{\sigma_1,\dots,\sigma_k\}\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}$ as follows.

Theorem 4.4.23. *If $n-k$ is even, then the operator $\tilde{T}[M]$ is strongly inverse positive in the space $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}_{\{\sigma_1,\dots,\sigma_k\}\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}$ if, and only if, one of the following assertions is satisfied.*

- *If $k > 1$ and $M \in (-\lambda_1, -\lambda_2]$, where:*
   * $\lambda_1 > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
   * $\lambda_2 < 0$ *is the maximum between:*
      · $\lambda^2_{\sigma_{\epsilon_1}} < 0$*, the biggest negative eigenvalue of operator* $\tilde{T}_n[0]$ *in the space* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_k\}}$.
      · $\lambda^2_{\varepsilon_{\kappa_1}} < 0$*, the biggest negative eigenvalue of operator* $\tilde{T}_n[0]$ *in the space* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.
- *If $k = 1$ and $M \in (-\lambda_1, -\lambda_2]$, where:*
   * $\lambda_1 > 0$ *is the least positive eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\sigma_1\}}$.
   * $\lambda_2 = \lambda^2_{\varepsilon_{\kappa_1}} < 0$ *is the biggest negative eigenvalue of operator* $\tilde{T}_n[0]$ *in the space* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-1}\}}_{\{\sigma_1|\alpha\}}$.

If $n-k$ is odd, then $\tilde{T}[M]$ is strongly inverse negative in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}_{\{\sigma_1,\dots,\sigma_k\}\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}$ if, and only if, one of the following assertions is satisfied.

- *If $1 < k < n-1$ and $M \in [-\lambda_2, -\lambda_1)$, where:*
   * $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.
   * $\lambda_2 > 0$ *is the minimum between:*
      · $\lambda^2_{\sigma_{\epsilon_1}} > 0$*, the least positive eigenvalue of operator* $\tilde{T}_n[0]$ *in the space* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_k\}}$.
      · $\lambda^2_{\varepsilon_{\kappa_1}} > 0$*, the least positive eigenvalue of operator* $\tilde{T}_n[0]$ *in the space* $X^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}$.

- *If* $k = 1 < n-1$ *and* $M \in [-\lambda_2, -\lambda_1)$*, where:*

  $*$ $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.

  $*$ $\lambda_2 = \lambda^2_{\varepsilon_{\kappa_1}} > 0$ *is the least positive eigenvalue of operator* $\tilde{T}_n[0]$ *in the space* $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-1}\}}$.

- *If* $1 < k = n-1$ *and* $M \in [-\lambda_2, -\lambda_1)$*, where:*

  $*$ $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.

  $*$ $\lambda_2 = \lambda^2_{\sigma_{\epsilon_1}} > 0$ *is the least positive eigenvalue of operator* $\tilde{T}_n[0]$ *in the space* $X_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_{n-1}\}}^{\{\varepsilon_1|\beta\}}$.

- *If* $n = 2$ *and* $M \in (-\infty, -\lambda_1)$*, where:*

  $*$ $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $\tilde{T}_n[0]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.

Proof. From Lemma 4.4.13, we only have to study the sign of $g_M(t,s)$, $x_M^{\sigma_{\epsilon_j}}$ for $j = 0,\dots,\ell$ and $z_M^{\varepsilon_{\kappa_i}}$ for $i = 0,\dots,h$.

First, let us see that if M belongs to the given intervals, then the operator is strongly inverse positive or negative in each case. And, finally, we will see that this interval cannot be increased.

Taking into account Theorem 3.7.1, $(-1)^{n-k} g_M(t,s) > 0$ on the given intervals.

Now, let us study the sign of $x_0^{\sigma_{\epsilon_j}}$ and $z_0^{\sigma_{\kappa_i}}$.

We know that $x_M^{\sigma_{\epsilon_j}}$ satisfies the boundary conditions (4.4.19) introduced in the proof of Proposition 4.4.19. Then, for $M = 0$, the maximal oscillation is satisfied. So, we can study the sign of $x_0^{\sigma_{\epsilon_j}\,(\sigma_{\epsilon_j})}$ taking into account that $x_0^{\sigma_{\epsilon_j}\,(\sigma_{\epsilon_j})}(a) = 1$.

If $\sigma_{\epsilon_j} < \alpha$, then $x_0^{\sigma_{\epsilon_j}} > 0$.

If $\sigma_{\epsilon_j} > \alpha$, from $\ell = \alpha$ to σ_{ϵ_j}, there are $\epsilon_j - 1 - \alpha$ zeros for $T_\ell x_0^{\sigma_{\epsilon_j}}(a)$.

Realise that, from the choice of ϵ_j, we have that $T_{\sigma_{\epsilon_j}} x_0^{\sigma_{\epsilon_j}}(a) > 0$. So, to have maximal oscillation, we need:

$$\begin{cases} T_\alpha x_0^{\sigma_{\epsilon_j}}(a) > 0, & \text{if } \sigma_{\epsilon_j} - \alpha - (\epsilon_j - 1 - \alpha) = \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is even,} \\[2ex] T_\alpha x_0^{\sigma_{\epsilon_j}}(a) < 0, & \text{if } \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is odd.} \end{cases}$$

These inequalities are also satisfied by $x_0^{\sigma_{\epsilon_j}\,(\alpha)}(a)$, thus:

$$\begin{cases} x_0^{\sigma_{\epsilon_j}} > 0 \text{ on } I, & \text{if } \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is even,} \\[2ex] x_0^{\sigma_{\epsilon_j}} < 0 \text{ on } I, & \text{if } \sigma_{\epsilon_j} - \epsilon_j + 1 \text{ is odd.} \end{cases} \tag{4.4.25}$$

Note that if $\sigma_{\epsilon_j} < \alpha$, then $\sigma_{\epsilon_j} = \epsilon_j - 1$. Hence, $\sigma_{\epsilon_j} - \epsilon_j + 1 = 2$ is an even number. Thus, equation (4.4.25) is satisfied for all σ_{ϵ_j}, with $j = 1,\dots,\ell$.

Moreover, from Propositions 4.4.17, 4.4.19 and 4.4.21, inequalities (4.4.25) are satisfied on the whole intervals given in the result. Thus, for those M, we have:

$$\begin{cases} (-1)^{n-\sigma_{\epsilon_j}-(k-j)+1} x_M^{\sigma_{\epsilon_j}} > 0 \text{ on } I, & \text{if } n-k \text{ is even,} \\ (-1)^{n-\sigma_{\epsilon_j}-(k-j)+1} x_M^{\sigma_{\epsilon_j}} < 0 \text{ on } I, & \text{if } n-k \text{ is odd.} \end{cases}$$

In an analogous way, we can study $z_M^{\varepsilon_{\kappa_i}}$ to conclude that for all M on the intervals given on the result, it is satisfied:

$$\begin{cases} (-1)^{n-k-\kappa_i+1} z_0^{\varepsilon_{\kappa_i}} > 0, & \text{if } n-k \text{ is even,} \\ (-1)^{n-k-\kappa_i+1} z_0^{\varepsilon_{\kappa_i}} < 0, & \text{if } n-k \text{ is odd.} \end{cases}$$

So, we have proved that if M belongs to those intervals, operator $\tilde{T}_n[M]$ is strongly inverse positive (negative). Moreover, from Theorem 3.7.1 if either $n-k$ is even and $M<0$ or $n-k$ is odd and $M>0$ the intervals cannot be increased, since g_M is not of constant sign. So, we only need to prove that if $n-k$ is even and $M>0$ or $n-k$ is odd and $M<0$ the intervals cannot be increased too.

To this end, we study the functions $x_M^{\sigma_{\epsilon_1}}$ and $z_M^{\varepsilon_{\kappa_1}}$. In particular, we will verify that if either $k \neq 1$ or $k \neq n-1$, one of them must necessarily change its sign for $M > -\lambda_2$ if $n-k$ is even or for $M < -\lambda_2$ if $n-k$ is odd.

If $\sigma_{\epsilon_1} = \sigma_k$ and $\varepsilon_{\kappa_1} = \varepsilon_{n-k}$ the result follows from Theorem 4.4.4. Otherwise, either $\lambda_2 = \lambda_{\sigma_{\epsilon_1}}$ or $\lambda_2 = \lambda_{\varepsilon_{\kappa_1}}$.

First, let us assume that $n-k$ is even. Suppose that there exists $M^* > -\lambda_2$ such that $\tilde{T}_n[M]$ is inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$. We will arrive to a contradiction as follows.

If $\lambda_2 = \lambda_{\sigma_{\epsilon_1}}$, let us consider the function $x_M^1(t) = (-1)^{n-\sigma_{\epsilon_j}-(k-j)+1} x_M^{\sigma_{\epsilon_1}}(t)$.

Trivially, $x_M^1 \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ and $\tilde{T}[M^*]x_{M^*}^1(t) = 0$. Then, we have that $x_{M^*}^1 \geqslant 0$ on I.

Let us see that necessarily $x_0^1 \geqslant x_{-\lambda_2}^1 \geqslant x_{M^*}^1$ on I.

Indeed, let us construct the following sequence:

$$\alpha_0 = x_0^1, \quad \tilde{T}_n[M^*]\,\alpha_{n+1} = (M^*+\lambda_2)\,\alpha_n,\ n \geqslant 0,$$

where $\alpha_n^{(\sigma_j)}(a) = 0$, if $j \neq \epsilon_1$ for $j = 1,\dots,k$, $\alpha_n^{(\sigma_{\epsilon_1})}(a) = (-1)^{n-\sigma_{\epsilon_1}-(k-\epsilon_1)+1}$ and $\alpha_n^{(\varepsilon_i)}(b) = 0$ for $i = 1,\dots,n-k$. In particular, we have $\alpha_n \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ for $n = 0,1,\dots$

It is obvious that this sequence is bounded from below by zero. Let us see that it is non-increasing.

$$\tilde{T}[M^*]\,\alpha_1 = (M^*+\lambda_2)\,x_0^1 \geqslant 0.$$

Since $\alpha_1 \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$ and we are working under the assumption that $\tilde{T}[M^*]$ is inverse positive in such set, we have that $\alpha_1 \geqslant 0$.

Now, $\tilde{T}_n[M^*](\alpha_0 - \alpha_1) = -\lambda_2\, x_0^1 \geqslant 0$.

We have $\dfrac{\mathrm{d}^{\sigma_j}}{\mathrm{d}\, t^{\sigma_j}}(\alpha_0 - \alpha_1)_{|t=a} = 0$ for $j = 1,\dots,k$ and $\dfrac{\mathrm{d}^{\varepsilon_i}}{\mathrm{d}\, t^{\varepsilon_i}}(\alpha_0 - \alpha_1)_{|t=b} = 0$ for $i = 1,\dots,n-k$, then $\alpha_0 - \alpha_1 \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$. So, $\alpha_0 \geqslant \alpha_1$.

Proceeding analogously for $n \geqslant 1$, we obtain that $\{\alpha_n\}$ is a non-increasing and non-negative sequence.

Now, let us consider the following sequence:

$$\beta_0 = x^1_{M^*}, \quad \tilde{T}_n[M^*]\,\beta_{n+1} = (M^* + \lambda_2)\,\beta_n,\ n \geqslant 0,$$

where $\beta_n^{(\sigma_j)}(a) = 0$, if $j \neq \epsilon_1$ for $j = 1,\dots,k$, $\beta_n^{(\sigma_{\epsilon_1})}(a) = (-1)^{n-\sigma_{\epsilon_1}-(k-\epsilon_1)+1}$ and $\beta_n^{(\varepsilon_i)}(b) = 0$ for $i = 1,\dots,n-k$. As a consequence, $\beta_n \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$ for $n = 0,1,\dots$

Let us see that this sequence is non-decreasing.

By definition, $\tilde{T}_n[M^*](\beta_1 - \beta_0) = (M^* + \lambda_2)x^1_{M^*} \geqslant 0$. In this case, it is fulfilled $\dfrac{\mathrm{d}^{\sigma_j}}{\mathrm{d}\, t^{\sigma_j}}(\beta_1 - \beta_0)_{|t=a} = 0$ for $j = 1,\dots,k$ and $\dfrac{\mathrm{d}^{\varepsilon_j}}{\mathrm{d}\, t^{\varepsilon_j}}(\beta_1 - \beta_0)_{|t=b} = 0$ for $i = 1,\dots,n-k$, then $\beta_1 - \beta_0 \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$. So, $\beta_1 \geqslant \beta_0$.

Analogously, for $n \geqslant 1$, we conclude that $\{\beta_n\}$ is a non-decreasing sequence. Moreover, by properties of the related Green's function, which is continuous on $I \times I$, it is bounded from above.

Since $\tilde{T}_n[-\lambda_2]$ is strongly inverse positive in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$, $x^1_{-\lambda_2}$ is the unique solution of $\tilde{T}_n[-\lambda_2]u(t) = 0$, coupled with the boundary conditions imposed to α_n and β_n. Thus, we can affirm that:

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = x^1_{-\lambda_2},$$

and $\alpha_0 = x_0^1 \geqslant x^1_{-\lambda_2} \geqslant x^1_{M^*} = \beta_0 \geqslant 0$ on I.

Repeating the previous arguments, we can conclude that for all $M \in [-\lambda_2, M^*]$, we have:

$$x^1_{-\lambda_2} \geqslant x^1_M \geqslant x^1_{M^*} \geqslant 0 \text{ on } I. \tag{4.4.26}$$

On the other hand, we know that ${x^1_{-\lambda_2}}^{(\beta)}(b) = 0$. From inequality (4.4.26), we have ${x^1_M}^{(\beta)}(b) = 0$ for all $M \in [-\lambda_2, M^*]$, which contradicts the discrete character of the spectrum of the operator $\tilde{T}_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}_{\{\sigma_1,\dots,\sigma_{\epsilon_1-1},\sigma_{\epsilon_1+1},\dots,\sigma_k\}}$. Thus, we arrive to a contradiction by supposing that there exists $M^* > -\lambda_2$ such that $\tilde{T}_n[M^*]$ is inverse positive in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}$.

Now, let us assume $\lambda_2 = \lambda_{\varepsilon_{\kappa_1}}$. In this case, we consider $z^1_M(t) = (-1)^{n-k-\kappa_1+1} z_M^{\varepsilon_{\kappa_1}}(t)$.

Since $z_M^1 \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ and $\tilde{T}[M^*]z_{M^*}^1(t) = 0$ on I, we have that $z_{M^*}^1 \geqslant 0$ on I.

Let us see that necessarily $z_0^1 \geqslant z_{-\lambda_2}^1 \geqslant z_{M^*}^1$ on I.

We proceed, arguing similarly to x_M^1, by constructing a non-increasing sequence and bounded from below by zero, $\{\alpha_n\}$, as follows:

$$\alpha_0 = z_0^1, \quad \tilde{T}_n[M^*]\,\alpha_{n+1} = (M^* + \lambda_2)\,\alpha_n,\ n \geqslant 0,$$

where $\alpha_n^{(\sigma_j)}(a) = 0$ for $j = 1, \dots, k$, $\alpha_n^{(\varepsilon_i)}(b) = 0$ if $i \neq \kappa_1$, for $i = 1, \dots, n-k$ and $\alpha_n^{(\kappa_i)}(b) = (-1)^{n-k+\kappa_i-1}$. In particular, $\alpha_n \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ for $n \geqslant 0$.

Now, let us consider the following sequence:

$$\beta_0 = z_{M^*}^1, \quad \tilde{T}_n[M^*]\,\beta_{n+1} = (M^* + \lambda_2)\,\beta_n,\ n \geqslant 0,$$

where $\beta_n^{(\sigma_j)}(a) = 0$ for $j = 1, \dots, k$, $\beta_n^{(\varepsilon_i)}(b) = 0$ if $i \neq \kappa_1$ for $i = 1, \dots, n-k$ and $\beta_n^{(\varepsilon_{\kappa_1})}(b) = (-1)^{n-k+\kappa_1-1}$. In particular, $\beta_n \in X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$ for $n \geqslant 0$.

Analogously as before, we conclude that $\{\beta_n\}$ is a non-decreasing sequence. Moreover, by properties of Green's function, it is bounded from above.

Since $\tilde{T}_n[-\lambda_2]$ is strongly inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$, we have that the function $z_{-\lambda_2}^1$ is the unique solution of $\tilde{T}_n[-\lambda_2]u(t) = 0$ with the boundary conditions given for α_n and β_n. Thus, we can affirm that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = z_{-\lambda_2}^1,$$

and $\alpha_0 = z_0^1 \geqslant z_{-\lambda_2}^1 \geqslant \beta_0 = x_{M^*}^1 \geqslant 0$ on I.

Repeating previous arguments, we can conclude that for all $M \in [-\lambda_2, M^*]$, we have:

$$z_{-\lambda_2}^1 \geqslant z_M^1 \geqslant z_{M^*}^1 \geqslant 0. \tag{4.4.27}$$

On another hand, we know that ${z_{-\lambda_2}^1}^{(\alpha)}(a) = 0$. From inequality (4.4.27), we have ${z_M^1}^{(\alpha)}(a) = 0$ for all $M \in [-\lambda_2, M^*]$, which contradicts the discrete character of the spectrum of the operator $\tilde{T}_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{\kappa_1-1},\varepsilon_{\kappa_1+1},\dots,\varepsilon_{n-k}\}}$. Again, we arrive to a contradiction under the assumption that there exists $M^* > -\lambda_2$ such that $\tilde{T}_n[M^*]$ is inverse positive in $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$.

Finally, we can proceed analogously when $n - k$ is odd to conclude that there is not any $M^* < -\lambda_2$ such that $\tilde{T}_n[M^*]$ is inverse negative in $X_{\{\sigma_1,\dots,\sigma_k\}_{\{\sigma_{\epsilon_1},\dots,\sigma_{\epsilon_\ell}\}}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}_{\{\varepsilon_{\kappa_1},\dots,\varepsilon_{\kappa_h}\}}}$. we can conclude that the intervals given in the result cannot be increased. □

Particular cases

This section is devoted to show the applicability of the previous results to some examples.

Realise that most of the examples given in Section 4.3 follow the structure given on this section. So, we will be able to obtain the characterisation of the strongly inverse positive (negative) character for those operators in different spaces with non-homogeneous boundary conditions.

$(k, n-k)$ boundary conditions

In this case $\mu = \max\{\alpha_2, \beta_2\} = -1$. So, since the biggest set where we can apply Theorem 4.4.23 is $X_{\{0,\dots,k-1\}_{\{k-1\}}}^{\{0,\dots,n-k-1\}_{\{n-k-1\}}}$, Theorem 4.4.23 is equivalent to Theorem 4.4.4.

However, in many cases, we can be under the conditions of Remark 4.4.15 which allows us to apply Theorem 4.4.23 in bigger sets with more non-homogeneous boundary conditions. In the sequel, we recall some of the examples given there where we can apply Theorem 4.4.23 for different non-homogeneous boundary conditions.

For instance, for the operators of the form $\dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n} + M$ the conditions of Remark 4.4.15 are fulfilled for any boundary conditions. In the sequel, we describe two of the examples considered at Section 4.3.

- $T_3^0[M] = \dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ in $X_{\{0\}_{\{0\}}}^{\{0,1\}_{\{0,1\}}}$.

For this particular case, we describe the set:

$$X_{\{0\}_{\{0\}}}^{\{0,1\}_{\{0,1\}}} = \Big\{u \in C^3(I) \;\big|\, u(a) \geqslant 0,\ u(b) \geqslant 0,\ u'(b) \leqslant 0\Big\}.$$

Realise that the results for this set can be extrapolated to the set $X_{\{0,1\}_{\{0,1\}}}^{\{0\}_{\{0\}}}$, just replacing the sign of the different eigenvalues.

The least positive eigenvalue in $X_{\{0\}}^{\{0,1\}}$ has been defined as $(\lambda_3^1)^3$, where $\lambda_3^1 \approxeq 4.233$ is the least positive solution of equation (2.1.4).

The biggest negative eigenvalue in $X_{\{0,1\}}^{\{1\}}$ is $\lambda_0^2 = -\left(m_8\right)^3$, where $m_8 \approxeq 3.017$ is the least positive solution of the next equation:

$$1 - e^{3\lambda/2}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}\lambda}{2}\right) + \cos\left(\frac{\sqrt{3}\lambda}{2}\right)\right) = 0.$$

Then, we conclude that:

- The operator $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is strongly inverse positive in $X_{\{0\}_{\{0\}}}^{\{0,1\}_{\{0,1\}}}$ if, and only if, $M \in \Big(-\left(\lambda_3^1\right)^3, \left(m_8\right)^3\Big] \approxeq \big(-4.233^3, 3.013^3\big]$.

- $T_6^0[M] = \dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} + M$ in $X_{\{0,1,2\}_{\{1,2\}}}^{\{0,1,2\}_{\{1,2\}}}$ or $X_{\{0,1,2\}_{\{0,1,2\}}}^{\{0,1,2\}_{\{0,1,2\}}}$.

The most general set which we are considering is $X_{\{0,1,2\}_{\{0,1,2\}}}^{\{0,1,2\}_{\{0,1,2\}}}$, which is defined by:

$$\Big\{u \in C^6(I) \;\big|\, u(a) \leqslant 0,\ u'(a) \leqslant 0,\ u''(a) \leqslant 0,\ u(b) \leqslant 0,\ u'(b) \geqslant 0,\ u''(b) \leqslant 0\Big\}.$$

Now, let us obtain the different eigenvalues.

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,1,2\}}^{\{0,1,2\}}$ is given by $-\left(\lambda_6^3\right)^6$, where $\lambda_6^3 \approxeq 6.28$ is the least positive solution of equation (2.1.9).

The least positive eigenvalue of operator $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,1,2,3\}}^{\{0,2\}}$ is given by $(m_9)^6$, where $m_9 \approxeq 6.09157$ is the least positive solution of:

$$2\sqrt{3}e^m\left(e^m+1\right)\sin\left(\frac{\sqrt{3}m}{2}\right)\big(\cosh(m)-2\big)+2\sqrt{3}e^{3m/2}\sin\left(\sqrt{3}m\right)$$
$$+3\left(e^m-1\right)\left(e^{2m}+1\right)\cos\left(\frac{\sqrt{3}m}{2}\right)-3e^{m/2}\left(e^{2m}-1\right)=0.$$

The least positive eigenvalue of operator $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,1,2,3\}}^{\{1,2\}}$ is given by $(m_{10})^6$, where $m_{10} \approxeq 5.54944$ is the least positive solution of:

$$e^{m/2}\left(\cos\left(\sqrt{3}m\right)-3\cosh(m)+6\right)+2e^m\cos\left(\frac{\sqrt{3}m}{2}\right)\big(\cosh(m)-2\big)$$
$$+2\cos\left(\frac{\sqrt{3}m}{2}\right)\big(\cosh(m)-2\big)=0.$$

Thus we conclude that:

- $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}+M$ is strongly inverse negative in $X_{\{0,1,2\}_{\{1,2\}}}^{\{0,1,2\}_{\{1,2\}}}$ if, and only if,

$$M\in\Big[-(m_9)^6,\left(\lambda_6^3\right)^6\Big)\approxeq\big[-6.09^6,6.28^6\big).$$

- $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}+M$ is a strongly inverse negative operator in $X_{\{0,1,2\}_{\{0,1,2\}}}^{\{0,1,2\}_{\{0,1,2\}}}$ if, and only if,

$$M\in\Big[-(m_{10})^6,\left(\lambda_6^3\right)^6\Big)\approxeq\big[-5.55^6,6.28^6\big).$$

Moreover, we can consider some operators with more coefficients non-nulls, provided that for several boundary conditions they are under the hypotheses of Remark 4.4.15.

- $T_6^1[M]=\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}-8\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}+M$ in $X_{\{0,1,2\}_{\{1,2\}}}^{\{0,1,2\}_{\{1,2\}}}$ or $X_{\{0,1,2\}_{\{0,1,2\}}}^{\{0,1,2\}_{\{0,1,2\}}}$.

$T_6^1[M]$ has been introduced in (2.1.12).

By using the characterisation given in Theorem 2.1.1, we can affirm that the following third order differential equation:

$$u^{(3)}(t)-8u(t)=0,$$

is disconjugate on $[0,1]$.

Thus, we can obtain the decomposition given in (3.3.1) considering:

$$v_1 = v_2 = v_3 \equiv 1 \quad \text{on} \;\; [0,1].$$

Then, we are in the hypotheses of Remark 4.4.15 whenever we consider the non-homogeneous boundary conditions of order less than 3. In particular we can study the operator $T_6^1[M]$ in the sets $X_{\{0,1,2\}_{\{1,2\}}}^{\{0,1,2\}_{\{1,2\}}}$ and $X_{\{0,1,2\}_{\{0,1,2\}}}^{\{0,1,2\}_{\{0,1,2\}}}$.

Let us obtain the different eigenvalues numerically.

The biggest negative eigenvalue of $T_6^1[0]$ in $X_{\{0,1,2\}}^{\{0,1,2\}}$ is given by -6.2835^6,

The least positive eigenvalue of $T_6^1[0]$ in $X_{\{0,1,2,3\}}^{\{0,2\}}$ is given by 6.10536^6,

The least positive eigenvalue of $T_6^1[0]$ in $X_{\{0,2\}}^{\{0,1,2,3\}}$ is given by 6.07764^6,

The least positive eigenvalue of $T_6^1[0]$ in $X_{\{0,1,2,3\}}^{\{1,2\}}$ is given by 5.55484^6,

The least positive eigenvalue of $T_6^1[0]$ in $X_{\{1,2\}}^{\{0,1,2,3\}}$ is given by 5.54416^6.

Thus we conclude that:

- $T_6^1[M]$ is strongly inverse negative in $X_{\{0,1,2\}_{\{1,2\}}}^{\{0,1,2\}_{\{1,2\}}}$ if, and only if,

$$M \in \left[-6.07764^6, 6.2835^6\right).$$

- $T_6^1[M]$ is a strongly inverse negative operator in $X_{\{0,1,2\}_{\{0,1,2\}}}^{\{0,1,2\}_{\{0,1,2\}}}$ if, and only if,

$$M \in \left[-5.54416^6, 6.2835^6\right).$$

∘ $T_3^1[M] = \dfrac{\mathrm{d}^3}{\mathrm{d}\,t^6} + \cos(10\,t)\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ in $X_{\{0\}_{\{0\}}}^{\{0,1\}_{\{0,1\}}}$.

For instance, consider the third order operator introduced in (2.1.16). In this case, since every first order equation is disconjugate, we have that $v_1 = v_2 \equiv 1$ on $[0,1]$ for the decomposition (3.3.1). Thus, we can study the strongly inverse positive character on the set $X_{\{0\}_{\{0\}}}^{\{0,1\}_{\{0,1\}}}$.

We have obtained numerically the least positive eigenvalue in $X_{\{0\}}^{\{0,1\}}$ in Section 2.1, it is given by 4.29055^3.

The biggest negative eigenvalue in $X_{\{0,1\}}^{\{1\}}$ is -3.06026^3.

Then, we conclude that:

- $T_3^1[M]$ is strongly inverse positive in $X_{\{0\}_{\{0\}}}^{\{0,1\}_{\{0,1\}}}$ if, and only if,

$$M \in \left(-4.29055^3, 3.06026^3\right].$$

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Operator $T_n^0[M]\,u(t) \equiv \dfrac{\mathrm{d}^n}{\mathrm{d}\,t^n} + M$**.**

In the sequel, we treat some of this kind of problems which have been introduced in Section 3.4.

- 2^{nd}-order

In second order, the only possibility is to consider $k = 1$. Then, the characterisation is obtained by applying Theorem 4.4.4 and the parameters set for the strongly inverse positive character is the same as in the homogeneous case which has been obtained in Section 3.4.

- 3^{rd}-order

Let us consider, for instance, $\{\sigma_1, \sigma_2\} = \{1, 2\}$ and $\{\varepsilon_1\} = \{0\}$.

In such a case, $\mu = \max\{\alpha_2, \beta_2\} = \max\{-1, 0\} = 0$. Then, we obtain the characterisation in $X_{\{1,2\}_{\{2\}}}^{\{0\}_{\{0\}}}$ from Theorem 4.4.4 or Theorem 4.4.23 equivalently.

But, from Remark 4.4.15, we are able to obtain the characterisation in $X_{\{1,2\}_{\{1,2\}}}^{\{0\}_{\{0\}}}$ given as follows:

- $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ is a strongly inverse negative operator in $X_{\{1,2\}_{\{1,2\}}}^{\{0\}_{\{0\}}}$ if, and only if,

$$M \in [-\lambda_2, -\lambda_1) \approxeq \big[-1.85^3, 1.85^3\big),$$

where $\lambda_1 = -m_4^3$, with $m_4 \approxeq 1.85$ the least positive solution of (4.3.1), is the biggest negative eigenvalue of $\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3}$ in $X_{\{1,2\}}^{\{0\}}$ and $\lambda_2 = m_4^3$ is the least positive eigenvalue of $T_3^0[0]$ in $X_{\{2\}}^{\{0,1\}}$.

- 4^{th}-order: $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ in $X_{\{0\}}^{\{1,2,3\}}$.

Let us consider again fourth order problems introduced in Section 3.4, $X_{\{0\}}^{\{1,2,3\}}$ and $X_{\{0,2\}}^{\{1,3\}}$. In the first case we cannot apply directly Theorem 4.4.23, since $\mu = 0$. However, with the same argument as in Remark 4.4.15, Theorem 4.4.23 is still true for $\sigma_{\epsilon_{\ell-1}} \geqslant \mu$ or $\varepsilon_{\kappa_{h-1}} \geqslant \mu$.

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0\}}^{\{1,2,3\}}$ is $\lambda_1 = -\dfrac{\pi^4}{4}$.

The least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,1\}}^{\{0,3\}}$ is $\lambda_1^2 = \pi^4$

The least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,1\}}^{\{1,3\}}$ is $\lambda_0^2 = m_1^4$, where $m_1 \approxeq 2.36502$ is the least positive solution of (3.6.2).

Thus, we conclude that:

- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ is strongly inverse negative in $X_{\{0\}_{\{0\}}}^{\{1,2,3\}_{\{2,3\}}}$ if, and only if, M belongs to the interval $\left[-\pi^4, \dfrac{\pi^4}{4}\right)$.

- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}+M$ is strongly inverse negative in $X_{\{0\}_{\{0\}}}^{\{1,2,3\}_{\{1,2,3\}}}$ if, and only if, M remains in the interval $\left[-m_1^4, \dfrac{\pi^4}{4}\right)$.

◦ 4^{th}-order: $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}+M$ in $X_{\{0,2\}}^{\{1,3\}}$.

For $X_{\{0,2\}}^{\{1,3\}}$, we have $\mu = \max\{1,2\} = 2$. Let us study the strongly inverse positive character in $X_{\{0,2\}_{\{0,2\}}}^{\{1,3\}_{\{1,3\}}}$.

The least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,2\}}^{\{1,3\}}$ is $\lambda_1 = \dfrac{\pi^4}{16}$.

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{2\}}^{\{0,1,3\}}$ is $\lambda_0^2 = -\dfrac{\pi^4}{4}$.

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,1,2\}}^{\{1\}}$ is $\lambda_1^2 = -4\,\pi^4$.

Thus, $\lambda_2 = -\dfrac{\pi^4}{4}$ and we can conclude that:

- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ is strongly inverse positive in $X_{\{0,2\}_{\{0,2\}}}^{\{1,3\}_{\{1,3\}}}$ if, and only if, $M \in \left(-\dfrac{\pi^4}{16}, \dfrac{\pi^4}{4}\right]$.

◦ Higher order: $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}+M$ in $X_{\{0,2,4\}}^{\{0,2,4\}}$.

Now, let us analyse the sixth order operator given in Subsection 3.4. That is, the operator $T_6^0[M] \equiv \dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}+M$ defined in $X_{\{0,2,4\}}^{\{0,2,4\}}$. In this case, $\mu = \max\{3,3\} = 3$, so we can apply Theorem 4.4.23 in different spaces.

Let us obtain the different eigenvalues:

The biggest negative eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,2,4\}}^{\{0,2,4\}}$ is $\lambda_1 = -\pi^6$.

The least positive eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,4\}}^{\{0,1,2,4\}}$ is $\lambda_2^2 = m_{11}^6$, where $m_{11} \approxeq 4.14577$ is the least positive solution of:

$$\begin{aligned}&\sqrt{3}e^{m/2}\left(e^{2m}+1\right)-3\left(e^m+1\right)^2\left(e^m-1\right)\sin\left(\frac{\sqrt{3}m}{2}\right)\\ &-2\sqrt{3}e^{3m/2}\cos\left(\sqrt{3}m\right)+\sqrt{3}\left(e^m+1\right)\left(e^m-1\right)^2\cos\left(\frac{\sqrt{3}m}{2}\right)=0.\end{aligned}$$

The least positive eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,1,2,4\}}^{\{0,4\}}$ is $\lambda_2^2 = m_{11}^6$.

The least positive eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{2,4\}}^{\{0,1,2,4\}}$ is $\lambda_0^2 = m_{12}^6$, where $m_{12} \approxeq 3.17334$ is the least positive solution of:

$$-\sqrt{3}e^{m/2}\left(e^{2m}+1\right) - 3\left(e^m+1\right)^2\left(e^m-1\right)\sin\left(\frac{\sqrt{3}m}{2}\right)$$
$$+2\sqrt{3}e^{3m/2}\cos\left(\sqrt{3}m\right) - \sqrt{3}\left(e^m+1\right)\left(e^m-1\right)^2\cos\left(\frac{\sqrt{3}m}{2}\right) = 0.$$

The least positive eigenvalue of $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6}$ in $X_{\{0,1,2,4\}}^{\{2,4\}}$ is $\lambda_0^2 = m_{12}^6$.

Thus, we conclude that:

- $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} + M$ is strongly inverse negative in $X_{\{0,2,4\}_{\{2,4\}}}^{\{0,2,4\}_{\{2,4\}}}$ if, and only if, M belongs to $\big[-m_{11}^6, \pi^6\big) \approxeq \big[-4.146^6, \pi^6\big)$.
- $\dfrac{\mathrm{d}^6}{\mathrm{d}\,t^6} + M$ is strongly inverse negative in $X_{\{0,2,4\}_{\{0,2,4\}}}^{\{0,2,4\}_{\{0,2,4\}}}$ if, and only if, M is in the interval $\big[-m_{12}^6, \pi^6\big) \approxeq \big[-3.173^6, \pi^6\big)$.

Operators with non-constant coefficients.

To finish this chapter we show an example where a fourth order operator with non-constant coefficients is considered.

- $T_4^{nc}[M] = \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + e^{2\,t}\sin(2\,t)\,\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + M$ in $X_{\{0,2\}}^{\{1,2\}}$.

Let us define the operator

$$T_4^{nc}[M]\,u(t) = u^{(4)} + e^{2\,t}\sin(2\,t)\,u'''(t) + M\,u(t), \quad t \in [0,1]$$

defined in $X_{\{0,2\}}^{\{1,2\}}$.

In such a space, we have $\mu = \max\{1,0\} = 1$, and the linear differential equation

$$u''(t) + e^{2\,t}\sin(2\,t)\,u'(t) = 0, \quad t \in [0,1],$$

is disconjugate on $[0,1]$, since it is a composition of two first order linear differential equations, see Theorem 1.1.7:

$$u'(t) + e^{2\,t}\sin(2\,t)\,u(t) = 0 \text{ and } u'(t) = 0, \quad t \in [0,1].$$

Thus, we can apply all previous results to characterise the strongly inverse positive character of $T_4^{nc}[M]$ in $X_{\{0,2\}}^{\{1,2\}}$.

First, we obtain numerically, by means of *Mathematica* program using similar procedures than the ones described in Appendix A for the $(k, n-k)$ boundary conditions, the different eigenvalues of $T_4^{nc}[0]$.

The least positive eigenvalue of $T_4^{nc}[0]$ in $X_{\{0,2\}}^{\{1,2\}}$ is given by 2.62355^4.

The biggest negative eigenvalue of $T_4^{nc}[0]$ in $X_{\{0,1,2\}}^{\{1\}}$ is given by -4.69621^4.

The biggest negative eigenvalue of $T_4^{nc}[0]$ in $X_{\{0\}}^{\{0,1,2\}}$ is given by -6.18170^4.

The biggest negative eigenvalue of $T_4^{nc}[0]$ in $X_{\{0,1,2\}}^{\{2\}}$ is given by -3.45041^4.

The biggest negative eigenvalue of $T_4^{nc}[0]$ in $X_{\{2\}}^{\{0,1,2\}}$ is given by -4.20409^4.

Thus, by means of Theorems 4.1.1 and 4.4.23, we conclude:

- $T_4^{nc}[M]$ is strongly inverse positive in $X_{\{0,2\}_{\{2\}}}^{\{1,2\}_{\{2\}}}$ if, and only if, M belongs to the interval $\left(-2.62355^4, 4.69621^4\right]$.
- $T_4^{nc}[M]$ is strongly inverse positive in $X_{\{0,2\}_{\{0,2\}}}^{\{1,2\}_{\{2\}}}$ if, and only if, M is in the interval $\left(-2.62355^4, 4.20409^4\right]$.
- $T_4^{nc}[M]$ is strongly inverse positive either in $X_{\{0,2\}_{\{2\}}}^{\{1,2\}_{\{1,2\}}}$ or $X_{\{0,2\}_{\{0,2\}}}^{\{1,2\}_{\{1,2\}}}$ if, and only if, $M \in \left(-2.62355^4, 3.45041^4\right]$.

Moreover, in order to use Theorem 4.2.1, we can obtain the necessary eigenvalues of $T_4^{nc}[0]$.

The least positive eigenvalue of $T_4^{nc}[0]$ in $X_{\{0,1\}}^{\{1,2\}}$ is given by 3.22872^4.

The least positive eigenvalue of $T_4^{nc}[0]$ in $X_{\{0,2\}}^{\{0,1\}}$ is given by 4.33768^4.

Thus, from Theorem 4.2.1, we conclude:

- If $T_4^{nc}[M]$ is a strongly inverse negative operator in $X_{\{0,2\}}^{\{1,2\}}$, then M remains in the interval $\left[-3.22872^4, -2.62355^4\right)$.

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Chapter 5
Simply supported beam

Contents

In this chapter, we consider a fourth order problem defined as follows:

$$u^{(4)}(t) + p_1(t)\,u^{(3)}(t) + p_2(t)\,u''(t) + c(t)\,u(t) = \sigma(t), \quad t \in I \equiv [a,b], \tag{5.0.1}$$

where $p_1 \in C^3(I)$, $p_2 \in C^2(I)$ and c, $\sigma \in C(I)$, coupled with the simply supported beam boundary conditions:

$$u(a) = u''(a) = u(b) = u''(b) = 0. \tag{5.0.2}$$

Before study the problem (5.0.1)-(5.0.2), we study the fourth order operator:

$$T_4[M]\,u(t) = u^{(4)}(t) + p_1(t)\,u^{(3)}(t) + p_2(t)\,u''(t) + M\,u(t), \quad t \in I, \tag{5.0.3}$$

where $M \in \mathbb{R}$ on the set $X_{\{0,2\}}^{\{0,2\}}$, which refers to the boundary conditions (5.0.2).

Notation 5.0.1. *Along this chapter, denote $X = X_{\{0,2\}}^{\{0,2\}}$, the set of functions which satisfy the boundary conditions* (5.0.2).

Realise that if the second order linear differential equation:

$$u''(t) + p_1(t)\,u'(t) + p_2(t)\,u(t) = 0, \quad t \in I, \tag{5.0.4}$$

is disconjugate, then we are under the hypotheses of Proposition 4.4.8. In particular we know that we can apply the results which we have obtained in Chapter 4 for this particular operator.

In addition, in this case, we will prove that the necessary condition on M given in Theorem 4.2.1 for the operator to be strongly inverse negative is also a sufficient one.

In the literature, many different fourth order boundary value problems coupled with the simply supported beam boundary conditions have been studied along time. Indeed, the study of this kind of problems is very important, since they are used to model different kinds of bridges. In [48], several examples of bridges and its mathematical models are shown.

In addition, in [53, Chapter 2] different models for suspension bridges are described. For instance, in [53, Section 2.6.3], it is considered a hinged beam (which represents the roadway) subject to non-linear forces along two-sided springs (the hangers of the suspension bridge). In such a case, the one dimension mathematical model for the vertical displacement of the roadway is given by:

$$\begin{cases} E\,I\,u^{(4)}(t) - T\,u''(t) + g\big(u(t)\big) = q(t), \quad t \in I, \\ u(a) = u(b) = u''(a) = u''(b) = 0, \end{cases} \tag{5.0.5}$$

where a and b are the extremes of the studied bridge, E and I are two positive constants given by the material of the beam (the Young's module and the moment of inertia), $T \geqslant 0$ is the constant strength tension, $q(t)$ is a downwards distributed load which acts on the beam and g is the restoring force. This model involves a non-linear part given by the function g. But in order to study it, it is very important to know first its linear part. Indeed, in some cases, for instance if the vertical displacement is small enough, we can consider g as a linear function in the way $g\big(u(t)\big) = k\,u(t)$, for a constant $k \in \mathbb{R}$. A more general problem consists on considering this restoring force as a non-autonomous function $g\big(t, u(t)\big) = f(t)\,u(t)$, where f is a continuous function on I. In particular the fact that the displacement of the bridge occurs in the same direction as the external force is fundamental in order to ensure the stability of the considered structure.

In Sections 5.1.1 and 5.2.4 we study the existence of constant sign solutions of the following fourth-order problem:

$$T[p,c]\,u(t) \equiv u^{(4)}(t) - p\,u''(t) + c(t)\,u(t) = h(t), \quad t \in I, \tag{5.0.6}$$

coupled with the boundary conditions:

$$u(a) = u(b) = u''(a) = u''(b) = 0. \tag{5.0.7}$$

Remark 5.0.2. *Realise that if, in* (5.0.5)*, we consider the non-autonomous function*

$$g\big(t, u(t)\big) = f(t)\,u(t),$$

and divide by $E\,I$, then problem (5.0.6)-(5.0.7) *is a particular case of problem* (5.0.5) *with* $p = \dfrac{T}{E\,I} \geq 0$, $c(t) = \dfrac{f(t)}{E\,I}$ *and* $h(t) = \dfrac{q(t)}{E\,I} \geqslant 0$ *(because q is a downwards load).*

In [22], it is characterised the strongly inverse negative character for the particular case of (5.0.3) with $p_1 = p_2 \equiv 0$ on I. Moreover, in [87] the strongly inverse positive character for that case is studied. The used tools are strongly involved with the constant sign of the related Greens functions. We have not found spectral relationship with operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in both references. In Chapter 4, we have also obtained the characterisation of the strongly inverse positive character of $T_4[M]$ in X by means of spectral theory by using Proposition 4.4.8 and Theorem 4.1.1.

Moreover, in [47], weaker sufficient conditions to ensure either the strongly inverse positive or negative character of problem (5.0.1)-(5.0.2) are proved, by considering $p_1 = p_2 \equiv 0$. In [46], there are also obtained sufficient conditions which ensure that this problem has a unique constant sign solution for $\sigma > 0$.

In [12], it is studied the operator:

$$\frac{\mathrm{d}^4}{\mathrm{d}\,t^4} - (\alpha^2 + \beta^2)\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + \alpha^2\,\beta^2,$$

defined in a complex domain, with $\alpha^2 \neq \beta^2$. In this case, some sufficient conditions to ensure the inverse positive character are proved.

In [68], some results which ensure the existence of one or multiple positive solutions of the problem:

$$u^{(4)}(t) = f\big(t, u(t), u'(t)\big), \quad t \in [0,1],$$

coupled with the simply supported beam boundary conditions (5.0.2) are shown.

In a first part of the chapter, we prove that the necessary condition given on Theorem 4.2.1 is also a sufficient one for the operator $T_4[M]$ in X. Then, following [46, 47], we obtain sufficient conditions to ensure that the problem (5.0.1)-(5.0.2), with $p_1 \equiv 0$ and, in addition, $p_2 \equiv -p \leqslant 0$ on I has a unique constant sign solution.

We have published the results here presented in [32, 33].

5.1 Operator $T_4[M] = \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + p_1(t)\,\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + p_2(t)\,\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$

Along this section, let us denote $X_{\{\sigma_1,\sigma_2\}}^{\{\varepsilon_1,\varepsilon_2\}}\big([c,d]\big)$ the set $X_{\{\sigma_1,\sigma_2\}}^{\{\varepsilon_1,\varepsilon_2\}}$ related to the interval $[c,d]$, that is:

$$X_{\{\sigma_1,\sigma_2\}}^{\{\varepsilon_1,\varepsilon_2\}}\big([c,d]\big) = \Big\{u \in C^4\big([c,d]\big) \mid u^{(\sigma_1)}(c) = u^{(\sigma_2)}(c) = u^{(\varepsilon_1)}(d) = u^{(\varepsilon_2)}(d) = 0\Big\}.$$

The aim of this section is to obtain a characterisation of the strongly inverse negative character of the operator $T_4[M]$, previously defined in (5.0.3), in X. The used techniques are similar to those applied in Chapters 3 and 4. Thus, let us see that some solutions of the linear differential equation:

$$T_4[M]\,u(t) = 0, \quad t \in I, \tag{5.1.1}$$

must be of constant sign.

Lemma 5.1.1. *If the second order linear differential equation* (5.0.4) *is disconjugate on an interval* $[c,d]$*, then the following assertions are fulfilled.*

- *Every non-trivial solution of* (5.1.1) *on* $[c,d]$ *satisfying the boundary conditions:*

$$u'(c) = u(d) = u''(d) = 0, \tag{5.1.2}$$

  does not have any zero on (c,d) *for* $M \in \Big[-\lambda'_{3[c,d]}, 0\Big]$*, where* $\lambda'_{3[c,d]}$ *is the least positive eigenvalue of* $T_4[0]$ *in* $X_{\{0,1\}}^{\{0,2\}}\big([c,d]\big)$.

- *Every non-trivial solution of* (5.1.1) *on* $[c,d]$ *satisfying the boundary conditions:*

$$u(c) = u''(c) = u'(d) = 0, \tag{5.1.3}$$

does not have any zero on (c,d) *for* $M \in \left[-\lambda''_{3\,[c,d]}, 0\right]$, *where* $\lambda''_{3\,[c,d]}$ *is the least positive eigenvalue of* $T_4[0]$ *in* $X_{\{0,2\}}^{\{0,1\}}\big([c,d]\big)$.

Proof. The result follows as a direct consequence of Propositions 4.4.17 and 4.4.18.

Consider the operator $T_4[M]$ defined in $X_{\{0,1\}}^{\{0,2\}}\big([c,d]\big)$. For the choice $\{\sigma_1,\sigma_2\} = \{0,1\}$, $\{\varepsilon_1,\varepsilon_2\} = \{0,2\}$ and $\sigma_{\epsilon_j} = 0 < 2 = \alpha$, we are under the hypotheses of Proposition 4.4.17 and the first par of the result follows directly.

Moreover, if we consider the operator $T_4[M]$ defined in $X_{\{0,2\}}^{\{0,1\}}\big([c,d]\big)$. It is obvious that for the choice $\{\sigma_1,\sigma_2\} = \{0,2\}$, $\{\varepsilon_1,\varepsilon_2\} = \{0,1\}$ and $\varepsilon_{\kappa_i} = 0 < 2 = \beta$, we are under the hypotheses of Proposition 4.4.18 and the proof is complete. □

As a consequence of this result we obtain a property for the existence of eigenvalues in some spaces.

Lemma 5.1.2. *If the second order linear differential equation* (5.0.4) *is disconjugate on* $[c,d]$, *then the following assertions are satisfied.*

- *For all* $M \in \left(-\lambda'_{3\,[c,d]}, 0\right]$ *and* $e \in [c,d)$, *the following problem has no non-trivial solution* $u \in C^4\big([e,d]\big)$*:*

$$T_4[M]\,u(t) = 0,\ t \in [e,d], \quad u(e) = u'(e) = u(d) = u''(d) = 0,$$

where $\lambda'_{3\,[c,d]}$ *is the least positive eigenvalue of* $T_4[0]$ *in* $X_{\{0,1\}}^{\{0,2\}}\big([c,d]\big)$.

- *For all* $M \in \left(-\lambda''_{3\,[c,d]}, 0\right]$ *and* $e \in (c,d]$, *the following problem has no non-trivial solution* $u \in C^4\big([e,d]\big)$*:*

$$T_4[M]\,u(t) = 0,\ t \in [c,e], \quad u(c) = u''(c) = u(e) = u'(e) = 0,$$

where $\lambda''_{3\,[c,d]}$ *is the least positive eigenvalue of* $T_4[0]$ *in* $X_{\{0,2\}}^{\{0,1\}}\big([c,d]\big)$.

Proof. Let us prove the first assertion, the proof of the second one is analogous.

To this end, we get the fundamental system of solutions $y_1[M](t)$, $y_2[M](t)$, $y_3[M](t)$, $y_4[M](t)$, where $y_i^{(i-1)}(d) = 1$ and $y_i^{(k)}(d) = 0$ for $k \in \{0,1,2,3\}$, $k \neq i-1$, and we construct the Wronskian:

$$W_M(t) = \begin{vmatrix} y_2[M](t) & y_4[M](t) \\ y_2'[M](t) & y_4'[M](t) \end{vmatrix},$$

which is a continuous function with respect to M.

It is obvious that the general solution of problem:

$$T_4[M]\,y(t) = 0, \quad t \in [c,d], \quad y(d) = y''(d) = 0,$$

is given by the expression:

$$y(t) = \alpha_1\,y_2[M](t) + \alpha_2\,y_4[M](t),$$

with $\alpha_1,\ \alpha_2 \in \mathbb{R}$.

Every non-trivial solution given by the previous expression, satisfies $y(e) = y'(e) = 0$ if, and only if, $W_M(e) = 0$.

Moreover, we have seen that there is not any non-trivial solution of:

$$T_4[0]\,u(t) = 0,\ t \in [c,d],$$

with more than three homogeneous boundary conditions on an interval $[c,e]$. So, $W_0[e] \neq 0$ for every $e \in [c,d)$.

Also, since we have seen that the first positive eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{0,2\}}\big([c,d]\big)$ is $\lambda'_{3\,[c,d]}$, we can affirm that $W_M(c) \neq 0$ for every $M \in (-\lambda'_{3\,[c,d]}, 0]$.

Then, there exist $M \in (-\lambda'_{3\,[c,d]}, 0]$ and $e \in [c,d]$ for which $W_M(e) = 0$ if, and only if, the following problem has a non-trivial solution:

$$T_4[M]\,u(t) = 0,\ t \in [e,d], \quad u(e) = u'(e) = u(d) = u''(d) = 0.$$

Since $W_M(t)$ is a continuous function of M, it must exist $\bar{M} \in (-\lambda'_{3\,[c,d]}, 0]$ and $\bar{e} \in [c,d]$ such that $W_{\bar{M}}(\bar{e}) = 0$ and $W'_{\bar{M}}(\bar{e}) = 0$, i.e.

$$\begin{vmatrix} y_2[\bar{M}](\bar{e}) & y_4[\bar{M}](\bar{e}) \\ \\ y'_2[\bar{M}](\bar{e}) & y'_4[\bar{M}](\bar{e}) \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} y_2[\bar{M}](\bar{e}) & y_4[\bar{M}](\bar{e}) \\ \\ y''_2[\bar{M}](\bar{e}) & y''_4[\bar{M}](\bar{e}) \end{vmatrix} = 0.$$

If we consider function:

$$y(t) = y_4[\bar{M}](\bar{e})\,y_2[\bar{M}]\,(t) - y_2[\bar{M}](\bar{e})\,y_4[\bar{M}]\,(t),$$

then it satisfies $T_4[M]\,y(t) = 0$ on $[\bar{e},d]$ coupled with the boundary conditions:

$$y(\bar{e}) = y'(\bar{e}) = y''(\bar{e}) = y(d) = y''(d) = 0.$$

In particular, it belongs to $X_{3\,[\bar{e},d]}$, with the same notation as in equation (2.2.1), and, as a consequence, there exists a positive eigenvalue of $T_4[0]$ in $X_{3\,[\bar{e},d]}$.

Now, since the linear differential equations (5.0.4) and $u''(t) = 0$ are disconjugate on $[\bar{e},d]$, we can apply Theorem 1.1.7 to affirm that $T_4[0]\,u(t) = 0$ is a disconjugate equation in $[\bar{e},d]$ too. So, since $n - k = 1$ is an odd number we attain a contradiction with Theorem 1.1.14. $\square$

As a direct corollary of previous property we obtain the following result.

Corollary 5.1.3. *If the second order linear differential equation* (5.0.4) *is disconjugate on* $[c, d]$*, then the following assertions hold.*

- *The least positive eigenvalue of* $T_4[0]$ *in* $X_{\{0,1\}}^{\{0,2\}}\big([e, d]\big)$*, denoted as* $\lambda'_{3\,[e,d]}$*, increases with respect to* $e \in [c, d)$*.*
- *The least positive eigenvalue of* $T_4[0]$ *in* $X_{\{0,2\}}^{\{0,1\}}\big([c, e]\big)$*, denoted as* $\lambda''_{3\,[c,e]}$*, decreases with respect to* $e \in (c, d]$*.*

Now, we state the correspondent to Theorem 4.1.1 to this particular case, where we characterise the strongly inverse positive character, and we also prove a characterization, by means of spectral theory, for the strongly inverse negative character.

Theorem 5.1.4. *If the second order linear differential equation* (5.0.4) *is disconjugate on* I*, then the following assertions are fulfilled.*

- $T_4[M]$ *is strongly inverse positive in* X *if, and only if,* $M \in (-\lambda_1, -\lambda_2]$*, where:*
  - * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T[0]$ *in* X*.*
  - * $\lambda_2 < 0$ *is the maximum between:*
    - $\cdot$ $\lambda'_2 < 0$*, the biggest negative eigenvalue of* $T[0]$ *in* X_1 $\left(\equiv X_{\{0\}}^{\{0,1,2\}}\right)$*.*
    - $\cdot$ $\lambda''_2 < 0$*, the biggest negative eigenvalue of* $T[0]$ *in* X_3 $\left(\equiv X_{\{0,1,2\}}^{\{0\}}\right)$*.*
- $T_4[M]$ *is strongly inverse negative in* X *if, and only if,* $M \in [-\lambda_3, -\lambda_1)$*, where:*
  - * $\lambda_3 > 0$ *is the minimum between:*
    - $\cdot$ $\lambda'_3 \equiv \lambda'_{3\,[a,b]} > 0$*, the least positive eigenvalue of* $T[0]$ *in* $X_{\{0,1\}}^{\{0,2\}}$*.*
    - $\cdot$ $\lambda''_3 \equiv \lambda''_{3\,[a,b]} > 0$*, the least positive eigenvalue of* $T[0]$ *in* $X_{\{0,2\}}^{\{0,1\}}$*.*

Proof. It is clear that the part related to the strongly inverse positive character is a particular case of Theorem 4.1.1.

Moreover, from Theorems 3.7.3, 3.7.5 and 3.2.6, we only need to prove that for every $t \in (a, b)$ the related Green's function, $g_M(t, s)$, is negative for a.e. $s \in I$. Indeed, we will prove that $g_M(t, s) < 0$ on $(a, b) \times (a, b)$ for $M \in [-\lambda_3, -\lambda_1)$.

For every $s \in (a, b)$, let us denote $u_M^s(t) \equiv g_M(t, s)$.

Now, let us suppose that the Green's function has a zero on $(a, b) \times (a, b)$ for some $\bar{M} \in [-\lambda_3, -\lambda_1)$, hence there exists $\bar{s} \in (a, b)$, such that $u_{\bar{M}}^{\bar{s}}$ has a zero on (a, b). Since $\bar{M} \in [-\lambda_3, -\lambda_1)$, from Theorem 3.7.5, we have that $u_{\bar{M}}^{\bar{s}}(t) < 0$ on a neighbourhood of $t = a$ and $t = b$.

So, under our assumption, there are two possibilities, either $u_{\bar{M}}^{\bar{s}}$ has a double zero or $u_{\bar{M}}^{\bar{s}}$ has at least two different zeros.

If $u_{\bar{M}}^{\bar{s}}$ has a double zero at a point $c \in (a, b)$, we have two possibilities either $c \geqslant \bar{s}$ or $c \leqslant \bar{s}$.

If $c \geqslant \bar{s}$, there exists an eigenvalue of $T_4[0]$ on the interval $[c,b]$, satisfying the correspondent boundary conditions to $X_{\{0,1\}}^{\{0,2\}}\big([c,b]\big)$ with $[c,b] \subset I$. And this eigenvalue is less than or equal to λ_3'', so this fact contradicts Lemma 5.1.2.

If $c \leqslant \bar{s}$, there exists an eigenvalue of $T_4[0]$ on $[a,c] \subset I$, satisfying the boundary conditions correspondent to $X_{\{0,2\}}^{\{0,1\}}$. And, again this value is less than or equal to λ_3', a fact which contradicts Lemma 5.1.2.

If $u_{\bar{M}}^{\bar{s}}$ has two different zeros at points $c_1, c_2 \in (a,b)$, such that $c_1 < c_2$. Since $u_{\bar{M}}^{\bar{s}}$ is a C^1 function on I, it must exist $d \in (c_1,c_2)$ such that ${u_{\bar{M}}^{\bar{s}}}'(d) = 0$. Again, we have two possibilities either $d \leqslant \bar{s}$ or $d \geqslant \bar{s}$.

If $d \leqslant \bar{s}$, then $u \in C^4\big([a,d]\big)$. From Corollary 5.1.3, we know that:

$$\lambda''_{3\,[a,d]} > \lambda''_{3\,[a,b]} \equiv \lambda_3'' \geqslant \lambda_3,$$

so $\bar{M} \in [-\lambda''_{3\,[a,d]}, -\lambda_1)$ and $u_{\bar{M}}^{\bar{s}}(a) = {u_{\bar{M}}^{\bar{s}}}''(a) = {u_{\bar{M}}^{\bar{s}}}'(d) = 0$. Hence, by applying Lemma 5.1.1, $u_{\bar{M}}^{\bar{s}}$ does not have any zero on (a,d), but $u_{\bar{M}}^{\bar{s}}(c_1) = 0$ and $c_1 \in (a,d)$, which is a contradiction.

Finally, if $d \geqslant \bar{s}$, we have that $u \in C^4([d,b])$. From Corollary 5.1.3, we know that

$$\lambda'_{3\,[d,b]} > \lambda'_{3\,[a,b]} \equiv \lambda_3' \geqslant \lambda_3,$$

so $\bar{M} \in [-\lambda'_{3\,[d,b]}, -\lambda_1)$ and ${u_{\bar{M}}^{\bar{s}}}'(d) = u_{\bar{M}}^{\bar{s}}(b) = {u_{\bar{M}}^{\bar{s}}}''(b) = 0$. Hence, by applying Lemma 5.1.1, $u_{\bar{M}}^{\bar{s}}$ does not have any zero on (d,b), a fact which contradicts that $u_{\bar{M}}^{\bar{s}}(c_2) = 0$ and $c_2 \in (d,b)$.

Hence, we arrive to a contradiction of supposing that while $M \in [-\lambda_3, -\lambda_1)$, $g_M(t,s)$ has a zero on $(a,b) \times (a,b)$. So, this together with Theorems 3.7.3 and 3.7.5, tell us that $T_4[M]$ is strongly inverse negative in X if, and only if, $M \in [-\lambda_3, -\lambda_1)$. $\square$

From Theorem 4.4.4 and Corollary 4.1.3, we obtain a direct result for the operator defined in (5.0.1) coupled with the following non-homogeneous boundary conditions:

$$u(a) = u(b) = 0, \quad u''(a) = d_1 \leqslant 0,\; u''(b) = d_2 \leqslant 0, \tag{5.1.4}$$

for d_1, d_2 real arbitrary numbers.

Let us denote, $\tilde{X} = \Big\{u \in C^4(I) \;\Big|\; u \text{ satisfies (5.1.4)}\Big\}$.

Corollary 5.1.5. *Consider the operator* $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + p_1(t)\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + p_2(t)\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + c(t)$, *with* $c \in C(I)$, *under the hypotheses of Theorem 5.1.4. Then, the following assertions are fulfilled.*

- *If* $-\lambda_1 < c(t) \leqslant -\lambda_2$ *for every* $t \in I$, *then* $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + p_1(t)\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + p_2(t)\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + c(t)$ *is strongly inverse positive in* $\tilde{X}$.
- *If* $-\lambda_3 \leqslant c(t) < -\lambda_1$ *for every* $t \in I$, *then* $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + p_1(t)\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + p_2(t)\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + c(t)$ *is strongly inverse negative in* $\tilde{X}$.

5.1.1 Particular cases

In this section, in order to see the usefulness of the given results, we show several examples where we can apply them.

- Operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$, with $p \geqslant 0$.

Now, let us consider the operator given by the expression $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$, with $p \geqslant 0$ defined on I.

In Theorem 2.1.1, we have proved that the second order linear differential equation $u''(t) + m\,u(t) = 0$ is disconjugate on I if, and only if, $m \in \left(-\infty, \left(\dfrac{\pi}{b-a}\right)^2\right)$, where $\lambda_1 = -\left(\dfrac{\pi}{b-a}\right)^2$ is the biggest negative eigenvalue of $\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in X_1. In particular,

$$u''(t) - p\,u(t) = 0,$$

is always a disconjugate equation on I for all $p \geqslant 0$, so we can apply Theorem 5.1.4.

In order to apply Theorem 5.1.4 we have to obtain the eigenvalues of this operator in X, X_3 $\left(X_{\{0,1,2\}}^{\{0\}}\right)$, X_1 $\left(X_{\{0\}}^{\{0,1,2\}}\right)$, $X_{\{0,1\}}^{\{0,2\}}$ and $X_{\{0,2\}}^{\{0,1\}}$.

In this particular case, the eigenvalues of the operator in X_1 and X_3 are the same, and also the eigenvalues in $X_{\{0,1\}}^{\{0,2\}}$ and $X_{\{0,2\}}^{\{0,1\}}$ coincide. Indeed, if we have $u \in X_1$ $\Big($resp. $u \in X_{\{0,1\}}^{\{0,2\}}\Big)$ such that $u^{(4)}(t) - pu''(t) + M\,u(t) = 0$, then $v(t) = u(1-t)$ satisfies $v^{(4)}(t) - pv''(t) + M\,v(t) = 0$ on I and $v \in X_3$ $\Big($resp. $v \in X_{\{0,2\}}^{\{0,1\}}\Big)$.

We consider the general expression of the solution of equation $u^{(4)}(t) - p\,u''(t) = \lambda\,u(t)$ to obtain the different eigenvalues.

So, the eigenvalues of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in X are given by:

$$\lambda = k^4 \left(\frac{\pi}{b-a}\right)^4 + k^2\,p\left(\frac{\pi}{b-a}\right)^2, \quad k = 1, 2, 3, \ldots$$

In particular, the least positive eigenvalue is given by $\lambda_1(p) = \left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2$.

The eigenvalues of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in X_3 are given as $-\lambda$, where λ is a positive solution of:

$$\frac{\tan\left(\dfrac{b-a}{2}\sqrt{2\sqrt{\lambda}-p}\right)}{\sqrt{2\sqrt{\lambda}-p}} = \frac{\tanh\left(\dfrac{b-a}{2}\sqrt{2\sqrt{\lambda}+p}\right)}{\sqrt{2\sqrt{\lambda}+p}}, \tag{5.1.5}$$

By denoting $\tilde{m}_1$ as the least positive solution of this equation, then $\lambda_2(p) = -\tilde{m}_1$ is the biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in X_3.

Finally, the eigenvalues of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in $X_{\{0,1\}}^{\{0,2\}}$ are given as the positive solutions of

$$\frac{\tan\left(\dfrac{(b-a)\sqrt{\sqrt{p^2+4\,\lambda}-p}}{\sqrt{2}}\right)}{\sqrt{\sqrt{p^2+4\,\lambda}-p}} = \frac{\tanh\left(\dfrac{(b-a)\sqrt{\sqrt{p^2+4\,\lambda}+p}}{\sqrt{2}}\right)}{\sqrt{\sqrt{p^2+4\,\lambda}+p}}. \tag{5.1.6}$$

Then the least positive solution of this equation, $\lambda_3(p)$, is the least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in $X_{\{0,1\}}^{\{0,2\}}$.

Now, we apply Theorem 5.1.4 to obtain that $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ is strongly inverse positive in X if, and only if $M \in \left(\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2, -\lambda_2(p)\right]$. And it is strongly inverse negative in X if, and only if, $M \in \left[-\lambda_3(p), -\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2\right)$.

- The operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$

As a particular case we can consider $p = 0$ and $[a,b] = [0,1]$, and we have the operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ on $[0,1]$. This result has already been studied in [22, 87], but in these cases the expression of Green's function was needed and they did not show any relationship with spectral theory.

In this case $\lambda_1(0) = \pi^4$ is the least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in X.

Moreover, $\lambda_2(0) = -\lambda^4$, where λ is the least positive zero of:

$$\tan\left(\frac{\lambda}{\sqrt{2}}\right) = \tanh\left(\frac{\lambda}{\sqrt{2}}\right),$$

i.e., $\lambda_2(0) \approxeq -5.55^4$ is the biggest negative eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in X_3.

Finally, $\lambda_3(0) = \lambda^4$, with λ is least positive root of:

$$\tanh(\lambda) = \tan(\lambda),$$

i.e., $\lambda_3(0) \approxeq 3.927^4$ is the least positive eigenvalue of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$ in $X_{\{0,1\}}^{\{0,2\}}$.

Then, we can use Theorem 5.1.4 to conclude that $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ is strongly inverse positive if, and only if $M \in \big(-\pi^4, -\lambda_2(0)\big] \approxeq \big(-\pi^4, 5.55^4\big]$ and $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$ is strongly inverse negative if, and only if $M \in \big[-\lambda_3(0), -\pi^4\big) \approxeq \big[-3.927^4, \pi^4\big)$. These results were already obtained in [22] (the inverse negative character) and in [87] (the positive one), but in our case we do not need to obtain the expression of Green's function.

Operators with non-constant coefficients

In this subsection we study some operators with non-constant coefficients, obtaining its eigenvalues numerically.

- The operator $\dfrac{d^4}{d\,t^4} - t^2 \dfrac{d^2}{d\,t^2} + M$

The first operator which we consider is $\dfrac{d^4}{d\,t^4} - t^2 \dfrac{d^2}{d\,t^2} + M$ on $[0,1]$. Its eigenvalues in the different spaces are obtained numerically by means of the representation of the corresponding Wronskians, following an analogous procedure to the one decribed in Appendix A for the particular $(k, n-k)$ boundary conditions, and are the following:

$\lambda_1 \approxeq 3.164^4$ is the least positive eigenvalue in X.
$\lambda_2'' \approxeq -5.554^4$ is the biggest negative eigenvalue in X_3.
$\lambda_2' \approxeq -5.5716^4$ is the biggest negative eigenvalue in X_1.
$\lambda_3'' \approxeq 3.934^4$ is the least positive eigenvalue in $X_{\{0,2\}}^{\{0,1\}}$.
$\lambda_3' \approxeq 3.946^4$ is the least positive eigenvalue in $X_{\{0,1\}}^{\{0,2\}}$.
In particular, $\lambda_2 \approxeq -5.554^4$ and $\lambda_3 \approxeq 3.934^4$.

By means of Theorem 2.1.1, we know that $u''(t) - t^2\,u(t) + M\,u(t) = 0$ is a disconjugate equation on $[0,1]$ if, and only if $M \in (-\infty, -\bar{\lambda})$, where $\bar{\lambda} \approxeq -10.16$. In particular, it is disconjugate for $M = 0$.

So we can apply Theorem 5.1.4 to conclude that:

- $\dfrac{d^4}{d\,t^4} - t^2 \dfrac{d^2}{d\,t^2} + M\,u$ is strongly inverse positive in X if, and only if, M belongs to the interval $(-\lambda_1, -\lambda_2] \approxeq (-3.164^4, 5.554^4]$.
- $\dfrac{d^4}{d\,t^4} - t^2 \dfrac{d^2}{d\,t^2} + M\,u$ is strongly inverse negative in X if, and only if M is in the interval $[-\lambda_3, -\lambda_1) \approxeq [-3.934^4, -3.164^4)$.

- The operator $\dfrac{d^4}{d\,t^4} + 16\,t^2 \dfrac{d^2}{d\,t^2} + M$

Now, let us study the operator $\dfrac{d^4}{d\,t^4} + 16\,t^2 \dfrac{d^2}{d\,t^2} + M\,u$. Using again Theorem 2.1.1, we can see that the second order linear differential equation $u''(t) + 16\,t^2\,u(t) + M\,u[t] = 0$ is disconjugate on $[0,1]$ if, and only if $M \in (-\infty, -\lambda_1^{2*})$, where $\lambda_1^{2*} \approxeq -5.05$. In particular, it holds for $M = 0$.

Then, in order to apply Theorem 5.1.4 we can obtain the eigenvalues of $\dfrac{d^4}{d\,t^4} + 16\,t^2 \dfrac{d^2}{d\,t^2}$ in the different spaces. As in the previous case we obtain them numerically by means of the study of the corresponding Wronskians.

$\lambda_1 \approxeq 2.662^4$ is the least positive eigenvalue in X.
$\lambda_2'' \approxeq -5.5334^4$ is the biggest negative eigenvalue in X_3.
$\lambda_2' \approxeq -5.2405^4$ is the biggest negative eigenvalue in X_1.
$\lambda_3'' \approxeq 3.555^4$ is the least positive eigenvalue in $X_{\{0,2\}}^{\{0,1\}}$.
$\lambda_3' \approxeq 3.809^4$ is the least positive eigenvalue in $X_{\{0,1\}}^{\{0,2\}}$.
So, $\lambda_2 \approxeq -5.2405^4$ and $\lambda_3 \approxeq 3.555^4$ and we can affirm that:

- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 16\,t^2\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ is strongly inverse positive in X if, and only if, M belongs to the interval $(-\lambda_1, -\lambda_2] \approxeq (-2.662^4, 5.2405^4]$.
- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 16\,t^2\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ is strongly inverse negative in X if, and only if, M is in the interval $[-\lambda_3, -\lambda_1) \approxeq [-3.555^4, -2.662^4)$.

∘ The operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 2\,t\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 2\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$

Finally, we consider operator $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 2\,t\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 2\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ on $[0,1]$.

The unique solution of $u''(t) + 2\,t\,u'(t) + 2\,u(t) = 0$ satisfying $u(0) = 0$ and $u'(0) = 1$, follows the expression:

$$y(t) = e^{-t^2}\int_0^t e^{s^2}\,\mathrm{d}\,s,$$

which does not vanish for every $t > 0$. Then the second order linear differential equation is disconjugate on every interval $[0, c]$, in particular on $[0,1]$. So we can apply Theorem 5.1.4. In order to do that, we obtain numerically the eigenvalues of $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 2\,t\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 2\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2}$ in the different spaces of definition as follows:
$\lambda_1 \approxeq 3.079^4$ is the least positive eigenvalue in X.
$\lambda_2'' \approxeq -5.595^4$ is the biggest negative eigenvalue in X_3.
$\lambda_2' \approxeq -5.606^4$ is the biggest negative eigenvalue in X_1.
$\lambda_3'' \approxeq 3.986^4$ is the least positive eigenvalue in $X_{\{0,2\}}^{\{0,1\}}$.
$\lambda_3' \approxeq 3.854^4$ is the least positive eigenvalue in $X_{\{0,1\}}^{\{0,2\}}$.

Then, $\lambda_2 \approxeq -5.595^4$ and $\lambda_3 \approxeq 3.854^4$ and we can conclude, applying Theorem 5.1.4, that:

- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 2\,t\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 2\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M$ is inverse positive in X if, and only if, M is in the interval $(-\lambda_1, -\lambda_2] \approxeq (-3.079^4, 5.595^4]$
- $\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + 2\,t\dfrac{\mathrm{d}^3}{\mathrm{d}\,t^3} + 2\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + M\,u$ is inverse negative in X if, and only if, M belongs to the interval $[-\lambda_3, -\lambda_1) \approxeq [-3.854^4, -3.079^4)$.

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

5.2 Operator $T_4[p,c] = \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - p\,\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} + c(t)$

This section is devoted to study the fourth order operator:

$$T_4[p,c]\,u(t) = u^{(4)}(t) - p\,u''(t) + c(t)\,u(t), \quad t \in I, \tag{5.2.1}$$

where $p \geqslant 0$ and c is a continuous function on I.

Thus, we study the related fourth order boundary value problem

$$\begin{aligned} T_4[p,c]\,u(t) = \sigma(t) \geqslant 0, \quad t \in I, \\ u(a) = u''(a) = u(b) = u''(b) = 0, \end{aligned} \tag{5.2.2}$$

with $\sigma \in C(I)$.

This problem has already been studied in previous section. Nevertheless, in that case, c must remain between two bounds given by two eigenvalues of operator $T_4[p,0]$ in different fixed sets, see Corollary 5.1.5. On the contrary, here we allow c to pass through that values in some sense which we will specify below.

Our study is done by means of the variational approach, thus in a previous subsection we obtain the related variational approach of the considered problem. This is a technique which will be more detailed in Chapter 7.

We obtain weaker conditions for c than those obtained in Corollary 5.1.5 to make sure either the strongly inverse positive character or strongly inverse negative character of $T_4[p,c]$ in X.

5.2.1 Variational approach

In this section we obtain the variational approach of problem (5.2.2) and some results which we will use in our main results.

First, we consider the Hilbert space $H := H^2(I) \cap H_0^1(I)$, where:

$$H^2(I) = \Big\{ u \in L^2(I) \;\big|\, u', u'' \in L^2(I) \Big\},$$

and:

$$H_0^1(I) = \Big\{ u \in L^2(I) \;\big|\, u' \in L^2(I),\ u(a) = u(b) = 0 \Big\}.$$

We say that $u \in H$ is a weak solution of (5.2.2) if it satisfies:

$$\int_a^b u''(t)\,v''(t)\,\mathrm{d}\,t + p\int_a^b u'(t)\,v'(t)\,\mathrm{d}\,t + \int_a^b c(t)\,u(t)\,v(t)\,\mathrm{d}\,t = \int_a^b \sigma(t)\,v(t)\,\mathrm{d}\,t, \tag{5.2.3}$$

for all $v \in H$.

For a function $f \in C(I)$. Let us denote:

$$f_m := \min_{t \in I} f(t) \qquad \text{and} \qquad f^m := \max_{t \in I} f(t),$$

and,

$$f^{\pm}(t) = \max\Big\{0, \pm f(t)\Big\}, \quad t \in I.$$

If $p = 0$ and $a = 0$, $b = 1$, we have the following result, see [93, 96].

Proposition 5.2.1. *Let $c(t) \neq -k^4\,\pi^4$ for any $k \in \mathbb{N}$ and all $t \in [0,1]$. Let $p = 0$, $a = 0$ and $b = 1$, then the problem* (5.2.2) *has a unique solution $u \in X$. Moreover, if $-\pi^4 < c_m < 0$, then:*

$$\|u\|_{C([0,1])} \leqslant \frac{\pi}{2\,(\pi^4 + c_m)}\,\|\sigma\|_{C([0,1])}\,.$$

Now, we enunciate an equivalent result to this Proposition, which refers to our case.

Proposition 5.2.2. *Let $c(t) \neq -k^4\left(\dfrac{\pi}{b-a}\right)^4 - k^2\,p\left(\dfrac{\pi}{b-a}\right)^2$ for any $k \in \{1,2,3,\dots\}$ and all $t \in I$. Then the problem* (5.2.2)*, with $p \geqslant 0$, has a unique solution $u \in X$.*

Moreover, if $-\left(\dfrac{\pi}{b-a}\right)^4 - p\left(\dfrac{\pi}{b-a}\right)^2 < c_m < 0$, then:

$$\|u\|_{C(I)} \leqslant \frac{\pi}{2\left(\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + c_m\right)}\,\|\sigma\|_{C(I)}\,.$$

Proof. If $c(t) \neq -k^4\left(\dfrac{\pi}{b-a}\right)^4 - p\,k^2\left(\dfrac{\pi}{b-a}\right)^2$ for any $k \in \{1,2,3,\dots\}$ and $t \in I$, it means that, since $c \in C(I)$, either there exists $k \in \{1,2,3\dots\}$ such that:

$$c(t) \in \left(-(k+1)^4\left(\frac{\pi}{b-a}\right)^4 - p\,(k+1)^2\left(\frac{\pi}{b-a}\right)^2, -k^4\left(\frac{\pi}{b-a}\right)^4 - p\,k^2\left(\frac{\pi}{b-a}\right)^2\right),$$

or that $c_m > -\left(\dfrac{\pi}{b-a}\right)^4 - p\left(\dfrac{\pi}{b-a}\right)^2$, i.e. there is no any eigenvalue of $T_4[0,p]$ between c_m and c^m. As a consequence, the existence of a unique solution of problem (5.2.2) is ensured. Now, let us see the boundedness.

We have the two following Wirtinger inequalities, for every $u \in H$, (see [78, 93]):

$$\|u\|_{L^2(I)} \leqslant \frac{b-a}{\pi}\,\|u'\|_{L^2(I)} \leqslant \left(\frac{b-a}{\pi}\right)^2\|u''\|_{L^2(I)}\,, \tag{5.2.4}$$

and,

$$\|u\|_{C(I)} \leqslant \frac{\sqrt{b-a}}{2}\,\|u'\|_{L^2(I)}\,. \tag{5.2.5}$$

Now, multiplying equation (5.2.2) by the unique solution $u \in X$ and integrating, we have:

$$\int_a^b u^{(4)}(t)\,u(t)\,\mathrm{d}\,t - p\int_a^b u''(t)\,u(t)\,\mathrm{d}\,t + \int_a^b c(t)\,u^2(t)\,\mathrm{d}\,t = \int_a^b \sigma(t)\,u(t)\,\mathrm{d}\,t,$$

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which is equivalent to:

$$\int_a^b u''^2(t)\,\mathrm{d}\,t + p\int_a^b u'^2(t)\,\mathrm{d}\,t = \int_a^b \sigma(t)\,u(t)\,\mathrm{d}\,t - \int_a^b c(t)\,u^2(t)\,\mathrm{d}\,t.$$

Taking into account the inequalities (5.2.4) and that $c_m \leqslant 0$, we have:

$$\|u''\|^2_{L^2(I)} + p\,\|u'\|^2_{L^2(I)} \geqslant \left(\frac{\pi}{b-a}\right)^2 \|u'\|^2_{L^2(I)} + p\,\|u'\|^2_{L^2(I)}\,,$$

and,

$$\begin{aligned}\int_a^b \sigma(t)\,u(t)\,\mathrm{d}\,t - \int_a^b c(t)\,u^2(t)\,\mathrm{d}\,t &\leqslant \|\sigma\|_{C(I)} \int_a^b \left|u(t)\right|\mathrm{d}\,t - c_m\,\|u\|^2_{L^2(I)}\\ &\leqslant \|\sigma\|_{C(I)}\,\sqrt{b-a}\,\|u\|_{L^2(I)} - c_m\,\|u\|^2_{L^2(I)}\\ &\leqslant \|\sigma\|_{C(I)}\,\sqrt{b-a}\frac{b-a}{\pi}\,\|u'\|_{L^2(I)} - c_m\left(\frac{b-a}{\pi}\right)^2\|u'\|^2_{L^2(I)}\,.\end{aligned}$$

So, combining the two last inequalities we arrive to:

$$\left(\left(\frac{\pi}{b-a}\right)^2 + p + c_m\left(\frac{b-a}{\pi}\right)^2\right)\|u'\|_{L^2(I)} \leqslant \|\sigma\|_{C(I)}\,\sqrt{b-a}\frac{b-a}{\pi},$$

which is equivalent to:

$$\|u'\|_{L_2(I)} \leqslant \frac{\pi}{\sqrt{b-a}}\frac{\|\sigma\|_{C(I)}}{\left(\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2 + c_m\right)},$$

this combined with the inequality (5.2.5) gives our result. □

Remark 5.2.3. *We note that previous inequality includes Proposition 5.2.1 as a particular case.*

For an arbitrary non-negative continuous function $r(t) \geqslant 0$ in I, we define the scalar product for all $u,\ v \in H$:

$$(u,v) = \int_a^b u''(t)\,v''(t)\,\mathrm{d}\,t + p\int_a^b u'(t)\,v'(t)\,\mathrm{d}\,t + \int_a^b r(t)\,u(t)\,v(t)\,\mathrm{d}\,t, \qquad (5.2.6)$$

and $\|u\| = (u,u)^{1/2}$ its associated norm.

We have the following inequality:

$$\begin{aligned}\left|u(t)-u(s)\right| = \left|\int_s^t u'(r)\,dr\right| &\leqslant \sqrt{t-s}\,\|u'\|_{L^2(I)} \leqslant \sqrt{t-s}\,\frac{b-a}{\pi}\,\|u''\|_{L^2(I)}\\ &\leqslant \sqrt{t-s}\,\frac{b-a}{\pi}\,\|u\|.\end{aligned}$$

Thus, we can affirm that the embedding of H into $C(I)$ is compact.

Let $f(t)$ and $\sigma(t)$ be continuous functions on I, following the arguments shown in [47], using the Riesz Representation Theorem we can define $S_f\colon H \to H$ and $\sigma^* \in H$ such that:

$$(S_f u, v) = \int_a^b f(t)\,u(t)\,v(t)\,\mathrm{d}\,t, \quad (\sigma^*, v) = \int_a^b \sigma(t)\,v(t)\,\mathrm{d}\,t,\; u, v \in H. \tag{5.2.7}$$

Now, let us introduce some results which make a relation between norm $\|\cdot\|$ and norms $\|\cdot\|_{C(I)}$ and $\|\cdot\|_{L^2(I)}$. Such a result generalizes [47, Lemma 7].

Lemma 5.2.4. *Let $u \in H$, $r \in C(I)$, $r \geqslant 0$ in I and $\|\cdot\|$ be the norm associated to the scalar product* (5.2.6)*. Then:*

$$\|u\|_{C(I)} \leqslant \frac{1}{\sqrt{\delta_1}}\,\|u\|,$$

and,

$$\|u\|_{L^2(I)} \leqslant \frac{\|u\|}{\sqrt{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\{r(t)\}}},$$

where:

$$\delta_1 = \max\left\{\frac{4\,p}{b-a}, \frac{4\,\pi^2}{(b-a)^{3/2}}\right\}. \tag{5.2.8}$$

Proof. Using the inequalities given in (5.2.4)-(5.2.5), we have that the two following inequalities are satisfied:

$$\begin{aligned}
p\,\|u\|^2_{C(I)} &\leqslant \frac{b-a}{4}\,p\int_a^b \left(u'(t)\right)^2 \mathrm{d}\,t\\
&\leqslant \frac{b-a}{4}\left(\int_a^b \left(u''(t)\right)^2 \mathrm{d}\,t + p\int_a^b \left(u'(t)\right)^2 \mathrm{d}\,t + \int_a^b r(t)\,u^2(t)\,\mathrm{d}\,t\right)\\
&= \frac{b-a}{4}\,\|u\|^2,
\end{aligned}$$

$$\begin{aligned}
\|u\|^2_{C(I)} &\leqslant \frac{b-a}{4}\int_a^b \left(u'(t)\right)^2 \mathrm{d}\,t \leqslant \frac{(b-a)^3}{4\,\pi^2}\int_a^b \left(u''(t)\right)^2 \mathrm{d}\,t\\
&\leqslant \frac{(b-a)^3}{4\,\pi^2}\left(\int_a^b \left(u''(t)\right)^2 \mathrm{d}\,t + p\int_a^b \left(u'(t)\right)^2 \mathrm{d}\,t + \int_a^b r(t)\,u^2(t)\,\mathrm{d}\,t\right)\\
&= \frac{(b-a)^3}{4\,\pi^2}\,\|u\|^2.
\end{aligned}$$

So, if $p \neq 0$:

$$\|u\|_{C(I)} \leqslant \min\left\{\sqrt{\frac{b-a}{4\,p}}, \frac{\sqrt{(b-a)^3}}{2\,\pi}\right\}\|u\| = \frac{1}{\sqrt{\delta_1}}\,\|u\|,$$

moreover, if $p=0$:

$$\|u\|_{C(I)} \leqslant \frac{\sqrt{(b-a)^3}}{2\,\pi}\,\|u\| = \frac{1}{\sqrt{\delta_1}}\,\|u\|.$$

On another hand, from equation (5.2.4), we have:

$$\begin{aligned}
\|u\|^2_{L^2(I)} &= \frac{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\{r(t)\}}{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\{r(t)\}} \int_a^b u^2(t)\,\mathrm{d}\,t \\
&\leqslant \frac{\displaystyle\int_a^b \big(u''(t)\big)^2\,\mathrm{d}\,t + p\int_a^b \big(u'(t)\big)^2\,\mathrm{d}\,t + \int_a^b r(t)\,u^2(t)\,\mathrm{d}\,t}{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\{r(t)\}} \\
&= \frac{\|u\|^2}{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\{r(t)\}}.
\end{aligned}$$

□

From classical arguments, see [13], we obtain the following result, where we see that a weak solution of (5.2.3) in H under suitable conditions is, indeed, a classical solution of (5.2.2) in X.

Proposition 5.2.5. *Suppose that $c,\sigma\in C(I)$ and $u\in H$ is a weak solution of* (5.2.3)*, then u is a classical solution of* (5.2.2) *in X.*

Proof. Since u is a weak solution of (5.2.3), we have:

$$\int_a^b u''(t)\,v''(t)\,\mathrm{d}\,t = \int_a^b \Big(\sigma(t) - c(t)\,u(t) - p\,u''(t)\Big)\,v(t)\,\mathrm{d}\,t, \quad \forall v\in H.$$

From this, we conclude that u'' has its second derivative in distributional sense, so we obtain, for every $v\in H$ with compact support, that:

$$\int_a^b u^{(4)}(t)\,v(t)\,\mathrm{d}\,t = \int_a^b \Big(\sigma(t) - c(t)\,u(t) - p\,u''(t)\Big)\,v(t)\,\mathrm{d}\,t, \quad \forall v\in H.$$

Hence, we conclude:

$$u^{(4)}(t) - p\,u''(t) + c(t)\,u(t) = \sigma(t), \quad \text{a.e. } t\in I.$$

From this, since $c,\sigma,u,u''\in C(I)$, we deduce that $u\in C^4(I)$ and the previous equality is fulfilled for all $t\in I$.

Trivially, since $u\in H$, we have $u(a)=u(b)=0$.

Now, by integrating (5.2.3), we obtain:

$$u''(a)\,v'(a) - u''(b)\,v'(b) + \int_a^b \Big(u^{(4)}(t) - p\,u''(t) + c(t)\,u(t)\Big)\,v(t)\,\mathrm{d}\,t = \int_a^b \sigma(t)\,v(t)\,\mathrm{d}\,t,$$

which is applicable for any arbitrary $v\in H$, then we conclude $u''(a)=u''(b)=0$. □

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Next result improves [47, Lemma 8].

Lemma 5.2.6. *Let $S_f \colon H \to H$ be the operator previously defined in* (5.2.7)*. Then,*

$$\|S_f\| \leqslant \frac{1}{\delta_1} \int_a^b |f(t)| \,\mathrm{d}\,t.$$

Proof. Using Lemma 5.2.4 we can deduce the following inequalities which prove the result:

$$\begin{aligned}\|S_f\| &= \sup_{\|u\|=1} \|S_f\, u\| = \sup_{\|u\|=1} \sup_{\|v\|=1} \left| \int_a^b f(t)\, u(t)\, v(t) \,\mathrm{d}\,t \right| \leqslant \sup_{\|u\|=1} \sup_{\|v\|=1} \int_a^b \big|f(t)\big| \, \big|u(t)\big| \, \big|v(t)\big| \,\mathrm{d}\,t \\ &\leqslant \sup_{\|u\|=1} \|u\|_{C(I)} \sup_{\|v\|=1} \|v\|_{C(I)} \int_a^b \big|f(t)\big| \,\mathrm{d}\,t \leqslant \frac{1}{\delta_1} \int_a^b \big|f(t)\big| \,\mathrm{d}\,t.\end{aligned}$$

□

Repeating the previous argument, we have:

$$\|S_f(u_n - u_m)\| \leqslant \frac{1}{\sqrt{\delta_1}} \int_a^b \big|f(t)\big| \,\mathrm{d}\,t \; \|u_n - u_m\|_{C(I)}\,.$$

Thus, from the compact embedding of H into $C(I)$, we can affirm that $S_f : H \to H$ is a compact operator.

The proof of next result is analogous to [47, Lemma 9].

Lemma 5.2.7. *Let $\sigma^* \in H$ previously defined in* (5.2.7)*. Then:*

$$\|\sigma^*\| \leqslant \sqrt{\frac{b-a}{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\big\{r(t)\big\}}}\; \|\sigma\|_{C(I)}\,.$$

Proof. Using the Hölder inequality and the second part of Lemma 5.2.4, we can achieve the following inequality which ensures the result.

$$\begin{aligned}\|\sigma^*\| &= \sup_{\|u\|=1} \left| \int_a^b \sigma(t)\, u(t) \,\mathrm{d}\,t \right| \leqslant \max_{t\in I} \sigma(t) \sup_{\|u\|=1} \int_a^b |u(t)| \,\mathrm{d}\,t \\ &\leqslant \|\sigma\|_{C(I)}\; (b-a)^{1/2} \sup_{\|u\|=1} \|u\|_{L^2(I)} \\ &\leqslant \sqrt{\frac{b-a}{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\big\{r(t)\big\}}}\; \|\sigma\|_{C(I)}\,.\end{aligned}$$

□

5.2.2 Strongly inverse positive (negative) character of $T_4[p,c]$ in X.

This section is devoted to prove maximum and anti-maximum principles for the previously introduced problem (5.2.2). These results generalise those obtained in [46, 47] for $p = 0$. The proofs follow similar arguments to the ones given in such articles. We point out that on them there is no reference to spectral theory. The results here presented are collected in [33].

First, we obtain the results for the homogeneous case and, once we have done that, we extrapolate them for the non-homogeneous boundary conditions (5.1.4) by means of Theorem 4.4.4.

The first of them ensures the existence of a unique solution of the problem under certain hypotheses and gives sufficient conditions to ensure that the operator (5.2.1) is strongly inverse positive in X.

Theorem 5.2.8. *Let $c, \sigma \in C(I)$ be such that:*

$$\int_a^b c^-(t)\,\mathrm{d}\,t < \delta_1,$$

where δ_1 has been defined in (5.2.8)*. Then problem* (5.2.2) *has a unique classical solution $u \in X$ and there exists $R > 0$ (depending on c and p) such that:*

$$\|u\|_{C(I)} \leqslant R\,\|\sigma\|_{C(I)}.$$

Moreover, if $c(t) \leqslant -\lambda_2^p$, for every $t \in I$, then $T_4[p,c]$ is strongly inverse positive in X.

Proof. First, we decompose $c(t) = c^+(t) - c^-(t)$. And, we write problem (5.2.2) as follows:

$$\begin{aligned} &u^{(4)}(t) - p\,u''(t) + c^+(t)\,u(t) = c^-(t)\,u(t) + \sigma(t), \quad t \in I,\\ &u(a) = u(b) = u''(a) = u''(b) = 0. \end{aligned}$$

If we denote $r(t) := c^+(t)$ and $f(t) := c^-(t)$, we have that the weak formulation of problem (5.2.2) is given in the following way:

$$u = S_{c^-}\,u + \sigma^*, \quad u \in H, \tag{5.2.9}$$

with the scalar product $(\cdot,\cdot)$ previously defined in (5.2.6).

Using Lemma 5.2.6 we have:

$$\|S_{c^-}\| \leqslant \frac{1}{\delta_1}\int_a^b \left|c^-(t)\right|\mathrm{d}\,t = \frac{1}{\delta_1}\int_a^b c^-(t)\,\mathrm{d}\,t < \frac{1}{\delta_1}\,\delta_1 = 1.$$

Hence, S_{c^-} is a contractive operator and there exists a unique weak solution $u \in H$. From Proposition 5.2.5, $u \in X$ is a classical solution of (5.2.2) in X.

Now, using (5.2.9), we obtain:

$$\|u\| = \|S_{c^-}\,u + \sigma^*\| \leqslant \|S_{c^-}\|\;\|u\| + \|\sigma^*\|,$$

then,

$$\|u\| \leqslant \frac{1}{1 - \|S_{c^-}\|}\;\|\sigma^*\|.$$

By another hand, using Lemmas 5.2.4 and 5.2.7:

$$\begin{aligned}\|u\|_{C(I)} &\leqslant \frac{1}{\sqrt{\delta_1}}\,\|u\| \leqslant \frac{1}{\sqrt{\delta_1}\big(1-\|S_{c^-}\|\big)}\,\|\sigma^*\| \\ &\leqslant \frac{\sqrt{b-a}}{\sqrt{\delta_1}\big(1-\|S_{c^-}\|\big)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\{c^+(t)\}}}\,\|\sigma\|_{C(I)} \\ &=: R\,\,\|\sigma\|_{C(I)}\,.\end{aligned}$$

Moreover, from Lemma 5.2.6, we know that:

$$R \leqslant \frac{\sqrt{b-a}}{\frac{1}{\sqrt{\delta_1}}\left(\delta_1 - \int_a^b c^-(t)\,\mathrm{d}\,t\right)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2 + \min\limits_{t\in I}\{c^+(t)\}}}\,.$$

The proof behind here is just the same as in the particular case of $p=0$, which is shown in [47, Theorem 4]. For convenience of the reader, we describe the different steps.

Let us see that if $c(t) \leqslant -\lambda_2^p$ for every $t\in[a,b]$, then $T_4[p,c]$ is strongly inverse positive in X.

We can assume that there exists $t\in[a,b]$ such that $c^-(t)\neq 0$, because if $c^-(t)=0$ for every $t\in[a,b]$, then the result follows from Corollary 5.1.5.

We construct an increasing sequence $\{u_n\}\in X$ using the following recurrence formula:

$$u_0 \equiv 0 \text{ and } T_4[p,c^+]\,u_{n+1}(t) = c^-(t)\,u_n(t) + \sigma(t) \text{ for } n = 0,1,2,\ldots$$

Since $c(t) \leqslant -\lambda_2^p$ for every $t\in[a,b]$, we can apply Corollary 5.1.5 to affirm that $T_4[p,c^+]$ is strictly inverse positive in X. Then, we have:

- For $n=0$, $T_4[p,c^+]u_1(t) = \sigma(t) \gneqq 0$, then $u_1(t)>0$ on (a,b) and, moreover, $u_1'(a)>0$ and $u_1'(b)<0$.
- For $n=1$, $T_4[p,c^+]u_2(t) = c^-(t)\,u_1(t)+\sigma(t) \gneqq 0$, so $u_2(t)>0$ on (a,b), $u_2'(a)>0$ and $u_2'(b)<0$.

Repeating this process, we arrive at the conclusion that for every $n\in\mathbb{N}$, $u_n(t)>0$ in (a,b), $u_n(a)>0$ and $u_n(b)<0$.

Moreover:

- $T_4[p,c^+]\big(u_2(t)-u_1(t)\big) = T_4[p,c^+]u_2(t) - T_4[p,c^+]u_1(t) = c^-(t)\,u_1(t) \gneqq 0$, then $u_2>u_1$ on (a,b), $u_2'(a)>u_1'(a)$ and $u_2'(b)<u_1'(b)$.
- $T_4[p,c^+]\big(u_3(t)-u_2(t)\big) = T_4[p,c^+]u_3(t) - T_4[p,c^+]u_2(t) = c^-(t)\big(u_2(t)-u_1(t)\big) \gneqq 0$, so $u_3>u_2$ on (a,b), $u_3'(a)>u_2'(a)$ and $u_3'(b)<u_2'(b)$.

Then, we arrive to $u_{n+1}>u_n$ on (a,b), $u_{n+1}'(a)>u_n'(a)$ and $u_{n+1}'(b)<u_{n+1}'(b)$, for every $n\geqslant 0$. Then $\{u_n\}_{n=0}^{+\infty}$ is an increasing sequence.

Now, let us see that it is a bounded sequence in H. We have that:

$$u_{n+1} = S_{c^-}\, u_n + \sigma^*, \text{ for } n = 0, 1, 2, \dots, \text{ with } u_0 = 0.$$

Then, we have:

- $u_1 = S_{c^-} u_0 + \sigma^* = \sigma^*$.
- $u_2 = S_{c^-} u_1 + \sigma^* = S_{c^-} \sigma^* + \sigma^*$.

Arguing as before, repeating the process, we arrive at:

$$u_{n+1} = S_{c^-}^n \sigma^* + \cdots + S_{c^-} \sigma^* + \sigma^*, \quad n = 0, 1, \dots$$

Hence,

$$\|u_{n+1}\| \leqslant \big(\|S_{c^-}\|^n + \cdots + \|S_{c^-}\| + 1 \big)\, \|\sigma^*\| = \frac{1 - \|S_{c^-}\|^{n+1}}{1 - \|S_{c^-}\|}\, \|\sigma^*\|,$$

since $\|S_{c^-}\| < 1$, we can conclude:

$$\|u_n\| \leqslant \frac{1}{1 - \|S_{c^-}\|}\, \|\sigma^*\|, \quad n = 1, 2, \dots$$

Then $\{u_n\}_{n=0}^{+\infty}$ is a bounded sequence with the norm of H. Since S_{c^-} is compact, this implies that $\{u_n\}_{n=0}^{+\infty}$ is relatively compact in H. The monotonicity of S_{c^-} implies that there exists $u \in H$ such that $\lim\limits_{n\to\infty} \|u_n - u\| = 0$ and this is the unique weak solution of (5.2.3) in H, but since $h, c \in C(I)$ this is a classic solution in X which satisfies $u > 0$ in (a, b) and, moreover $u'(a) > 0$ and $u'(b) < 0$, then $T_4[p, c]$ is strongly inverse positive in X. □

Now, we introduce a result which also gives sufficient conditions to ensure the existence of solution for our problem and, moreover, it makes sure that operator $T_4[p, c]$ is strongly inverse negative in X.

Theorem 5.2.9. *Let $c, \sigma \in C(I)$ be such that:*

$$-16 \left(\frac{\pi}{b-a}\right)^4 - 4\,p \left(\frac{\pi}{b-a}\right)^2 < c_m < -\left(\frac{\pi}{b-a}\right)^4 - p \left(\frac{\pi}{b-a}\right)^2,$$

and,

$$\int_a^b \big(c(t) - c_m\big)\, \mathrm{d}\,t < \delta_1\, \delta_2,$$

where δ_1 is defined on (5.2.8) *and:*

$$\delta_2 = \min\left\{ -1 - \frac{c_m}{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}, 1 + \frac{c_m}{16\left(\frac{\pi}{b-a}\right)^4 + 4\,p\left(\frac{\pi}{b-a}\right)^2} \right\}. \quad (5.2.10)$$

Then problem (5.2.2) *has a unique classical solution on X and there exists $R > 0$ (depending on c and p) such that:*

$$\|u\|_{C(I)} \leqslant R\,\|\sigma\|_{C(I)}\,.$$

Moreover, if $c_m \geqslant -\lambda_3^p$, then $T_4[p,c]$ is strongly inverse negative in X.

Proof. We rewrite problem (5.2.2) in the following way:

$$\begin{aligned} u^{(4)}(t) - p\,u''(t) + c_m\,u(t) &= \big(c_m - c(t)\big)\,u(t) + \sigma(t), \quad t \in I,\\ u(a) = u(b) = u''(a) = u''(b) &= 0. \end{aligned}$$

In this case, we consider $r(t) \equiv 0$ and we have that the weak formulation is equivalent to

$$u + S_{c_m}\,u = S_{c_m - c}\,u + \sigma^*. \tag{5.2.11}$$

Since $c_m \in \left(-16\left(\dfrac{\pi}{b-a}\right)^4 - 4\,p\left(\dfrac{\pi}{b-a}\right)^2, -\left(\dfrac{\pi}{b-a}\right)^4 - p\left(\dfrac{\pi}{b-a}\right)^2\right)$, we have that $I + S_{c_m}$ is invertible in H. Then, we can write:

$$u = (I + S_{c_m})^{-1}\,(S_{c_m - c}\,u + \sigma^*). \tag{5.2.12}$$

Moreover $\left\|(I + S_{c_m})^{-1}\right\| = \dfrac{1}{\delta_2}$, where δ_2 is the distance from -1 to the spectrum of S_{c_m}, see [85], i.e., δ_2 is defined by (5.2.10). Since $\displaystyle\int_a^b \big(c(t) - c_m\big)\,\mathrm{d}\,t < \delta_1\,\delta_2$ and using Lemma 5.2.6, we have:

$$\|S_{c_m - c}\| \leqslant \frac{1}{\delta_1}\int_a^b \big(c(t) - c_m\big)\,\mathrm{d}\,t < \frac{\delta_1\,\delta_2}{\delta_1} = \delta_2,$$

so,

$$\left\|(I + S_{c_m})^{-1}\,S_{c_m - c}\right\| \leqslant \left\|(I + S_{c_m})^{-1}\right\|\,\|S_{c_m - c}\| < \frac{\delta_2}{\delta_2} = 1.$$

Thus, as in Theorem 5.2.8, we can use the contractive character of $(I + S_{c_m})^{-1}\,S_{c_m - c}$, to ensure that there exists a unique weak solution of (5.2.2), $u \in H$, from Proposition 5.2.5, it is also a classical solution $u \in X$.

Now, using (5.2.12), we have:

$$\|u\| \leqslant \left\|(I + S_{c_m})^{-1}\right\|\,\|S_{c_m - c}\,u + \sigma\| \leqslant \frac{1}{\delta_2}\,\|S_{c_m - c}\|\,\|u\| + \frac{1}{\delta_2}\,\|\sigma^*\|.$$

As a consequence, we deduce that:

$$\|u\| \leqslant \frac{\|\sigma^*\|}{\delta_2 - \|S_{c_m - c}\|},$$

and, combining this inequality with Lemmas 5.2.4 and 5.2.7, we have:

$$\|u\|_{C(I)} \leqslant \frac{1}{\sqrt{\delta_1}}\,\|u\| \leqslant \frac{\|\sigma^*\|}{\sqrt{\delta_1}\big(\delta_2 - \|S_{c_m-c}\|\big)}$$

$$\leqslant \frac{1}{\sqrt{\delta_1}\big(\delta_2 - \|S_{c_m-c}\|\big)} \sqrt{\frac{b-a}{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}}\; \|\sigma\|_{C(I)}$$

$$:= R\;\; \|\sigma\|_{C(I)}\,.$$

Moreover, from Lemma 5.2.6:

$$R \leqslant \frac{1}{\frac{1}{\sqrt{\delta_1}}\left(\delta_1\delta_2 - \displaystyle\int_a^b \big(c(t)-c_m\big)\,\mathrm{d}\,t\right)} \sqrt{\frac{b-a}{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}}.$$

The proof of the fact that while $-\lambda_3^p \leqslant c_m \leqslant -\left(\frac{\pi}{b-a}\right)^4 - \left(\frac{\pi}{b-a}\right)^2$, $T_4[p,c]$ is strongly inverse negative is just the same as [47, Theorem 5].

In the sequel, we describe the steps there shown.

We use the recurrence formula:

$$u_0 \equiv 0 \text{ and } T_4[p,c_m]u_{n+1}(t) = \big(c_m - c(t)\big)\,u_n(t) + \sigma(t),\; \text{ for } n = 0,1,2,\dots$$

Applying Corollary 5.1.5, we know that $T_4[p,c_m]$ is inverse negative in X.

Since $c_m - c(t) \leqslant 0$, for every $t \in [a,b]$, we obtain, as in Theorem 5.2.8, the decreasing sequence with negative terms in (a,b), $\{u_n\}$, satisfying $u_n'(a) < 0$ and $u_n'(b) > 0$.

For every $n = 0,1,2,\dots$, we can express:

$$u_{n+1} = (I + S_{c_m})^{-1}\,(S_{c_m-c}u_n + \sigma^*).$$

Now, let us see that it is a bounded sequence in H.

- For $n = 0$, $u_1 = (I + S_{c_m})^{-1}h^*$, we have:

$$\|u_1\| = \big\|(I+S_{c_m})^{-1}\sigma^*\big\| \leqslant \big\|(I+S_{c_m})^{-1}\big\|\,\|\sigma^*\| = \frac{\|\sigma^*\|}{\delta_2}.$$

- For $n = 1$, $u_2 = \big(I + S_{c_m})^{-1}\,(S_{c_m-c}(I+S_{c_m})^{-1}\sigma^* + \sigma^*\big)$. So, we have:

$$\begin{aligned}\|u_2\| &= \Big\|(I+S_{c_m})^{-1}\,\big(S_{c_m-c}(I+S_{c_m})^{-1}\sigma^* + \sigma^*\big)\Big\| \\ &\leqslant \big\|(I+S_{c_m})^{-1}\big\|\,\Big(\,\|S_{c_m-c}\|\,\big\|(I+S_{c_m})^{-1}\big\|\,\|\sigma^*\| + \|\sigma^*\|\,\Big) \\ &= \frac{1}{\delta_2}\left(\|S_{c_m-c}\|\,\frac{1}{\delta_2} + 1\right)\|\sigma^*\|,\end{aligned}$$

If we repeat this process up to the term $n+1$, we have:

$$\begin{aligned}\|u_{n+1}\| &\leqslant \left(\frac{1}{\delta_2^n}\,\|S_{c_m-c}\|^n + \cdots + \frac{1}{\delta_2}\,\|S_{c_m-c}\| + 1\right)\frac{1}{\delta_2}\,\|\sigma^*\| \\ &= \frac{1-\dfrac{1}{\delta_2^{n+1}}\,\|S_{c_m-c}\|^{n+1}}{1-\dfrac{1}{\delta_2}\,\|S_{c_m-c}\|}\,\frac{1}{\delta_2}\,\|\sigma^*\| \leqslant \frac{1}{1-\dfrac{1}{\delta_2}\,\|S_{c_m-c}\|}\,\frac{1}{\delta_2}\,\|\sigma^*\|,\end{aligned}$$

where we have used that $\|S_{c_m-c}\| < \delta_2$.

Then $\{u_n\}$ is a bounded sequence in H. Hence, using the compactness of the operator $(I+S_{c_m})^{-1}\,S_{c_m-c}$, we can affirm that there exists $u \in H$, such that $\lim\limits_{n\to\infty}\|u_n-u\| = 0$ and it is the unique weak solution of (5.2.3) in H. Since $c,\,h \in C(I)$, it is also the unique classical solution of (5.2.2) in X. Moreover, $u<0$ in (a,b), $u'(a)<0$ and $u'(b)>0$. Then $T_4[p,c]$ is strongly inverse negative and the result is proved. □

From Lemma 4.4.3 and Theorem 4.4.4, we can now state two results for the non-homogeneous case.

$$\begin{aligned} T_4[p,c]\,u(t) = \sigma(t) \geqslant 0, \quad t\in I,\\ u(a)=u(b)=0,\ u''(a)\leqslant 0,\ u''(b)\leqslant 0,\end{aligned} \tag{5.2.13}$$

with $\sigma \in C(I)$.

Theorem 5.2.10. *Let $c,\sigma \in C(I)$ be such that:*

$$\int_a^b c^-(t)\,\mathrm{d}\,t < \delta_1,$$

where δ_1 has been defined in (5.2.8). *Then problem* (5.2.13) *has a unique classical solution $u \in \tilde{X}$.*

Moreover, if $c(t) \leqslant -\lambda_2^p$, for every $t\in I$, then $T_4[p,c]$ is strongly inverse positive in $\tilde{X}$.

Theorem 5.2.11. *Let $c,\sigma \in C(I)$ be such that:*

$$-16\left(\frac{\pi}{b-a}\right)^4 - 4\,p\left(\frac{\pi}{b-a}\right)^2 < c_m < -\left(\frac{\pi}{b-a}\right)^4 - p\left(\frac{\pi}{b-a}\right)^2,$$

and

$$\int_a^b \left(c(t)-c_m\right)\mathrm{d}\,t < \delta_1\,\delta_2,$$

where δ_1 and δ_2 are defined in (5.2.8) *and* (5.2.10)*, respectively.*

Then problem (5.2.13) *has a unique classical solution on $\tilde{X}$.*

Moreover, if $c_m \geqslant -\lambda_3^p$, then $T_4[p,c]$ is strongly inverse negative in $\tilde{X}$.

5.2.3 Maximum and anti-maximum principles for problem (5.2.2)

In this section, even though we are not able to ensure the strongly inverse positive character of the operator $T_4[p,c]$ on X, we can make sure that, under the hypothesis that $\sigma(t) > 0$ for every $t \in I$, then problem (5.2.2) has a unique positive (resp. negative) solution in X. The proofs follow similar steps to the ones given in [46] and they are also published in [33].

Theorem 5.2.12. *Let $\sigma \in C(I)$ be a function such that $0 < \sigma_m \leqslant \sigma^m$. Let $c \in C(I)$ be a function that satisfies one of the two following hypotheses:*

(1) $-\left(\frac{\pi}{b-a}\right)^4 - p\left(\frac{\pi}{b-a}\right)^2 < c_m \leqslant 0$ *and* $c^m \leqslant -\lambda_2^p + \frac{\sigma_m}{\sigma^m}\,\frac{2}{\pi}\left(\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2 + c_m\right)$.

(2) $\displaystyle\int_a^b c^-(t)\,\mathrm{d}\,t < \delta_1$, *with δ_1 defined in equation* (5.2.8), *and:*

$$c^m \leqslant -\lambda_2^p + \frac{\sigma_m}{\sigma^m}\frac{1}{\sqrt{\delta_1}}\left(\delta_1 - \int_a^b c^-(t)\,\mathrm{d}\,t\right)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\left\{c^+(t), -\lambda_2^p\right\}},$$

then problem (5.2.2) *has a unique positive solution in X.*

Proof. The existence of a unique solution follows from Proposition 5.2.2 on the first case and from Theorem 5.2.8 on the second one.

Now, let us see that this solution, u, is positive on (a,b).

Let us assume that $c^m > -\lambda_2^p$. If $c^m \leqslant -\lambda_2^p$, we can apply either Corollary 5.1.5 $\big($in case (1)$\big)$ or Theorem 5.2.8 $\big($in (2)$\big)$ to affirm that $T_4[p,c]$ is strongly inverse positive on X.

Let $d(t) := \min\left\{c(t), -\lambda_2^p\right\}$ be a continuous function such that $c_m \leqslant d(t) \leqslant -\lambda_2^p$. We transform the equation (5.2.2) in the following equivalent one:

$$T_4[p,d]\,u(t) = \sigma(t) - \big(c(t) - d(t)\big)\,u(t), \quad t \in I,$$

and we consider the next recurrence formula:

$$T_4[p,d]\,u_{n+1} = \sigma(t) - \big(c(t) - d(t)\big)\,u_n, \quad t \in I,\ n = 0,1,2,\dots$$

Since, $d(t) \leqslant -\lambda_2^p$ for every $t \in I$, $T_4[p,d]$ is a strongly inverse positive operator in X.

We choose $u_0 \equiv 0$ and we have $T_4[p,d]\,u_1 = \sigma(t)$. Since $T_4[p,d]$ is strongly inverse positive, $u_1 > 0$ in (a,b), $u_1'(a) > 0$ and $u_1'(b) < 0$.

Now, using that $u_1 \in X$ is the unique solution of $T_4[p,d]\,u(t) = \sigma(t)$ for $t \in I$, we deduce that:

- If c satisfies (1), we can apply Proposition 5.2.2 to affirm:

$$\|u_1\|_{C(I)} \leqslant \frac{\pi}{2\left(\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2 + c_m\right)}\,\sigma^m.$$

Since $c(t) - d(t) \leqslant c^m + \lambda_2^p$, using the hypotheses, we have:

$$\begin{aligned} T_4[p,d]\,u_2 = \sigma(t) - \big(c(t) - d(t)\big)\,u_1 &\geqslant \sigma_m - \big(c^m + \lambda_2^p\big)\,\|u_1\|_{C(I)} \\ &\geqslant \sigma_m - \frac{\sigma_m}{\sigma^m}\,\frac{1}{R}\,\sigma^m\,R = 0, \end{aligned} \tag{5.2.14}$$

where R is defined by:

$$R := \frac{\pi}{2\left(\left(\dfrac{\pi}{b-a}\right)^4 + \left(\dfrac{\pi}{b-a}\right)^2 + c_m\right)}.$$

- If c satisfies (2), we look at the proof of the Theorem 5.2.8 to conclude that:

$$\|u_1\|_{C(I)} \leqslant \frac{\sigma^m}{\dfrac{1}{\sqrt{\delta_1}}\left(\delta_1 - \displaystyle\int_a^b c^-(t)\,\mathrm{d}\,t\right)\sqrt{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \displaystyle\min_{t\in I}\left\{c^+(t), -\lambda_2^p\right\}}}.$$

Proceeding as in (1), we can see that (5.2.14) is fulfilled, in this case, for:

$$R := \frac{1}{\dfrac{1}{\sqrt{\delta_1}}\left(\delta_1 - \displaystyle\int_a^b c^-(t)\,\mathrm{d}\,t\right)\sqrt{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2 + \displaystyle\min_{t\in I}\left\{c^+(t), -\lambda_2^p\right\}}}.$$

From here, the proof is equal than in the case that $p = 0$, see [46, Theorem 4].

Despite this, for convenience of the reader, we describe the used arguments.

Since $u_1(0) = 0$ and $\sigma(0) > 0$, we have $T_4[p,d]\,u_2(t) \gneqq 0$, for every $t \in [a,b]$, then, from the strongly inverse positive character, $u_2 > 0$ on (a,b), $u_2'(a) > 0$ and $u_2'(b) < 0$.

Moreover:

$$T_4[p,d]\,\big(u_1(t) - u_2(t)\big) = \big(c(t) - d(t)\big)u_1(t) \gneqq 0, \quad t \in I,$$

hence, $u_1(t) > u_2(t)$ for $t \in (a,b)$, $u_1'(a) > u_2'(a)$ and $u_1'(b) < u_2'(b)$.

Now,

$$T_4[p,d]\,u_3(t) = \sigma(t) - \big(c(t) - d(t)\big)\,u_2(t) > \sigma(t) - \big(c(t) - d(t)\big)\,u_1(t) \gneqq 0, \quad t \in I,$$

so, $u_3 > 0$ on (a,b), $u_3'(a) > 0$ and $u_3'(b) < 0$.

It is satisfied:

$$T_4[p,d]\,\big(u_3(t) - u_2(t)\big) = \big(c(t) - d(t)\big)\big(u_1(t) - u_2(t)\big) \gneqq 0, \quad t \in I,$$

then, $u_3(t) > u_2(t)$ for $t \in (a,b)$, $u_3'(a) > u_2'(a)$ and $u_3'(b) < u_2'(b)$.

Similarly:

$$T_4[p,d]\,\big(u_1(t) - u_3(t)\big) = \big(c(t) - d(t)\big)u_2(t) \gneqq 0, \quad t \in I,$$

then, $u_1(t) > u_3(t)$ for $t \in (a,b)$, $u_1'(a) > u_3'(a)$ and $u_1'(b) < u_3'(b)$.

We continue by induction applying the recurrence formula and we obtain a sequence $\{u_n\}_{n=0}^{+\infty} \subset X$ such that:

$$0 = u_0 < u_2 < u_4 < \ldots < u_5 < u_3 < u_1, \quad t \in (a,b),$$
$$\begin{cases} 0 < u_{2k}'(a) < u_{2k+2}'(a), \\ u_{2k+2}'(b) < u_{2k}'(b) < 0, \\ 0 < u_{2k+1}'(a) < u_{2k-1}'(a), \\ u_{2k-1}'(b) < u_{2k+1}'(b) < 0, \end{cases} \quad k = 1, 2, \ldots\ldots$$

The sub-sequences $\left\{u_{2k}\right\}_{k=0}^{+\infty}$ and $\left\{u_{2k+1}\right\}_{k=0}^{+\infty}$ are strictly monotone and bounded in $C(I)$.

Since the embedding of X into $C(I)$ is compact, we have that $T_4[p,c]^{-1}$ is compact. As a consequence, there exist $\underline{u}$ and $\overline{u}$ in X such that:

$$\lim_{k\to+\infty} \|u_{2k} - \underline{u}\|_{C(I)} = 0 \quad \text{and} \quad \lim_{k\to+\infty} \|u_{2k+1} - \overline{u}\|_{C(I)} = 0.$$

Moreover, since we know that there is a unique solution of (5.2.2) in X, we can conclude that $u := \underline{u} = \overline{u}$ and it satisfies $u > 0$ on (a,b), $u'(a) > 0$ and $u'(b) < 0$.

Hence, the result is proved. $\square$

Theorem 5.2.13. *Let $\sigma \in C(I)$ be a function such that $0 < \sigma_m \leqslant \sigma^m$. Let $c \in C(I)$ be a function that satisfies:*

$$\int_a^b \big(c(t) - c_m\big)\,\mathrm{d}\,t < \delta_1\,\delta_2,$$

where δ_1 and δ_2 have been defined in equations (5.2.8) and (5.2.10), respectively, and:

$$-\lambda_3^p - \frac{\sigma_m}{\sigma^m}\left(\frac{1}{\sqrt{\delta_1}}\left(\delta_1\,\delta_2 - \int_a^b \big(c(t)-c_m\big)\,\mathrm{d}\,t\right)\right)\sqrt{\frac{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}{b-a}} \leqslant c_m < -\left(\frac{\pi}{b-a}\right)^4 - \left(\frac{\pi}{b-a}\right)^2,$$

then problem (5.2.2) *has a unique negative solution in X.*

Proof. The existence of a unique solution, $u \in X$, is given by Theorem 5.2.9.

To see that $u < 0$ in (a,b), we assume that $c_m < -\lambda_3^p$. On the contrary, if $c_m \geqslant -\lambda_3^p$, using Theorem 5.2.9 we know that $T_4[p,c]$ is strongly inverse negative and the result follows directly.

Let $e(t) := \max\left\{c(t), -\lambda_3^p\right\}$ be a continuous function on I such that $-\lambda_3^p \leqslant e(t) \leqslant c^m$, and we write the equation (5.2.2) in an equivalent form:

$$T_4[p,e]\,u(t) = \sigma(t) - \big(c(t) - e(t)\big)\,u(t), \quad t \in I,$$

which, from Theorem 5.2.9, is an strongly inverse negative operator on X, and we consider the recurrence formula:

$$T_4[p,e]\,u_{n+1}(t) = \sigma(t) - \big(c(t) - e(t)\big)\,u_n(t), \quad t \in I,\ n = 0, 1, 2, \ldots$$

As in the proof of Theorem 5.2.12, we choose $u_0 \equiv 0$ and we have:

$$T_4[p,e]\,u_1(t) = \sigma(t) > 0, \quad t \in I,$$

then $u_1 < 0$ on (a,b), $u_1'(a) < 0$ and $u_1'(b) > 0$.

Since, u_1 is the unique solution of problem $T_4[p,e]\,u = \sigma$ in X, using Theorem 5.2.9, we have:

$$\|u_1\|_{C(I)} \leqslant \frac{\sigma^m}{\dfrac{1}{\sqrt{\delta_1}}\left(\delta_1\,\delta_2 - \displaystyle\int_a^b (c(t) - c_m)\,\mathrm{d}\,t\right)}\sqrt{\frac{b-a}{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2}}.$$

Fixing:

$$R := \frac{1}{\dfrac{1}{\sqrt{\delta_1}}\left(\delta_1\,\delta_2 - \displaystyle\int_a^b (c(t) - c_m)\,\mathrm{d}\,t\right)}\sqrt{\frac{b-a}{\left(\dfrac{\pi}{b-a}\right)^4 + p\left(\dfrac{\pi}{b-a}\right)^2}},$$

and, taking into account that $0 \geqslant c(t) - e(t) \geqslant c_m + \lambda_3^p$, for all $t \in I$, and $u_1 < 0$ on I, using the inferior bound of c_m, we arrive to:

$$\begin{aligned} T_4[p,e]u_2(t) &= \sigma(t) - \big(c(t) - e(t)\big)u_1(t) \geqslant \sigma_m - \big(c_m + \lambda_3^p\big)u_{1m} \\ &= \sigma_m + \big(c_m + \lambda_3^p\big)\,\|u_1\|_{C(I)} \geqslant \sigma_m + \big(c_m + \lambda_3^p\big)\frac{\sigma^m}{R} = 0, \quad t \in I. \end{aligned}$$

Since $u_1(a) = 0$ and $h(a) > 0$, we have that $u_2 > 0$ on (a,b) and $u_2'(a) < 0$, $u_2'(b) > 0$.

Now, repeating the arguments of Theorem 5.2.12 we obtain a sequence $\{u_n\}_{n=0}^{+\infty} \subset X$ such that:

$$u_1 < u_3 < u_5 < \ldots < u_4 < u_2 < u_0 = 0, \quad t \in (a,b),$$

$$\begin{cases} u_{2k+2}'(a) < u_{2k}'(a) < 0, \\ 0 < u_{2k}'(b) < u_{2k+2}'(b), \\ u_{2k-1}'(a) < u_{2k+1}'(a) < 0, \\ 0 < u_{2k+1}'(b) < u_{2k-1}'(b). \end{cases} \quad k = 1, 2, \ldots$$

So, with the same arguments of Theorem 5.2.12 we conclude that the unique solution of (5.2.2) in X satisfies $u < 0$ on (a,b), $u'(a) < 0$ and $u'(b) > 0$. □

Remark 5.2.14. *Realise that in this case we cannot extrapolate in a direct way the results to the set $\tilde{X}$. This is due to the fact that we are not able to ensure the existence and constant sign of functions $x_{p,c}$ and $z_{p,c}$, defined as in equations* (4.4.5) *and* (4.4.6)*, respectively, in this new situation.*

However in Theorem 5.2.12 (1), since we do not reach any eigenvalue, we can apply Lemma 4.4.3 to make sure the existence of unique solution for the non-homogeneous problem. Moreover, from Proposition 5.2.2 we can deduce the boundedness needed and the result remains true for $T_4[p,c]$ in $\tilde{X}$.

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5.2.4 A particular case

To finish this section, we present an example where we show the applicability of the previous results.

Let us consider a steel bridge of 1 km length (for steel, the Young's module is given by $E = 2.1 \cdot 10^{11}$ Pa$= 2.1 \cdot 10^{14}$ mPa).

We represent the transversal section of the roadway in Figure 5.2.1. By using the on-line calculator *skyciv.com* [101], we have that the moment of inertia is $I = 1.2575 \cdot 10^{-9}$ km^4. Thus, $E\,I = 264075 \dfrac{\text{km}^3\text{kg}}{\text{s}^2}$.

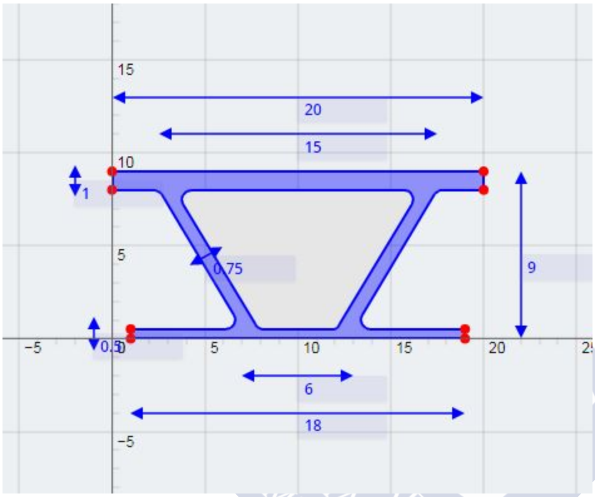

Figure 5.2.1: Transversal section of the roadway

Moreover, let us assume that the strength tension of the cables is given by $T = 528150$ kN. So, for $p = \dfrac{T}{E\,I} = 2$, $c(t) = \dfrac{f(t)}{264075}$ and $h(t) = \dfrac{q(t)}{264075} \geqslant 0$, we consider the problem which models our suspension bridge as we have mentioned in Remark 5.0.2:

$$\begin{cases} u^{(4)}(t) - 2\,u''(t) + c(t)\,u(t) = h(t), & t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{5.2.15}$$

Then, by using the previous results, we can obtain conclusions on the dead load, $q(t)$, to ensure that the vertical oscillations of the roadway will be positive (downwards) for every live load, $f(t)$.

Remark 5.2.15. *Realise that the units which appear in problem* (5.2.15) *are not in the International System of Units (SI) because we use km instead of m.*

Thus, the unit of strength will be kN instead of N and the unit of tension will be mPa instead of Pa.

Now, in order to apply the results, let us obtain the different eigenvalues given in Section 5.1.1 for $p = 2$.

- Clearly, $\bar{\lambda}_1 = \pi^4 + 2\,\pi^2$ and $\bar{\lambda}_1' = 16\,\pi^4 + 8\,\pi^2$ are the first and second positive eigenvalues of the related operator in X.
- $\bar{\lambda}_2 \approxeq -5.62^4$, the opposed of the least positive solution of (5.1.5) for $a = 0$, $b = 1$ and $p = 2$, is the biggest negative eigenvalue in X_1 and X_3.
- $\bar{\lambda}_3 \approxeq 4.018^4$, the least positive solution of (5.1.6) for $a = 0$, $b = 1$ and $p = 2$, is the least positive eigenvalue in $X_{\{0,2\}}^{\{0,1\}}$ and $X_{\{0,1\}}^{\{0,2\}}$.

From Theorem 5.2.8, we conclude that if $\displaystyle\int_0^1 f^-(t)\,\mathrm{d}\,t < 1056300\,\pi^2$, then the problem (5.2.15) has a unique solution. If, in addition, $f(t) \leqslant 2.63424 \cdot 10^8 \dfrac{\mathrm{kN}}{\mathrm{km}}$, the vertical displacement of the bridge is downwards for every live load, $q(t)$.

Moreover, if we consider a constant and positive load, $q > 0$, from Theorem 5.2.12, we have:

(1) If $-264075\,\pi^2(\pi^2+2) < f_m \leqslant 0$ and

$$f(t) \leqslant 2.63434 \cdot 10^8 + 528150\,\pi^2\left(\pi^2 + 2 + \frac{f_m}{264075\,\pi^2}\right),$$

or,

(2) If $\displaystyle\int_0^1 f^-(t)\,\mathrm{d}\,t < 1056300\,\pi^2$ and

$$f(t) \leqslant 2.63434{\cdot}10^8 + \left(528150\,\pi + \int_0^1 \frac{f^-(t)}{2\,\pi}\,\mathrm{d}\,t\right)\sqrt{\pi^4 + 2\,\pi^2 + \min_{t\in[0,1]}\left\{\frac{f^+(t)}{264075}, 5.62^4\right\}},$$

then (5.2.15) has a unique positive solution, which means that the vertical displacement is downwards.

For this example, the negative solution has no physical meaning. However, if we think in an abstract way on problem (5.2.15), we can prove the existence of negative solutions as follows.

From Theorem 5.2.9, if $-2112600\,\pi^2(2\,\pi^2+1) < f_m < -264075\,\pi^2(\pi^2+2)$ and

$$\int_0^1 (f(t) - f_m)\,\mathrm{d}\,t < 1056300\,\pi^2\delta_2, \tag{5.2.16}$$

where $\delta_2 = \min\left\{-1 - \dfrac{f_m}{264075\,\pi^2(\pi^2+2)}, 1 + \dfrac{f_m}{2112600\,\pi^2(2\,\pi^2+1)}\right\}$, then the problem (5.2.15) has a unique solution. If, in addition,

$$f(t) \geqslant -6.8828 \cdot 10^7 (> -2112600\,\pi^2(2\,\pi^2+1) \approxeq -4.32423 \cdot 10^8),$$

the unique solution is negative for every $q(t) \geqslant 0$.

Finally, if $q > 0$ is constant and (5.2.16) is fulfilled coupled with:

$$-6.8828{\cdot}10^7 - \sqrt{\pi^2+2}\left(528105\,\pi^2\delta_2 - \int_0^1 \frac{f(t) - f_m}{2}\,\mathrm{d}\,t\right) \leqslant f_m < -264075\,\pi^2(\pi^2+2),$$

then the problem (5.2.15) has a unique negative solution.

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Chapter 6
Existence results for non-linear problems via constant sign Green's function

Contents

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In this chapter, we use some of the properties of the linear problems studied in Chapter 3 and of the related Green's function to prove the existence of solutions of different non-linear two-point boundary value problems.

We consider the non-linear boundary value problem of order n:

$$\begin{cases} T_n[M]\,u(t) = f\big(t, u(t)\big),\ t \in I \equiv [a,b], \\ u^{(\sigma_1)}(a) = u^{(\sigma_2)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = u^{(\varepsilon_2)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0, \end{cases} \tag{6.0.1}$$

where $T_n[M]$ was defined in (1.0.2) and $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ were introduced in Chapter 3. Moreover, f is a continuous function in $\mathbb{R}^2$ with additional restrictions which will be detailed below.

As we have noticed before, solutions of problem (6.0.1) correspond to the fixed points of the integral operator:

$$\mathcal{L}[M]\,u(t) = \int_a^b g_M(t,s)\,f\big(s, u(s)\big)\,\mathrm{d}\,s, \quad t \in I, \tag{6.0.2}$$

where $g_M(t,s)$ is the related Green's function of operator $T_n[M]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.

Thus, if we ensure the existence of a fixed point of the operator $\mathcal{L}[M]$ we will have also proved the existence of a solution of problem (6.0.1).

The use of different kinds of fixed point theorems has been shown to be a very useful tool to obtain the existence of solutions of non-linear boundary value problems.

For instance, in [20] it is proved that the following second order system:

$$\begin{cases} u_1''(t) + \lambda_1 \, f_1\big(u_1(t), u_2(t)\big) = 0, & t \in (0,1), \\ u_2''(t) + \lambda_2 \, f_2\big(u_1(t), u_2(t)\big) = 0, & t \in (0,1), \\ u_1'(0) = u_1(1) + u_1'(1) = 0, & \\ u_2'(0) = u_2(1) + \varepsilon \, u_2(\eta) = 0, & \eta, \; \varepsilon \in (0,1), \end{cases}$$

has a solution for every λ_1, $\lambda_2 > 0$, by applying some previously obtained fixed point results to a related system of integral operators.

In [52], under suitable conditions on the functions f and g, and $\lambda \in \mathbb{R}$, it is ensured the existence of at least two strictly positive solutions for the second order boundary value problem:

$$\begin{cases} u''(t) + \lambda \, g(t) \, f\big(u(t)\big) = 0, & t \in (0,1), \\ u'(0) = \sigma \, u'(1) + u(\eta) = 0, & \eta \in [0,1], \end{cases}$$

which models the behaviour of a thermostat. Again, the existence result is obtained by applying a fixed point theorem.

In [27], we have studied a fourth order boundary value problem coupled with the cantilever beam boundary conditions:

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), & t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. & \end{cases}$$

In such a work, two combined techniques on the existence of solutions are used: the critical and the fixed point theory. Even though this work is not collected in this Thesis, it indicates a possible future way to follow, combining different techniques in the research of new existence results for non-linear problems.

In [3], it is obtained a result in the line of the Leggett-Williams Fixed Point Theorem (see [64]) that makes sure the existence of at least a positive fixed point on different sets defined by means of suitable functionals. As a direct application of this result, it is proved the existence of solution for the following second order non-linear differential boundary value problem:

$$\begin{cases} u''(t) + f\big(u(t)\big) = 0, & t \in (0,1), \\ u(0) = u'(1) = 0. & \end{cases}$$

Moreover, in [8], by means of a new fixed point theorem, it is obtained a different existence result for this problem.

In [99], an extension of the Leggett-Williams Fixed Point Theorem given in [64] is proved and, as an application, it is obtained a result which ensures the existence of multiple solutions of the following third order boundary value problem:

$$\begin{cases} u'''(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0, & t \in (0,1), \\ u(0) = \displaystyle\sum_{i=1}^{m-2} k_i\, u(\xi_i),\ u'(0) = u'(1) = 0, & \end{cases}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $k_i \in \mathbb{R}^+$ for $i = 1, \ldots, m-2$ and $0 < \displaystyle\sum_{i=1}^{m-2} k_i < 1$.

In [9], as an application of the Leggett-Williams Fixed Point Theorem, it is proved the existence of at least one solution for the problem:

$$\begin{cases} u^{(4)}(t) = f\big(u(t)\big), & t \in (0,1), \\ u(0) = u'(0) = u(1) = u'(1) = 0, & \end{cases}$$

under suitable conditions of f.

As we have previously said, we want to find fixed points of the integral operator $\mathcal{L}[M]$ defined in (6.0.2). We will achieve this by studying a more general operator. In Sections 6.2 and 6.3, we prove the existence of one or multiple positive fixed points of an integral operator defined as follows:

$$\begin{aligned} \mathcal{L}\colon C(I) &\longrightarrow C(I) \\ u &\longmapsto \mathcal{L}\, u(t) := \int_a^b \mathcal{G}(t,s)\, f\big(s, u(s)\big)\, \mathrm{d}\, s, \end{aligned} \tag{6.0.3}$$

where:

$$\begin{aligned} f\colon I \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (t,u) &\longmapsto f(t,u), \end{aligned}$$

is a constant sign continuous function and:

$$\begin{aligned} \mathcal{G}\colon I \times I &\longrightarrow \mathbb{R} \\ (t,s) &\longmapsto \mathcal{G}(t,s), \end{aligned}$$

is an integral kernel satisfying either condition $\big(P_{g_1}\big)$ or $\big(N_{g_1}\big)$, two properties which we have defined in Section 3.7.

In Chapter 3, Theorem 3.7.1, we have obtained the parameter set for which the related Green's function of problem (6.0.1) satisfies either property $\big(P_{g_1}\big)$ or $\big(N_{g_1}\big)$ on a neighbourhood of $\bar{M}$ $\Big($being $\bar{M} \in \mathbb{R}$ such that $T_n[\bar{M}]$ satisfies condition (T_d) in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}\Big)$.

Moreover, Theorem 3.7.3 gives an upper bound on the interval for which either property $\big(N_{g_1}\big)$ or $\big(P_{g_1}\big)$ is fulfilled for $M < \bar{M} - \lambda_1$ or $M > \bar{M} - \lambda_1$, respectively. Taking into account Remark 3.7.8, this interval can be the optimal one in some cases. For instance, in the problem considered in Example 3.7.9 or in the problems coupled with simply supported beam boundary conditions which have in Chapter 5.

In the last section of this chapter, as an application of the different fixed point theorems, we prove the existence of one or multiple constant sign solutions for different particular problems.

First, we study the particular case which has been developed along Chapters 3 and 4, that is, the fourth order operator:

$$T_4^0[M]\,u(t) = u^{(4)}(t) + M\,u(t),\; t \in [0,1],$$

coupled with the boundary conditions:

$$u(0) = u''(0) = u'(1) = u''(1) = 0.$$

After that, we study different problems with the $(k, n-k)$ boundary conditions. Finally, we show two examples coupled with the simply supported beam boundary conditions.

It is important to note that (cf. [16, 42, 60, 63, 95] and references therein) several results for the existence of one or multiple fixed points of integral operators are obtained by imposing similar conditions to property (P_{g_1}) to the kernel $\mathcal{G}$. Indeed the imposed conditions in [95] are equivalent to (P_{g_1}).

In this chapter, we use the well-known Krasnosel'skiĭ's Fixed Point Theorem, collected in [61]. Moreover, in order to obtain the existence of two or three fixed points we use two results due to Avery and Henderson, [10], and Avery, [7], respectively. The structure that we follow is the one given in [4], where these results are used to prove the existence of one or multiple solutions for a problem on time scales.

The reader can see the results shown here in [34, 82].

This chapter is structured as follows: in next section, we describe the studied problem and show some preliminary results which are used along the chapter. In Section 6.2, we obtain some results that ensure the existence of one or multiple fixed points by using the Krasnoseslkiĭ's Fixed Point Theorem given in [61]. Then, in Section 6.3, following the results of Avery and Henderson, [10], and Avery, [7], we obtain the existence of at least two or three fixed points, respectively. Finally, in Section 6.4, as an application of these fixed point theorems, we deduce the existence and multiplicity of solutions of different kind of non-linear boundary value problems.

6.1 Description of the problem and some previous fixed point existence results

Our interest consists on studying the existence of some fixed points of the integral operator described in (6.0.3) in an appropriate cone.

First, let us define the concept of cone.

Definition 6.1.1. *Let $\mathcal{B}$ be a real Banach space. A non-empty closed convex set $\mathcal{P} \subset \mathcal{B}$ is called a cone if it satisfies the following two conditions:*

1. *$\lambda\, x \in \mathcal{P}$ for all $x \in \mathcal{P}$ and $\lambda \geqslant 0$.*
2. *If $x \in \mathcal{P}$ and $-x \in \mathcal{P}$, then $x = 0$.*

In the sequel, we describe the cone where the fixed points are located, as well as some constants which are used along the chapter.

Now, let us assume that property (P_{g_1}), introduced in Chapter 3, is fulfilled, that is:

(P_{g_1}) There are three continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a, b)$ and $0 < k_1(t) < k_2(t)$ for all $t \in (a, b)$, satisfying:

$$\phi(s)\, k_1(t) \leqslant \mathcal{G}(t, s) \leqslant \phi(s)\, k_2(t), \quad \text{for all } (t, s) \in I \times I.$$

Let us consider a subinterval $I_1 = [a_1, b_1] \subset I$ such that $\left|k_1(t)\right| > 0$ for all $t \in I_1$. Then, we denote:

$$K_1 = \max_{t \in I} \left|k_1(t)\right| > 0, \quad m_1 = \min_{t \in I_1} \left|k_1(t)\right| > 0, \quad K_2 = \max_{t \in I} \left|k_2(t)\right| > 0. \tag{6.1.1}$$

Consider the Banach space $\mathcal{B} = C(I)$ coupled with the norm:

$$\|u\|_{C(I)} = \max_{t \in I} |u(t)|,$$

and the cone:

$$\mathcal{P} = \left\{ u \in \mathcal{B} \;\middle|\; u(t) \geqslant \frac{k_1(t)}{K_2} \|u\|_{C(I)}, \quad t \in I \right\}.$$

Remark 6.1.2. *By assuming (N_{g_1}) is fulfilled, or which is the same:*

(N_{g_1}) *There are three continuous functions ϕ, k_1 and k_2 such that $\phi(s) > 0$ for all $s \in (a, b)$ and $k_2(t) < k_1(t) < 0$ for all $t \in (a, b)$, satisfying:*

$$\phi(s)\, k_2(t) \leqslant \mathcal{G}(t, s) \leqslant \phi(s)\, k_1(t), \quad \textit{for all } (t, s) \in I \times I.$$

We consider K_1, m_1 and K_2 as in equation (6.1.1) *and the cone:*

$$\mathcal{Q} = \left\{ u \in \mathcal{B} \;\middle|\; u(t) \leqslant \frac{k_1(t)}{K_2} \|u\|_{C(I)}, \quad t \in I \right\}.$$

In the sequel, to make the chapter more readable, we show some preliminary results which we will use along it.

Definition 6.1.3. *A bounded linear operator between two Banach spaces is called completely continuous if it maps weakly convergent sequences into norm-convergence sequences.*

Remark 6.1.4. *Compacts operators (linear operators which maps bounded sets into relatively compacts subsets) are always completely continuous operators.*

If the Banach space is reflexive (see Definition 7.1.13), then the reciprocal is true.

First, let us consider the Krasnosel'skiĭ's Fixed Point Theorem, [61]:

Theorem 6.1.5. *Let $\mathcal{B}$ be a Banach space, $\mathcal{P} \subset \mathcal{B}$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of $\mathcal{B}$ centred at the origin, with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $\mathcal{L} \colon \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ is a completely continuous operator such that either:*

$$\|\mathcal{L}u\| \leqslant \|u\|, \quad u \in \mathcal{P} \cap \partial\,\Omega_1, \quad \textit{and} \quad \|\mathcal{L}u\| \geqslant \|u\|, \quad u \in \mathcal{P} \cap \partial\,\Omega_2,$$

or,

$$\|\mathcal{L}u\| \geqslant \|u\|, \quad u \in \mathcal{P} \cap \partial\,\Omega_1, \quad \textit{and} \quad \|\mathcal{L}u\| \leqslant \|u\|, \quad u \in \mathcal{P} \cap \partial\,\Omega_2,$$

holds. Then $\mathcal{L}$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Definition 6.1.6. *A map α is said to be a non-negative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space $\mathcal{B}$ if $\alpha:\ \mathcal{P} \to [0,+\infty)$ is continuous and:*

$$\alpha(t\,x+(1-t)\,y) \geqslant t\,\alpha(x)+(1-t)\,\alpha(y), \quad \forall\, x,\, y \in \mathcal{P}, \quad t\in[0,1].$$

Similarly, a map β is said to be a non-negative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space $\mathcal{B}$ if $\beta:\ \mathcal{P} \to [0,+\infty)$ is continuous and:

$$\beta(t\,x+(1-t)\,y) \leqslant t\,\beta(x)+(1-t)\,\beta(y), \quad \forall\, x,\, y \in \mathcal{P}, \quad t\in[0,1].$$

Now, let us consider β, γ and θ, non-negative continuous convex functionals on the cone $\mathcal{P}$, and α and ψ, non-negative concave functionals on $\mathcal{P}$.

For non-negative real numbers d, p and q, we define the following subspaces of the cone $\mathcal{P}$:

$$\begin{aligned}
\mathcal{P}(\gamma,r) &= \Big\{u\in\mathcal{P} \;\Big|\; \gamma(u)<r\Big\},\\
\mathcal{P}(\gamma,\alpha,p,r) &= \Big\{u\in\mathcal{P} \;\Big|\; p\leqslant\alpha(u),\ \gamma(u)\leqslant r\Big\},\\
\mathcal{Q}(\gamma,\beta,d,r) &= \Big\{u\in\mathcal{P} \;\Big|\; \beta(u)\leqslant d,\ \gamma(u)\leqslant r\Big\},\\
\mathcal{P}(\gamma,\theta,\alpha,p,q,r) &= \Big\{u\in\mathcal{P} \;\Big|\; p\leqslant\alpha(u),\ \theta(u)\leqslant q,\ \gamma(u)\leqslant r\Big\}.
\end{aligned}$$

In the sequel we introduce a result, proved in [10], which ensures the existence of two fixed points on the cone $\mathcal{P}$.

Theorem 6.1.7. *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. Let α and γ be increasing and non-negative continuous functionals on $\mathcal{P}$. Let θ be a non-negative continuous functional on $\mathcal{P}$ with $\theta(0)=0$ such that, for some positive constants r and M,*

$$\alpha(u)\leqslant\theta(u)\leqslant\gamma(u) \quad \textit{and} \quad \|u\|\leqslant M\,\alpha(u), \quad \textit{for all } u\in\overline{\mathcal{P}(\alpha,r)}.$$

Assume that there exist two positive numbers p and q with $p<q<r$ such that:

$$\theta(\lambda\,u)\leqslant\lambda\,\theta(u), \quad \textit{for all } 0\leqslant\lambda\leqslant 1 \quad \textit{and} \quad u\in\partial\mathcal{P}(\theta,q).$$

Suppose that $\mathcal{L}\colon \overline{\mathcal{P}(\alpha,r)} \to \mathcal{P}$ is a completely continuous operator satisfying:

$i)$ $\alpha(\mathcal{L}\,u)>r$ for all $u\in\partial\mathcal{P}(\alpha,r)$,

$ii)$ $\theta(\mathcal{L}\,u)<q$ for all $u\in\partial\mathcal{P}(\theta,q)$,

$iii)$ $\mathcal{P}(\gamma,p)\neq\emptyset$ and $\gamma(\mathcal{L}\,u)>p$ for all $u\in\partial\mathcal{P}(\gamma,p)$.

Then, $\mathcal{L}$ has at least two fixed points u_1 and u_2 such that:

$$p<\gamma(u_1), \quad \textit{with} \quad \theta(u_1)<q,$$

and,

$$q<\theta(u_2), \quad \textit{with} \quad \alpha(u_2)<r.$$

Finally, we introduce a result, proved in [7], which ensures the existence of three fixed points of the operator $\mathcal{L}$ on the cone $\mathcal{P}$.

Theorem 6.1.8. *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$, and let r and M be positive numbers. Assume that α and ψ are non-negative, continuous and concave functionals on $\mathcal{P}$, and γ, β and θ are non-negative, continuous and convex functionals on $\mathcal{P}$ with:*

$$\alpha(u) \leqslant \beta(u) \quad \textit{and} \quad \|u\| \leqslant M\,\gamma(u), \quad \forall\, u \in \overline{\mathcal{P}(\gamma, r)}.$$

Suppose that $\mathcal{L}\colon \overline{\mathcal{P}(\gamma,r)} \to \overline{\mathcal{P}(\gamma,r)}$ is a completely continuous operator and there exist non-negative numbers h, d, p, q, with $0 < d < p$, such that:

a) $\Big\{u \in \mathcal{P}(\gamma,\theta,\alpha,p,q,r) \;\big|\; \alpha(u) > p\Big\} \neq \emptyset$ *and* $\alpha(\mathcal{L}\,u) > p$ *for* $u \in \mathcal{P}(\gamma,\theta,\alpha,p,q,r)$*,*

b) $\Big\{u \in \mathcal{P}(\gamma,\beta,\psi,h,d,r) \;\big|\; \beta < d\Big\} \neq \emptyset$ *and* $\beta(\mathcal{L}\,u) < d$ *for* $u \in \mathcal{P}(\gamma,\beta,\psi,h,d,r)$*,*

c) $\alpha(\mathcal{L}\,u) > p$ *for* $u \in \mathcal{P}(\gamma,\alpha,p,r)$ *with* $\theta(\mathcal{L}\,u) > q$*,*

d) $\beta(\mathcal{L}\,u) < d$ *for* $u \in \mathcal{Q}(\gamma,\beta,d,r)$ *with* $\Psi(\mathcal{L}\,u) < h$*.*

Then $\mathcal{L}$ has at least three fixed points u_1, u_2, $u_3 \in \overline{\mathcal{P}(\gamma,r)}$ such that:

$$\beta(u_1) < d < \beta(u_2) \quad \textit{and} \quad \alpha(u_2) < p < \alpha(u_3).$$

6.2 Existence of fixed points by means of Krasnosel'skiĭ's Fixed Point Theorem

The aim of this section consists on ensuring the existence of at least a fixed point of operator $\mathcal{L}$, defined in (6.0.3). Such an existence result follows as an application of Theorem 6.1.5. Consider K_1, K_2 and m_1 defined in (6.1.1). let us define the following conditions on f:

(H_1) There exists $p > 0$ such that

$$f(t,u) \leqslant \frac{p}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\,s}, \quad \forall\, t \in I,\ \forall\, u \in [0,p].$$

(H_2) There exists $q > 0$ such that

$$f(t,u) \geqslant \frac{K_2\,u}{K_1 \displaystyle\int_{a_1}^{b_1} \big|k_1(s)\big|\,\phi(s)\,\mathrm{d}\,s}, \quad \forall\, t \in I_1,\ \forall\, u \in \left[\frac{m_1}{K_2}\,q, q\right].$$

Following the same steps as in [4, Theorem 2.3], we prove the following result.

Theorem 6.2.1. *Suppose that there exist two positive numbers $p \neq q$ such that condition (H_1) is satisfied with respect to p and condition (H_2) is satisfied with respect to q. Then, provided that the integral kernel $\mathcal{G}$ satisfies $\left(P_{g_1}\right)$, operator $\mathcal{L}$, defined in* (6.0.3)*, has a fixed point, $u \in \mathcal{P}$, such that $\|u\|_{C(I)}$ lies between p and q.*

Proof. First, let us see that $\mathcal{L}(\mathcal{P}) \subset \mathcal{P}$.

Let $u \in \mathcal{P}$, we have:

$$\begin{aligned}
\mathcal{L}\,u(t) :=& \int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \geqslant \int_a^b k_1(t)\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \\
=&\frac{k_1(t)}{K_2}\int_a^b K_2\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \geqslant \frac{k_1(t)}{K_2}\int_a^b \sup_{t\in I}\Big\{\mathcal{G}(t,s)\Big\}\, f\big(s,u(s)\big)\,\mathrm{d}\,s \\
\geqslant&\frac{k_1(t)}{K_2}\sup_{t\in I}\left\{\int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s\right\} = \frac{k_1(t)}{K_2}\,\|\mathcal{L}\,u\|_{C(I)}\,, \quad \forall t\in I.
\end{aligned}$$

Thus, since f is a continuous function on $I\times\mathbb{R}$, $\mathcal{L}\,u \in \mathcal{P}$ and $\mathcal{L}\colon \mathcal{P}\to\mathcal{P}$ is a completely continuous operator because it is compact.

Now, let us define the open balls centred at the origin as follows:

$$\Omega_p = \Big\{u \in C(I) \mid \|u\|_{C(I)} < p\Big\} \quad \text{and} \quad \Omega_q = \Big\{u \in C(I) \mid \|u\|_{C(I)} < q\Big\}.$$

From $\left(P_{g_1}\right)$ and the positiveness of f for all $u \in \mathcal{P}$, we have that the following inequality is fulfilled.

$$\|\mathcal{L}\,u\|_{C(I)} = \sup_{t\in I}\left\{\int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s\right\} \leqslant K_2\int_a^b \phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s. \tag{6.2.1}$$

On the other hand, for $u \in \mathcal{P}\cap\partial\Omega_p$, we have that $\|u\|_{C(I)} = p$, so, from (6.2.1) and (H_1), we have:

$$\begin{aligned}
\|\mathcal{L}\,u\|_{C(I)} \leqslant& K_2\int_a^b \phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \leqslant K_2\int_a^b \phi(s)\frac{p}{K_2\displaystyle\int_a^b \phi(\ell)\,\mathrm{d}\,\ell}\,\mathrm{d}\,s \\
=& p = \|u\|_{C(I)}.
\end{aligned} \tag{6.2.2}$$

Thus, $\|\mathcal{L}\,u\|_{C(I)} \leqslant \|u\|_{C(I)}$ for all $u \in \mathcal{P}\cap\partial\Omega_p$.

Now, using $\left(P_{g_1}\right)$ again, we have, for all $u \in \mathcal{P}$:

$$\begin{aligned}
\|\mathcal{L}\,u\|_{C(I)} =&\sup_{t\in I}\left\{\int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s\right\} \geqslant \sup_{t\in I}\left\{\int_a^b k_1(t)\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s\right\} \\
=&K_1\int_a^b \phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s.
\end{aligned} \tag{6.2.3}$$

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Let $v \in \mathcal{P} \cap \partial\Omega_q$, then:

$$\min_{t\in I_1} v(t) \geqslant \min_{t\in I_1} \frac{k_1(t)}{K_2} \|v\|_{C(I)} = \frac{m_1}{K_2}\, q.$$

So, from (6.2.3) and (H_2):

$$\begin{aligned}\|\mathcal{L}\, v\|_{C(I)} &\geqslant K_1 \int_a^b \phi(s)\, f\big(s, v(s)\big)\, \mathrm{d}\, s \geqslant K_1 \int_{a_1}^{b_1} \phi(s)\, f\big(s, v(s)\big)\, \mathrm{d}\, s \\ &\geqslant K_1 \int_{a_1}^{b_1} \phi(s)\, \frac{K_2\, v(s)}{K_1 \displaystyle\int_{a_1}^{b_1} k_1(\ell)\, \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s \\ &\geqslant \int_{a_1}^{b_1} \phi(s)\, \frac{K_2 \dfrac{k_1(s)}{K_2}\, \|v\|_{C(I)}}{\displaystyle\int_{a_1}^{b_1} k_1(\ell)\, \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s = \|v\|_{C(I)} = q.\end{aligned}$$

Hence, $\|\mathcal{L}\, v\|_{C(I)} \geqslant \|v\|_{C(I)}$ for all $v \in \mathcal{P} \cap \partial\Omega_q$.

Then, from Theorem 6.1.5, we conclude that $\mathcal{L}$ has a fixed point in $\mathcal{P}$ such that $\|u\|_{C(I)}$ lies between p and q. □

Remark 6.2.2. *In case that the integral kernel $\mathcal{G}$ satisfies property $\big(N_{g_1}\big)$, we have two possibilities to deal with the studied operator $\mathcal{L}$.*

The first of them is to rewrite the operator in the following way:

$$\mathcal{L}\, u(t) = \int_a^b \Big(-\mathcal{G}(t,s)\Big)\Big(-f\big(s, u(s)\big)\Big)\, \mathrm{d}\, s.$$

It is clear that $\mathcal{G}$ satisfies property $\big(N_{g_1}\big)$ if, and only if, property $\big(P_{g_1}\big)$ is fulfilled by $-\mathcal{G}$. Thus, by imposing the hypotheses of Theorem 6.2.1 to $-f$, instead of f, we have ensured the existence of positive fixed points of $\mathcal{L}$.

On the other hand, if $\mathcal{G}$ satisfies property $\big(N_{g_1}\big)$ and $g(t,u) = f(t,-u)$ fulfils conditions (H_1) and (H_2), we will prove the existence of negative fixed points of $\mathcal{L}$ in $\mathcal{Q}$. We will detail this case below.

Theorem 6.2.3. *Consider $g(t,u) = f(t,-u)$ such that there exist two positive numbers $p \neq q$ for which condition (H_1) is satisfied with respect to p and condition (H_2) is satisfied with respect to q. Then, provided that the integral kernel $\mathcal{G}$ satisfies $\big(N_{g_1}\big)$, operator $\mathcal{L}$, defined in* (6.0.3)*, has a fixed point, $u \in \mathcal{Q}$, such that $\|u\|_{C(I)}$ lies between p and q.*

Proof. The proof is analogous to the one of Theorem 6.2.1, taking into account that $\mathcal{G}$ is a negative function and that property $\big(N_{g_1}\big)$ is fulfilled instead of $\big(P_{g_1}\big)$. For convenience of the reader, we will detail it below.

At first, we prove that $\mathcal{L}(\mathcal{P}) \subset \mathcal{P}$.

Let $u \in \mathcal{Q}$, we have:

$$\begin{aligned}\mathcal{L}\, u(t) := \int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s \leqslant \int_a^b k_1(t)\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s\\ = \frac{k_1(t)}{K_2}\int_a^b K_2\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s,\end{aligned}$$

taking into account that $k_1(t) \leqslant 0$ and $0 \leqslant -\mathcal{G}(t,s) \leqslant K_2\,\phi(s)$ for all $t \in I$, we have:

$$\begin{aligned}\mathcal{L}\, u(t) &\leqslant \frac{k_1(t)}{K_2}\int_a^b \sup_{t\in I}\Big\{-\mathcal{G}(t,s)\Big\}\, f\big(s,u(s)\big)\,\mathrm{d}\, s\\ &\leqslant \frac{k_1(t)}{K_2}\sup_{t\in I}\left\{\int_a^b -\mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s\right\} = \frac{k_1(t)}{K_2}\,\|\mathcal{L}\, u\|_{C(I)}\,, \quad \forall t \in I.\end{aligned}$$

Thus, as in Theorem 6.2.1, $\mathcal{L}\, u \in \mathcal{Q}$ and $\mathcal{L}\colon \mathcal{Q} \to \mathcal{Q}$ is a completely continuous operator.

Now, let us define the open balls centred at the origin as follows:

$$\Omega_p = \Big\{u \in C(I) \;\big|\; \|u\|_{C(I)} < p\Big\} \quad \text{and} \quad \Omega_q = \Big\{u \in C(I) \;\big|\; \|u\|_{C(I)} < q\Big\}.$$

From $\big(N_{g_1}\big)$ and the positiveness of f for all $u \in \mathcal{Q}$, we have that the following inequality is fulfilled.

$$\|\mathcal{L}\, u\|_{C(I)} = \sup_{t\in I}\left\{\int_a^b -\mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s\right\} \leqslant K_2\int_a^b \phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s. \tag{6.2.4}$$

On the other hand, for $u \in \mathcal{P}\cap\partial\Omega_p$, we have that $\|u\|_{C(I)} = p$, so $u \in [-p,0]$. Hence from (6.2.4) and (H_1) for $f(t,-u)$, we have (6.2.2) fulfilled.

Thus, $\|\mathcal{L}\, u\|_{C(I)} \leqslant \|u\|_{C(I)}$ for all $u \in \mathcal{Q}\cap\partial\Omega_p$.

Now, using $\big(N_{g_1}\big)$ again, we have, for all $u \in \mathcal{P}$:

$$\begin{aligned}\|\mathcal{L}\, u\|_{C(I)} =&\sup_{t\in I}\left\{\int_a^b -\mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s\right\} \geqslant \sup_{t\in I}\left\{\int_a^b -k_1(t)\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s\right\}\\ =&K_1\int_a^b \phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\, s.\end{aligned} \tag{6.2.5}$$

Let $v \in \mathcal{Q}\cap\partial\Omega_q$, then:

$$\max_{t\in I_1} v(t) \leqslant \max_{t\in I_1}\frac{k_1(t)}{K_2}\|v\|_{C(I)} = \frac{-m_1}{K_2}\, q.$$

So, from (6.2.5) and (H_2) for $f(t,-u)$, taking into account that $v(t) \in \left[-q, \dfrac{-m_1}{K_2}q\right]$

and $\left|k_1(t)\right| = -k_1(t)$ for $t \in I$, we have:

$$\begin{aligned}
\|\mathcal{L}\, v\|_{C(I)} &\geqslant K_1 \int_a^b \phi(s)\, f\big(s, v(s)\big)\,\mathrm{d}\, s \geqslant K_1 \int_{a_1}^{b_1} \phi(s)\, f\big(s, v(s)\big)\,\mathrm{d}\, s \\
&\geqslant K_1 \int_{a_1}^{b_1} \phi(s)\, \frac{K_2\left(-v(s)\right)}{K_1 \displaystyle\int_{a_1}^{b_1} -k_1(\ell)\,\phi(\ell)\,\mathrm{d}\,\ell}\,\mathrm{d}\, s \\
&\geqslant \int_{a_1}^{b_1} \phi(s)\, \frac{K_2 \dfrac{-k_1(s)}{K_2}\, \|v\|_{C(I)}}{\displaystyle\int_{a_1}^{b_1} -k_1(\ell)\,\phi(\ell)\,\mathrm{d}\,\ell}\,\mathrm{d}\, s = \|v\|_{C(I)} = q.
\end{aligned}$$

Hence, $\|\mathcal{L}\, v\|_{C(I)} \geqslant \|v\|_{C(I)}$ for all $v \in \mathcal{Q} \cap \partial\Omega_q$.

Then, from Theorem 6.1.5, we conclude, as in Theorem 6.2.1, that $\mathcal{L}$ has a fixed point in $\mathcal{Q}$ such that $\|u\|_{C(I)}$ lies between p and q. □

Define, for every $t \in I$, the following functions:

$$f_0^+(t) := \limsup_{u \to 0^+} \frac{f(t,u)}{u}, \quad f_\infty^+(t) := \limsup_{u \to \infty} \frac{f(t,u)}{u},$$
$$f_0^-(t) := \liminf_{u \to 0^+} \frac{f(t,u)}{u}, \quad f_\infty^-(t) := \liminf_{u \to \infty} \frac{f(t,u)}{u}.$$

Analogously to [4, Corollary 2.4], we deduce the following existence result as a corollary of Theorem 6.2.1.

Corollary 6.2.4. *If $\mathcal{G}$ satisfies $\left(P_{g_1}\right)$, then operator $\mathcal{L}$, defined in* (6.0.3)*, has a fixed point in $\mathcal{P}$, provided that one of two following conditions holds.*

(H_3) $f_0^+(t) < \dfrac{1}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\, s}$, $\forall\, t \in I$ *and* $f_\infty^-(t) > \dfrac{K_2}{K_1 \displaystyle\int_{a_1}^{b_1} \left|k_1(s)\right| \phi(s)\,\mathrm{d}\, s}$, $\forall\, t \in I_1$,

(H_4) $f_\infty^+(t) < \dfrac{1}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\, s}$, $\forall\, t \in I$ *and* $f_0^-(t) > \dfrac{K_2}{K_1 \displaystyle\int_{a_1}^{b_1} \left|k_1(s)\right| \phi(s)\,\mathrm{d}\, s}$, $\forall\, t \in I_1$.

In particular, there is one fixed point if $f_0^+(t) = 0$ and $f_\infty^-(t) = \infty$ $\big($reciprocally, $f_\infty^+(t) = 0$ and $f_0^- = \infty(t)\big)$ for all $t \in I$.

Proof. First, let us assume that (H_3) is fulfilled.

Then, there exist $p > 0$, small enough, and $q > 0$, big enough, such that:

- $\dfrac{f(t,u)}{u} \leqslant \dfrac{1}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\, s}$ for all $t \in I$ and $0 < u \leqslant p$.

- $\dfrac{f(t,u)}{u} \geqslant \dfrac{K_2}{K_1 \displaystyle\int_{a_1}^{b_1} k_1(s)\,\phi(s)\,\mathrm{d}\,s}$ for all $t \in I_1$ and $u \geqslant q$.

Thus, (H_1) and (H_2) are fulfilled for $p > 0$ and $\dfrac{K_2}{m_1} q > 0$, respectively. So, from Theorem 6.2.1, $\mathcal{L}$ has a fixed point on $\mathcal{P}$ such that $\|u\|_{C(I)}$ lies between p and $\dfrac{K_2}{m_1} q$.

Now, suppose that (H_4) is satisfied.

Then, there exist $0 < p < q$, such that:

- $\dfrac{f(t,u)}{u} \geqslant \dfrac{K_2}{K_1 \displaystyle\int_{a_1}^{b_1} k_1(s)\,\phi(s)\,\mathrm{d}\,s}$ for all $t \in I_1$ and $0 < u \leqslant p$.
- $\dfrac{f(t,u)}{u} \leqslant \dfrac{1}{K_2 \displaystyle\int_{a}^{b} \phi(s)\,\mathrm{d}\,s}$ for all $t \in I$ and $u \geqslant q$.

Hence, (H_2) is fulfilled for $p > 0$.

Let us see that (H_1) is also satisfied.

First, let us assume that $f(t,u)$ is a bounded function. That is, there exists $N > 0$ such that $0 \leqslant f(t,u) \leqslant N$ for all $t \in I$ and $0 \leqslant u < \infty$. Then, let us choose

$$r \geqslant N\,K_2 \int_a^b \phi(s)\,\mathrm{d}\,s,$$

such that

$$f(t,u) \leqslant N \leqslant \frac{r}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\,s}, \quad \forall t \in I,\ 0 \leqslant u \leqslant \infty.$$

In particular, the previous inequalities are fulfilled for $0 \leqslant u \leqslant r$. Thus, (H_1) holds for this $r > 0$.

Now, suppose that f is not bounded. Then, there exists $t_0 \in I$ and $\bar{r} \geqslant q$ such that $f(t,u) \leqslant f(t_0,\bar{r})$ for all $t \in I$ and $0 \leqslant u \leqslant \bar{r}$. Thus, by using the hypotheses, we have:

$$f(t,u) \leqslant f(t_0,\bar{r}) \leqslant \frac{\bar{r}}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\,s}, \quad \forall t \in I,\ 0 \leqslant u \leqslant \bar{r}.$$

Therefore, (H_1) is fulfilled for such $\bar{r} > 0$, and the result follows again from Theorem 6.2.1. □

As in Theorem 6.2.3, we have the analogous result if $\mathcal{G}$ satisfies $\left(N_{g_1}\right)$.

Corollary 6.2.5. *Consider $g(t,u) = f(t,-u)$. If $\mathcal{G}$ satisfies $\left(N_{g_1}\right)$, then operator $\mathcal{L}$, defined in* (6.0.3)*, has a fixed point in $\mathcal{Q}$, provided that either (H_3) or (H_4) holds related to g.*

In particular, there is one fixed point if $g_0^+(t) = 0$ and $g_\infty^-(t) = \infty$ $\big($reciprocally, if $g_\infty^+(t) = 0$ and $g_0^-(t) = \infty\big)$ for $t \in I$.

To finish this section, we obtain the existence of at least two fixed points of the integral operator $\mathcal{L}$.

Let us consider the following conditions on f:

(H_1^*) There exists $p > 0$ such that (H_1) is fulfilled and:

$$f(t,p) < \frac{p}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\,s}, \quad \forall t \in I.$$

(H_2^*) There exists $q > 0$ such that (H_2) is fulfilled and:

$$f\,(t,q) > \frac{K_2\, q}{K_1 \displaystyle\int_{a_1}^{b_1} \left|k_1(s)\right| \phi(s)\,\mathrm{d}\,s}, \quad \forall t \in I_1.$$

Remark 6.2.6. *Realise that conditions (H_1^*) and (H_2^*) are a small restriction of (H_1) and (H_2), respectively. As we will see, this restriction allows us to prove that the two fixed points that we find are, in fact, different.*

Theorem 6.2.7. *If $\mathcal{G}$ satisfies $\left(P_{g_1}\right)$, then the operator $\mathcal{L}$ has at least two fixed points, u_1 and $u_2 \in \mathcal{P}$, provided that $f_0^-(t) = f_\infty^-(t) = \infty$ for $t \in I$ and (H_1^*) holds. In such a case, $0 < \|u_1\|_{C(I)} < p < \|u_2\|_{C(I)}$ $\big(p$ given in $(H_1^*)\big)$.*

Proof. As in Corollary 6.2.4, we can see that $f_0^-(t) = f_\infty^-(t) = \infty$ implies that there exist positive numbers q_1 and q_2, with $0 < q_1 < p < q_2$ such that (H_2) is fulfilled with respect to q_1 and q_2, respectively.

Since (H_1^*) is a restriction of (H_1) we have that, in particular, (H_1) holds for p.

Hence, from Theorem 6.2.1, we conclude that there exist two fixed points, u_1 and u_2, such that $0 < q_1 \leqslant \|u_1\|_{C(I)} \leqslant p \leqslant \|u_2\|_{C(I)} \leqslant q_2$.

To finish the proof, we need to ensure that u_1 and u_2 are, in fact, different. To this end, let us prove that if $u \in \mathcal{P} \cap \partial\Omega_p$, then u cannot be a fixed point of $\mathcal{L}$.

Let $u \in \mathcal{P} \cap \partial\Omega_p$, in particular, $\|u\|_{C(I)} = p$. From (H_1^*) and property $\left(P_{g_1}\right)$, we have:

$$\begin{aligned}
\|\mathcal{L}\,u\|_{C(I)} &= \sup_{t\in I} \int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \leqslant \sup_{t\in I} \int_a^b k_2(t)\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \\
&= K_2 \int_a^b \phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \leqslant K_2 \int_a^b \phi(s) \frac{p}{K_2 \displaystyle\int_a^b \phi(\ell)\,\mathrm{d}\,\ell}\,\mathrm{d}\,s \\
&= p = \|u\|_{C(I)}.
\end{aligned} \tag{6.2.6}$$

Since $\|u\|_{C(I)} = p$ we have that there exists $t_0 \in I$ such that $u(t_0) = p$. Now, since f is a continuous function and $\|u\|_{C(I)} = p$, from (H_1^*), there exists a neighbourhood of t_0 such that the strict inequality given in (H_1^*) is fulfilled, hence the last inequality in (6.2.6) is strict and $\|\mathcal{L}\,u\|_{C(I)} < \|u\|_{C(I)}$. Thus, u cannot be a fixed point. So, there exist two fixed points, u_1 and u_2, such that $0 < q_1 \leqslant \|u_1\|_{C(I)} < p < \|u_2\|_{C(I)}$ and the result is proved. □

In an analogous way, we can prove the following "dual" result:

Theorem 6.2.8. *If $\mathcal{G}$ satisfies $\left(P_{g_1}\right)$, then the operator $\mathcal{L}$ has at least two fixed points, u_1 and $u_2 \in \mathcal{P}$, provided that $f_0^+(t) = f_\infty^+(t) = 0$ and (H_2^*) holds. In such a case, we have $0 < \|u_1\|_{C(I)} < q < \|u_2\|_{C(I)}$ $\big(q$ given in $(H_2^*)\big)$.*

Moreover, for $\mathcal{G}$ satisfying $\left(N_{g_1}\right)$, we have the two following results.

Theorem 6.2.9. *Consider $g(t,u) = f(t,-u)$. If $\mathcal{G}$ satisfies $\left(N_{g_1}\right)$, then the operator $\mathcal{L}$ has at least two fixed points, u_1 and $u_2 \in \mathcal{Q}$, provided that $g_0^-(t) = g_\infty^-(t) = \infty$ for $t \in I$ and (H_1^*) holds for g. In such a case, $0 < \|u_1\|_{C(I)} < p < \|u_2\|_{C(I)}$ $\big(p$ given in $(H_1^*)\big)$.*

Theorem 6.2.10. *Consider $g(t,u) = f(t,-u)$. If $\mathcal{G}$ satisfies $\left(N_{g_1}\right)$, then the operator $\mathcal{L}$ has at least two fixed points, u_1 and $u_2 \in \mathcal{Q}$, provided that $g_0^+(t) = g_\infty^+(t) = 0$ and (H_2^*) holds for g. In such a case, $0 < \|u_1\|_{C(I)} < q < \|u_2\|_{C(I)}$ $\big(q$ given in $(H_2^*)\big)$.*

6.3 Existence of multiple fixed points

In this section, by means of Theorems 6.1.7 and 6.1.8, we prove the existence of two or three non-trivial fixed points, respectively, of operator $\mathcal{L}$ defined in (6.0.3). We follow the steps given in [4], however we impose slightly weaker conditions on f than the ones given in that reference.

Theorem 6.3.1. *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r,$$

and assume that function f satisfies the following conditions:

(i) $f(t,u) \geqslant \dfrac{u}{m_1 \displaystyle\int_{a_1}^{b_1} \phi(s)\,\mathrm{d}\,s}$ for all $t \in I_1$ and $u \in \left[r, \dfrac{K_2}{m_1}\,r\right]$, being the inequality strict at $u = r$,

(ii) $f(t,u) \leqslant \dfrac{q}{K_2 \displaystyle\int_{a}^{b} \phi(s)\,\mathrm{d}\,s}$ for all $t \in I$ and $u \in \left[0, \dfrac{K_2}{m_1}\,q\right]$, being the inequality strict at $u = q$,

(iii) $f(t,u) > \dfrac{K_2\, u}{K_1 \displaystyle\int_{a_1}^{b_1} \left|k_1(s)\right| \phi(s)\,\mathrm{d}\,s}$ for all $t \in I_1$ and $u \in \left[\dfrac{m_1}{K_2}\,p, p\right]$.

Then, if $\mathcal{G}$ satisfies $\left(P_{g_1}\right)$, the operator $\mathcal{L}$ has at least two fixed points, u_1 and u_2, such that:

$$p < \|u_1\|_{C(I)}, \quad \max_{t \in I_1} u_1(t) < q < \max_{t \in I_1} u_2(t), \quad \min_{t \in I_1} u_2(t) < r.$$

Proof. The proof is based on Theorem 6.1.7.

Consider:

$$\alpha(u) := \min_{t\in I_1} \left|u(t)\right|, \tag{6.3.1}$$

$$\theta(u) := \max_{t\in I_1} \left|u(t)\right|, \tag{6.3.2}$$

and

$$\gamma(u) := \|u\|_{C(I)}. \tag{6.3.3}$$

Clearly, $\alpha(u) \leqslant \theta(u) \leqslant \gamma(u)$ for all $u \in \mathcal{P}$.

Since $u \in \mathcal{P}$, then:

$$\alpha(u) = \min_{t\in I_1} u(t) \geqslant \min_{t\in I_1} \frac{k_1(t)}{K_2} \|u\|_{C(I)} = \frac{m_1}{K_2}\gamma(u),$$

that is, $\gamma(u) \leqslant \dfrac{K_2}{m_1}\alpha(u)$ for all $u \in \mathcal{P}$.

Moreover, for all $\lambda \in \mathbb{R}$ and $u \in \mathcal{P}$, we have:

$$\theta(\lambda\, u) = \max_{t\in I_1} \left\{\lambda\, u(t)\right\} = \lambda \max_{t\in I_1} u(t) = \lambda\,\theta(u).$$

As we have noticed, $\mathcal{L}\colon \mathcal{P} \longrightarrow \mathcal{P}$ is a completely continuous operator.

Moreover, let us consider $u \in \partial\mathcal{P}(\alpha, r)$, i.e. $\min_{t\in I_1} u(t) = r$, we have that:

$$\alpha(u) = r \geqslant \frac{m_1}{K_2}\|u\|_{C(I)}.$$

Thus, from (i), the following inequalities are satisfied:

$$\begin{aligned}\alpha(\mathcal{L}\,u) &= \min_{t\in I_1}\left\{\int_a^b \mathcal{G}(t,s)\,f\big(s,u(s)\big)\,\mathrm{d}\,s\right\} \geqslant \min_{t\in I_1}\left\{\int_a^b k_1(t)\,\phi(s)\,f\big(s,u(s)\big)\,\mathrm{d}\,s\right\}\\ &= \min_{t\in I_1} k_1(t)\int_{a_1}^{b_1}\phi(s)\,f\big(s,u(s)\big)\,\mathrm{d}\,s \geqslant m_1\int_{a_1}^{b_1}\phi(s)\,\frac{u(s)}{m_1\displaystyle\int_{a_1}^{b_1}\phi(\ell)\,\mathrm{d}\,\ell}\,\mathrm{d}\,s.\end{aligned} \tag{6.3.4}$$

On the other hand $\alpha(u) = r$ implies that there exists $t_1 \in I_1$ such that $u(t_1) = r$. Since f is a continuous function on $I_1 \subset I$, from (i), there exists $I_0 \subset I_1$ a non-trivial subinterval where the inequality given on (i) is strict. Thus, the last inequality of (6.3.4) is also strict and we have for all $u \in \partial\mathcal{P}(\alpha, r)$:

$$\alpha(\mathcal{L}\,u) > m_1\int_{a_1}^{b_1}\phi(s)\,\frac{u(s)}{m_1\displaystyle\int_{a_1}^{b_1}\phi(\ell)\,\mathrm{d}\,\ell}\,\mathrm{d}\,s \geqslant r.$$

Now, for $u \in \partial\mathcal{P}(\theta, q)$, i.e. $\max_{t\in I_1} u(t) = q$, we have that:

$$\gamma(u) \geqslant \theta(u) = q \geqslant \alpha(u) \geqslant \frac{m_1}{K_2}\gamma(u).$$

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Thus, $q \leqslant \|u\|_{C(I)} \leqslant \dfrac{K_2}{m_1} q$ and from (ii), we have:

$$\begin{aligned} \theta(\mathcal{L}\, u) &= \max_{t \in I_1} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\, \mathrm{d}\, s \right\} \leqslant \max_{t \in I_1} \left\{ \int_a^b k_2(t)\, \phi(s)\, f\big(s,u(s)\big)\, \mathrm{d}\, s \right\} \\ &\leqslant K_2 \int_a^b \phi(s)\, f\big(s,u(s)\big)\, \mathrm{d}\, s \leqslant K_2 \int_a^b \phi(s) \frac{q}{K_2 \displaystyle\int_a^b \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s. \end{aligned} \tag{6.3.5}$$

Repeating the previous arguments, $\theta(u) = q$ implies that there exists $t_1 \in I_1 \subset I$ such that $u(t_1) = q$. Since f is a continuous function on I, from (ii), there exists $I_0 \subset I_1$ a non-trivial subinterval where the inequality given on (ii) is strict. Thus, the last inequality of (6.3.5) is also strict and we have for all $u \in \partial \mathcal{P}(\theta, q)$:

$$\theta(\mathcal{L}\, u) < K_2 \int_a^b \phi(s) \frac{q}{K_2 \displaystyle\int_a^b \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s = q.$$

Finally, $\mathcal{P}(\gamma, p) = \left\{ u \in \mathcal{P} \;\middle|\; \|u\|_{C(I)} < p \right\} \neq \emptyset$ and for all $u \in \partial \mathcal{P}(\gamma, p)$, we have that $\|u\|_{C(I)} = p$ and $\alpha(u) \geqslant \dfrac{m_1}{K_2}\, p$.

Thus, from (iii), we obtain:

$$\begin{aligned} \gamma(\mathcal{L}\, u) &= \sup_{t \in I} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\, \mathrm{d}\, s \right\} \geqslant \sup_{t \in I} \left\{ \int_a^b k_1(t)\, \phi(s)\, f\big(s,u(s)\big)\, \mathrm{d}\, s \right\} \\ &= K_1 \int_{a_1}^{b_1} \phi(s)\, f\big(s,u(s)\big)\, \mathrm{d}\, s \geqslant K_1 \int_{a_1}^{b_1} \phi(s) \frac{K_2\, u(s)}{K_1 \displaystyle\int_{a_1}^{b_1} k_1(\ell) \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s \\ &\geqslant \int_{a_1}^{b_1} \phi(s) \frac{K_2 \dfrac{k_1(s)}{K_2} \|u\|_{C(I)}}{\displaystyle\int_{a_1}^{b_1} k_1(\ell)\, \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s = p. \end{aligned}$$

So, we conclude that $\gamma(\mathcal{L}\, u) > p$ for $u \in \partial \mathcal{P}(\gamma, p)$.

Hence, all the hypotheses of Theorem 6.1.7 are fulfilled. Thus, $\mathcal{L}$ has at least two fixed points on $\mathcal{P}$, u_1 and u_2, such that $p < \gamma(u_1) = \|u\|_{C(I)}$ and $q > \theta(u_1) = \max\limits_{t \in I_1} u_1(t)$. Moreover, $q < \theta(u_2) = \max\limits_{t \in I_1} u_2(t)$ and $r > \alpha(u_2) = \min\limits_{t \in I_1} u_2(t)$ and the proof is complete. □

Remark 6.3.2. *Realise that in the third item of Theorem 6.3.1, we cannot avoid the strict inequality on the whole interval, since $\|u\|_{C(I)} = p$ implies that there exists $t_1 \in I$ such that $u(t_1) = p$, however we cannot ensure that $t_1 \in I_1$.*

Now, as an application of Theorem 6.1.8, we obtain the next result that ensures the existence of at least three critical points of operator $\mathcal{L}$.

Theorem 6.3.3. *Let p, q and r be positive numbers satisfying the relation:*

$$0 < p < \frac{K_2}{m_1}\, p < q < \frac{K_2}{m_1} q \leqslant r.$$

Assume, moreover, that the function f satisfies the following conditions:

(a) $f(t,u) \leqslant \dfrac{r}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\,s}$ *for all $t \in I$ and $u \in [0,r]$,*

(b) $f(t,u) < \dfrac{p}{K_2 \displaystyle\int_a^b \phi(s)\,\mathrm{d}\,s}$ *for all $t \in I$ and $u \in \left[0, \dfrac{K_2}{m_1} p\right]$,*

(c) $f(t,u) \geqslant \dfrac{u}{m_1 \displaystyle\int_{a_1}^{b_1} \phi(s)\,\mathrm{d}\,s}$ *for all $t \in I_1$ and $u \in \left[q, \dfrac{K_2}{m_1} q\right]$, being the inequality strict for $u = q$.*

Then, if $\mathcal{G}$ satisfies $\left(P_{g_1}\right)$, the operator $\mathcal{L}$ has at least three fixed points u_1, u_2, u_3 which belong to $\left\{u \in \mathcal{P} \;\middle|\; \|u\|_{C(I)} \leqslant r\right\}$ such that:

$$\max_{t \in I_1} u_1(t) < p < \max_{t \in I_1} u_2(t) \;\; \textit{and} \;\; \min_{t \in I_1} u_2 < q < \min_{t \in I_1} u_3(t).$$

Proof. The proof follows from Theorem 6.1.8.

We consider α, θ and γ as in (6.3.1)–(6.3.3), $\Psi(u) = \alpha(u)$ and $\beta(u) = \theta(u)$. Clearly α and Ψ are concave and non-negative functionals in $\mathcal{P}$. In addition, β, θ and γ are convex and non-negative functionals in $\mathcal{P}$.

We have proved in Theorem 6.2.1 that $\mathcal{L}(\mathcal{P}) \subset \mathcal{P}$ and it is completely continuous.

Let us see now that $\mathcal{L}\left(\overline{\mathcal{P}(\gamma,r)}\right) \subset \overline{\mathcal{P}(\gamma,r)}$. Indeed, let $u \in \overline{\mathcal{P}(\gamma,r)}$ (i.e. $\|u\|_{C(I)} \leqslant r$), from (a) we have:

$$\begin{aligned}
\|\mathcal{L}\,u\|_\infty &= \sup_{t\in I}\left\{\int_a^b \mathcal{G}(t,s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s\right\} \leqslant \sup_{t\in I}\left\{\int_a^b k_2(t)\,\phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s\right\}\\
&= K_2 \int_a^b \phi(s)\, f\big(s,u(s)\big)\,\mathrm{d}\,s \leqslant K_2 \int_a^b \phi(s)\, \frac{r}{K_2 \displaystyle\int_a^b \phi(\ell)\,\mathrm{d}\,\ell}\,\mathrm{d}\,s = r,
\end{aligned}$$

thus, $\mathcal{L}\,u \in \overline{\mathcal{P}(\gamma,r)}$ and we conclude that $\mathcal{L}\left(\overline{\mathcal{P}(\gamma,r)}\right) \subset \overline{\mathcal{P}(\gamma,r)}$.

Obviously, $\alpha(u) \leqslant \beta(u)$ and $\gamma(u) = \|u\|_{C(I)}$.

We consider $u_q(t) = \dfrac{K_2}{m_1}\, q$, it is obvious that u_q belongs to the following set:

$$\left\{ u \in \mathcal{P}\left(\gamma, \theta, \alpha, q, \frac{K_2}{m_1}\, q, r\right) \,\middle|\, \alpha(u) > q \right\},$$

described as follows:

$$\left\{ u \in \mathcal{P} \,\middle|\, q < \min_{t \in I_1} u(t),\ \max_{t \in I_1} u(t) \leqslant \frac{K_2}{m_1} q,\ \|u\|_{C(I)} \leqslant r \right\} \neq \emptyset.$$

Let $u \in \mathcal{P}\left(\gamma, \theta, \alpha, q, \dfrac{K_2}{m_1}\, q, r\right)$, then using (c), we obtain:

$$\begin{aligned} \alpha(\mathcal{L}\, u) &= \min_{t \in I_1} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s, u(s)\big)\, \mathrm{d}\, s \right\} \geqslant \min_{t \in I_1} \left\{ \int_a^b k_1(t)\, \phi(s)\, f\big(s, u(s)\big)\, \mathrm{d}\, s \right\} \\ &\geqslant m_1 \int_{a_1}^{b_1} \phi(s)\, f\big(s, u(s)\big)\, \mathrm{d}\, s \geqslant m_1 \int_{a_1}^{b_1} \phi(s)\, \frac{u(s)}{m_1 \displaystyle\int_{a_1}^{b_1} \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s. \end{aligned} \tag{6.3.6}$$

If there exists $s_1 \in I_1$ such that $u(s_1) > q$, then it must exist a non-trivial subinterval of I_1 where $u(s) > q$ for all s in such subinterval. Then, directly from (6.3.6), we have $\alpha(\mathcal{L}\, u) > q$.

Now, if $u(s) = q$ for all $s \in I_1$, from (c) and (6.3.6), we obtain:

$$\alpha(\mathcal{L}\, u) \geqslant m_1 \int_{a_1}^{b_1} \phi(s)\, f(s, q)\, \mathrm{d}\, s > m_1 \int_{a_1}^{b_1} \phi(s)\, \frac{q}{m_1 \displaystyle\int_{a_1}^{b_1} \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s = q.$$

So, the hypothesis $a)$ on Theorem 6.1.8 is fulfilled. Now, let us see $b)$.

If we consider the function $u_p(t) = \dfrac{m_1}{K_2}\, p$, it is clear that it belongs to the set:

$$\left\{ u \in \mathcal{P}\left(\gamma, \beta, \Psi, \frac{m_1}{K_2}\, p, p, r\right) \,\middle|\, \beta(u) < p \right\},$$

which is described as follows:

$$\left\{ u \in \mathcal{P} \,\middle|\, \frac{m_1}{K_2}\, p \leqslant \min_{t \in I_1} u(t),\ \max_{t \in I_1} u(t) < p,\ \|u\|_{C(I)} \leqslant r \right\} \neq \emptyset.$$

Let $u \in \mathcal{P}\left(\gamma, \beta, \Psi, \dfrac{m_1}{K_2}\, p, p, r\right)$, since $u \in \mathcal{P}$, we have that $\|u\|_{C(I)} \leqslant \dfrac{K_2}{m_1}\, p$. Thus, from (b), we have:

$$\begin{aligned} \beta(\mathcal{L}\, u) &= \max_{t \in I_1} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s, u(s)\big)\, \mathrm{d}\, s \right\} \leqslant \max_{t \in I_1} \left\{ \int_a^b K_2\, \phi(s)\, f\big(s, u(s)\big)\, \mathrm{d}\, s \right\} \\ &< K_2 \int_a^b \phi(s)\, \frac{p}{K_2 \displaystyle\int_a^b \phi(\ell)\, \mathrm{d}\, \ell}\, \mathrm{d}\, s = p. \end{aligned}$$

Thus, $\beta(\mathcal{L}\,u) < p$ for all $u \in \mathcal{P}\left(\gamma, \beta, \Psi, \dfrac{m_1}{K_2}\,p, p, r\right)$ and, as a consequence, condition $b)$ in Theorem 6.1.8 is satisfied.

Now, let $u \in \mathcal{P}(\gamma, \alpha, q, r)$ be such that $\theta(\mathcal{L}\,u) > q$. Then:

$$\begin{aligned}
\alpha(\mathcal{L}\,u) &= \min_{t \in I_1} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \right\} \geqslant \min_{t \in I_1} \left\{ \int_a^b k_1(t)\,\phi(s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \right\} \\
&= \frac{m_1}{K_2} \int_a^b K_2\,\phi(s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \geqslant \frac{m_1}{K_2} \int_a^b \max_{t \in I_1} \Big\{ \mathcal{G}(t,s) \Big\}\, f\big(s, u(s)\big)\,\mathrm{d}\,s \\
&\geqslant \frac{m_1}{K_2} \max_{t \in I_1} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \right\} = \frac{m_1}{K_2}\theta(\mathcal{L}\,u) > \frac{m_1}{K_2}\,q.
\end{aligned}$$

Finally, take $u \in \mathcal{Q}(\gamma, \beta, p, r)$ such that $\Psi(\mathcal{L}\,u) < \dfrac{m_1}{K_2} p$, then:

$$\begin{aligned}
\beta(\mathcal{L}\,u) &= \max_{t \in I_1} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \right\} \leqslant \int_a^b K_2\,\phi(s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \\
&= \frac{K_2}{m_1} \int_a^b m_1\,\phi(s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \leqslant \frac{K_2}{m_1} \int_a^b \min_{t \in I_1} \Big\{ \mathcal{G}(t,s) \Big\}\, f\big(s, u(s)\big)\,\mathrm{d}\,s \\
&\leqslant \frac{K_2}{m_1} \min_{t \in I_1} \left\{ \int_a^b \mathcal{G}(t,s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s \right\} = \frac{K_2}{m_1}\Psi(\mathcal{L}\,u) < \frac{K_2}{m_1}\frac{m_1}{K_2}\,p = p.
\end{aligned}$$

Therefore, all the hypotheses of Theorem 6.1.8 are fulfilled and we have ensured the existence of at least three critical points such that:

$$\begin{aligned}
\beta(u_1) = \max_{t \in I_1} u_1(t) < p < \beta(u_2) = \max_{t \in I_1} u_2(t), \\
\alpha(u_2) = \min_{t \in I_1} u_2(t) < q < \alpha(u_3) = \min_{t \in I_1} u_3(t).
\end{aligned}$$

Then, we have completely proved the result. □

As in Section 6.2, we obtain the analogous results in the case $\mathcal{G}$ satisfies property $\left(N_{g_1}\right)$. They are shown below.

Theorem 6.3.4. *Consider $g(t,u) = f(t,-u)$. Let p, q, r and g as in Theorem 6.3.1. Then, if $\mathcal{G}$ satisfies $\left(N_{g_1}\right)$, the operator $\mathcal{L}$ has at least two fixed points, u_1 and $u_2 \in \mathcal{Q}$, such that:*

$$p < \|u_1\|_{C(I)}, \quad \min_{t \in I_1} u_1(t) > -q > \min_{t \in I_1} u_2(t), \quad \max_{t \in I_1} u_2(t) > -r.$$

Theorem 6.3.5. *Consider $g(t,u) = f(t,-u)$. Let p, q, r and g as in Theorem 6.3.3. Then, if $\mathcal{G}$ satisfies $\left(N_{g_1}\right)$, the operator $\mathcal{L}$ has at least three fixed points u_1, u_2, u_3 which belong to $\Big\{ u \in \mathcal{Q} \mid \|u\|_{C(I)} \leqslant r \Big\}$ and:*

$$\min_{t \in I_1} u_1(t) > -p > \min_{t \in I_1} u_2(t) \;\; \textit{and} \;\; \max_{t \in I_1} u_2 > -q > \max_{t \in I_1} u_3(t).$$

6.4 Existence results for non-linear boundary value problems

In this section, we use the fixed point theorems which we have obtained previously, coupled with the fact that the existence of a solution of problem (6.0.1) is equivalent to the existence of a fixed point of operator $\mathcal{L}[M]$, defined in (6.0.2).

In Chapter 3, we have studied the parameter set for which g_M satisfies either property $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$, see Theorems 3.7.1 and 3.7.3.

Moreover, in Chapter 4 (Section 4.3), we have provided several examples, with different boundary conditions, where we obtain the exact intervals for which the related Green's functions satisfies $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$.

In addition, in Chapter 5, we have studied different examples coupled with the simply supported beam boundary conditions.

Thus, we can apply the previous results to all of those examples. The main difficulty of obtaining the functions k_1 and k_2 lies in the expression of the related Green's function.

The construction of k_1 and k_2 from $g_M(t,s)$ has been described in the proof of Theorem 3.7.1. In the positive case, they are given by:

$$\begin{aligned} k_1(t) &= \min_{s \in I} \tilde{v}_M^t(s),\ t \in (a,b), \\ k_2(t) &= \max_{s \in I} \tilde{v}_M^t(s),\ t \in (a,b), \end{aligned} \tag{6.4.1}$$

where, for each $t \in (a,b)$, $\tilde{v}_M^t$ is the continuous extension of:

$$v_M(t,s) = \frac{g_M(t,s)}{(s-a)^\eta (b-s)^\gamma},$$

to I, with η and γ defined in expressions (3.4.7)-(3.4.8).

On the other hand, whenever g_M satisfies $\left(N_{g_1}\right)$, k_1 and k_2 are given by:

$$\begin{aligned} k_1(t) &= \max_{s \in I} \tilde{v}_M^t(s),\ t \in (a,b), \\ k_2(t) &= \min_{s \in I} \tilde{v}_M^t(s),\ t \in (a,b). \end{aligned} \tag{6.4.2}$$

In both cases, $\phi(s) = (s-a)^\eta (b-s)^\gamma$.

In the sequel, along this section, let us denote $\tilde{u}(t,s) = \tilde{v}_M^t(s)$.

Sometimes, it is really difficult to find the exact functions k_1 and k_2 as they are defined above. However, we can consider some approximations, ℓ_1 and ℓ_2, such that:

$$\left|\ell_1(t)\right| \leqslant \left|k_1(t)\right| \text{ and } \left|\ell_2(t)\right| \geqslant \left|k_2(t)\right|, \quad \forall\, t \in [0,1]. \tag{6.4.3}$$

Clearly, the bounds obtained for ℓ_1 and ℓ_2 are still suitable for the results obtained in Sections 6.2 and 6.3.

Moreover, once that either property $\left(P_{g_1}\right)$ or $\left(N_{g_1}\right)$ are fulfilled, we can apply Corollaries 6.2.4 and 6.2.5, respectively. Thus, we can prove the existence of at least a fixed point (which means a solution of boundary value problem (6.0.1)) for some particular f, without knowing the expression of the Green's function. In particular, f is either sub-linear:

$$\lim_{u\to 0^+}\frac{f(t,u)}{u}=+\infty \ \text{ and } \ \lim_{u\to +\infty}\frac{f(t,u)}{u}=0$$

or, on the other hand, super-linear:

$$\lim_{u\to 0^+}\frac{f(t,u)}{u}=0 \ \text{ and } \ \lim_{u\to +\infty}\frac{f(t,u)}{u}=+\infty.$$

This is a useful property to ensure the existence of solutions for a wide range of non-linear boundary value problems, however we do not localise where such a solution is placed. In the sequel, we will describe several examples for which k_1 and k_2 are obtained. Despite the calculus are hard in some cases, the information obtained will be very interesting, since we prove the existence of multiple solutions and, in addition, we obtain some properties of such solutions.

6.4.1 Operator $T_4^0[M]=\frac{\mathrm{d}^4}{\mathrm{d}\,t^4}+M$ in $X_{\{0,2\}}^{\{1,2\}}$

Along Chapters 3 and 4, we have studied operator $T_4^0[M]$ in the set $X_{\{0,2\}}^{\{1,2\}}$ with $I=[0,1]$.

In particular, in Examples 3.7.2 and 3.7.9 we have obtained the following conclusions for the related Green's function, $g_M(t,s)$. It satisfies de property $\left(P_{g_1}\right)$ if, and only if, $M\in\left(-m_1^4,4\pi^4\right]$, where $m_1\approxeq 2.36502$ has been defined as the least positive solution of (3.6.2). Moreover, $g_M(t,s)$ satisfies property $\left(N_{g_1}\right)$ if, and only if, $M\in\left[-\pi^4,-m_1^4\right)$.

In particular, $g_0(t,s)$ satisfies $\left(P_{g_1}\right)$ and $g_{-\pi^4}(t,s)$, $\left(N_{g_1}\right)$. Let us study these two particular cases.

- $T_4^0[0]=\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$

We want to study the existence of solutions of the problem:

$$\begin{cases} u^{(4)}(t)=f\big(t,u(t)\big),\ t\in[0,1],\\ u(0)=u''(0)=0,\\ u'(1)=u''(1)=0.\end{cases} \tag{6.4.4}$$

From the adjoint boundary conditions, given in Example 3.4.4, we have that $\eta=\gamma=1$ and:

$$\phi(s)=s\,(1-s). \tag{6.4.5}$$

Using the *Mathematica* program, developed in [21], we calculate the related Green's function:

$$g_0(t,s)=\begin{cases}\dfrac{1}{6}s\Big(t\left(t^2-3t+3\right)-s^2\Big), & 0\leqslant s\leqslant t\leqslant 1,\\[2ex] \dfrac{1}{6}(1-s)\,t\left(3s-t^2\right), & 0<t<s\leqslant 1.\end{cases}$$

This expression has been previously defined in Example 3.3.13. Realise that the functions k_1 and k_2 for that example are different from the ones which we need here because, in that example, we have obtained them to prove property (P_g) and, now, we want to calculate functions which appear on the definition of property $\left(P_{g_1}\right)$, which is slightly different to (P_g).

So, we have:

$$\tilde{u}(t,s) = \begin{cases} \dfrac{\left(t\left(t^2-3t+3\right)-s^2\right)}{6\,(1-s)}, & 0 \leqslant s \leqslant t < 1, \\ \dfrac{1}{6}t\left(3-\dfrac{t^2}{s}\right), & 0 < t < s \leqslant 1,\ t \neq 1, \end{cases}$$

where $\tilde{u}(t,s)$ is the continuous extension of $\dfrac{g_0(t,s)}{\phi(s)}$ to $(0,1)\times[0,1]$.

Directly, we obtain:

$$\frac{\partial \tilde{u}(t,s)}{\partial s} = \begin{cases} \dfrac{s^2-2\,s+t\left(3-3\,t+t^2\right)}{6\,(1-s)^2}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{t^3}{6s^2}, & 0 < t < s \leqslant 1, \end{cases}$$

which is positive for all $(t,s)\in(0,1)\times[0,1]$.

Thus, $\tilde{u}(t,s)$ is increasing as a function of s for all $t\in(0,1)$. So:

$$k_1(t) = \tilde{u}(t,0) = \frac{1}{6}\,t\left(3-3\,t+t^2\right), \tag{6.4.6}$$

$$k_2(t) = \tilde{u}(t,1) = \frac{1}{6}\,t\,\left(3-t^2\right). \tag{6.4.7}$$

In Figure 6.4.1, we represent the function $\tilde{u}(t,s) = \dfrac{g(t,s)}{\phi(s)}$, coupled with the bounds, $k_2(t)$ and $k_1(t)$. In addition, in Figure 6.4.2, it appears the same representation considering the constant values $t_0 = \dfrac{1}{3}$ and $s_0 = \dfrac{1}{3}$, respectively.

From the expression of k_1 and k_2, given in (6.4.6)-(6.4.7), we have:

$$K_1 = k(1) = \frac{1}{6}, \text{ and } K_2 = k_2(1) = \frac{1}{3}.$$

Since k_1 is an increasing function on $[0,1]$, the best choice for I_1 are intervals which are of the form $[c,1]$, where $c\in(0,1)$. In this case,

$$m_1 = k_1(c) = \frac{1}{6}\,c\left(3-3\,c+c^2\right).$$

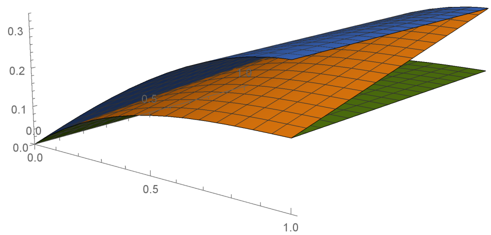

Figure 6.4.1: $\tilde{u}(t,s)$ (orange) bounded from above by $k_2(t)$ (blue) and from below by $k_1(t)$ (green).

Figure 6.4.2: Figure 6.4.1 for $t_0 = \dfrac{1}{3}$ on the left and for $s_0 = \dfrac{1}{3}$ on the right.

Now, we calculate the expressions which are necessary in order to have the correspondent results of Sections 6.2 and 6.3 for this case:

$$\int_0^1 \phi(s)\,\mathrm{d}\,s = \frac{1}{6},\quad \int_c^1 \phi(s)\,\mathrm{d}\,s = \frac{1}{6}\,(1-c)^2(1+2\,c), \tag{6.4.8}$$

and,

$$\int_c^1 k_1(s)\,\phi(s)\,\mathrm{d}\,s = \frac{5\,c^6 - 24\,c^5 + 45\,c^4 - 30\,c^3 + 4}{180}.$$

Hence, the needed bounds are the following ones:

$$\frac{p}{K_2 \displaystyle\int_0^1 \phi(s)\,\mathrm{d}\,s} = 18\,p, \tag{6.4.9}$$

$$\frac{K_2\,u}{K_1 \displaystyle\int_c^1 k_1(s)\,\phi(s)\,\mathrm{d}\,s} = \frac{360\,u}{5\,c^6 - 24\,c^5 + 45\,c^4 - 30\,c^3 + 4}, \tag{6.4.10}$$

$$\frac{u}{m_1 \displaystyle\int_c^1 \phi(s)\,\mathrm{d}\,s} = \frac{36\,u}{c\,(1-c)^2(1+2\,c)\,(3-3\,c+c^2)}. \tag{6.4.11}$$

We can easily see that (6.4.10) is an increasing function of c and (6.4.11) attains its minimum for $c \in [0.339, 0.34]$. We try to make these bounds as small as possible. However,

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we cannot minimise them simultaneously. We choose, for instance, $c = \dfrac{1}{3}$. In this case, $m_1 = \dfrac{19}{162}$ and (6.4.10) and (6.4.11) are expressed in the following way:

$$\frac{K_2\, u}{K_1 \displaystyle\int_c^1 k_1(s)\, \phi(s)\, \mathrm{d}\, s} = \frac{63610\, u}{611}, \tag{6.4.12}$$

$$\frac{u}{m_1 \displaystyle\int_c^1 \phi(s)\, \mathrm{d}\, s} = \frac{6561\, u}{95}. \tag{6.4.13}$$

So, we can transform the previous results for this particular case. First, let us write the correspondent (H_1) and (H_2).

(H_1) There exists $p > 0$ such that:

$$f(t,u) \leqslant 18\, p, \quad \forall\, t \in [0,1],\ \forall\, u \in [0,p].$$

(H_2) There exists $q > 0$ such that:

$$f(t,u) \geqslant \frac{63610\, u}{611}, \quad \forall\, t \in \left[\frac{1}{3}, 1\right],\ \forall\, u \in \left[\frac{19}{54}\, q, q\right].$$

Finally, let us write the results collected in Section 6.3. Using (6.4.9), (6.4.12) and (6.4.13), we state Theorem 6.3.1 for this case.

Theorem 6.4.1. *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r,$$

and assume that function f satisfies the following conditions:

(i) $f(t,u) \geqslant \dfrac{6561\, u}{95}$ *for all* $t \in \left[\dfrac{1}{3}, 1\right]$ *and* $u \in \left[r, \dfrac{54}{19}\, r\right]$*, being the inequality strict at* $u = r$*,*

(ii) $f(t,u) \leqslant 18\, q$ *for all* $t \in [0,1]$ *and* $u \in \left[0, \dfrac{54}{19}\, q\right]$*, being the inequality strict at* $u = q$*,*

(iii) $f(t,u) > \dfrac{63610\, u}{611}$ *for all* $t \in \left[\dfrac{1}{3}, 1\right]$ *and* $u \in \left[\dfrac{19}{54}\, p, p\right]$.

Then, the problem (6.4.4) *has at least two positive solution, u_1 and u_2, such that:*

$$p < \|u_1\|_{C(I)}, \quad \max_{t \in \left[\frac{1}{3},1\right]} u_1(t) < q < \max_{t \in \left[\frac{1}{3},1\right]} u_2(t), \quad \min_{t \in \left[\frac{1}{3},1\right]} u_2(t) < r. \tag{6.4.14}$$

Now, consider the following continuous function:

$$f(t,u)=\begin{cases} 54\,t, & u\leqslant \dfrac{1}{54},\\[2ex] \dfrac{t}{u}, & u\in\left(\dfrac{1}{54},3\right),\\[2ex] \dfrac{t\,u^2}{27}, & u\geqslant 3. \end{cases} \tag{6.4.15}$$

Let us choose $p=\dfrac{1}{19}$, $q=3$ and $r=5595$. So, we have:

(i) For $u\geqslant r$, $f(t,u)\geqslant \dfrac{t\,5595\,u}{27}\geqslant \dfrac{1865\,u}{27}>\dfrac{6531\,u}{95}$ for every $t\in\left[\dfrac{1}{3},1\right]$,

(ii) For all $t\in[0,1]$ and $u\in\left[0,\dfrac{162}{19}\right]$ we have:

$$\begin{aligned} f(t,u)\leqslant \max\left\{f\,(t,0),\, f\left(t,\frac{162}{19}\right)\right\} &= \max\left\{54\,t,\frac{972}{361}t\right\}\\ = 54\,t\leqslant 54 = 3\cdot 18, \end{aligned}$$

being the inequality strict for $u>\dfrac{1}{54}$. In particular, the strict inequality is fulfilled for $u=3$.

(iii) $f(t,u)=\dfrac{t}{u^2}u\geqslant \dfrac{19^2}{3}\,u>\dfrac{63610\,u}{611}$ for every $t\in\left[\dfrac{1}{3},1\right]$ and $u\in\left[\dfrac{1}{54},\dfrac{1}{19}\right]$.

So, we can ensure the existence of at least two positive solutions of problem (6.4.4), with f defined in (6.4.15), satisfying (6.4.14).

Now, we have the next result which ensures the existence of three solutions.

Theorem 6.4.2. *Let p, q and r be positive numbers satisfying the relation:*

$$0<p<\frac{54}{19}<q<\frac{54}{19}q\leqslant r.$$

Assume, moreover, that the function f satisfies the following conditions:

(a) $f(t,u)\leqslant 18\,r$ *for all* $t\in[0,1]$ *and* $u\in[0,r]$,

(b) $f(t,u)<18\,p$ *for all* $t\in[0,1]$ *and* $u\in\left[0,\dfrac{54}{19}\,p\right]$,

(c) $f(t,u)\geqslant\dfrac{6561\,u}{95}$ *for all* $t\in\left[\dfrac{1}{3},1\right]$ *and* $u\in\left[q,\dfrac{54}{19}q\right]$, *being the inequality strict for* $u=q$.

Then, problem (6.4.4) *has at least three solutions,* u_1, u_2, u_3 *such that:* $\|u_i\|_{C(I)} \leqslant r$ *for* $i = 1, 2, 3$ *and*

$$\max_{t\in\left[\frac{1}{3},1\right]} u_1(t) < p < \max_{t\in\left[\frac{1}{3},1\right]} u_2(t), \qquad \min_{t\in\left[\frac{1}{3},1\right]} u_2(t) < q < \min_{t\in\left[\frac{1}{3},1\right]} u_3(t). \tag{6.4.16}$$

Consider the continuous function:

$$f(t,u) = \begin{cases} 105\,t\,u^2, & u \leqslant 6, \\[2ex] \dfrac{3780\,t}{u-5}, & u > 6. \end{cases} \tag{6.4.17}$$

If we choose $r = 210$, $p = \dfrac{19}{324}$ and $q = 2$, we have:

(a) $f(t,u) \leqslant f(t,6) = 3780\,t \leqslant 3780 = 18 \cdot 210$ for all $t \in [0,1]$ and $u \in [0,210]$,

(b) $f(t,u) \leqslant f\left(t, \dfrac{1}{6}\right) \leqslant \dfrac{35}{2} \cdot \dfrac{1}{6} < 18 \cdot \dfrac{1}{6}$ for all $t \in [0,1]$ and $u \in \left[0, \dfrac{1}{6}\right]$,

(c) $f(t,u) \geqslant \dfrac{105 \cdot 2\,u}{3} = 70\,u > \dfrac{6561\,u}{95}$ for all $t \in \left[\dfrac{1}{3}, 1\right]$ and $u \in \left[2, \dfrac{108}{19}\right]$.

Thus, by using the previous result, we conclude that problem (6.4.4) has at least three solutions for f defined in (6.4.17), satisfying (6.4.16).

- $T_4^0[-\pi^4] = \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} - \pi^4$

In this case, by using the second choice of Remark 6.2.2, we want to prove the existence of negative solutions of the problem:

$$\begin{cases} u^{(4)}(t) - \pi^4 u(t) = f\big(t, u(t)\big), & t \in [0,1], \\ u(0) = u''(0) = 0, \\ u'(1) = u''(1) = 0. \end{cases} \tag{6.4.18}$$

Clearly, ϕ is the same as the one given in (6.4.5).

By means of the *Mathematica* program, described in [21], we obtain the expression of the related Green's function, $g_{-\pi^4}(t,s)$:

$$\begin{cases} \dfrac{\operatorname{csch}(\pi)\Big(\sinh(\pi s)\big(\sin(\pi t) - \sinh(\pi - \pi t)\big) - \sin(\pi s)\big(\sinh(\pi t) - \sinh(\pi)\cos(\pi t) + \cosh(\pi)\sin(\pi t)\big)\Big)}{2\pi^3}, & 0 \leqslant s \leqslant t \leqslant 1, \\[3ex] \dfrac{\operatorname{csch}(\pi)\Big(\sin(\pi t)\big(\sinh(\pi s) + \sinh(\pi)\cos(\pi s) - \cosh(\pi)\sin(\pi s)\big) - \sinh(\pi t)\big(\sin(\pi s) + \sinh(\pi - \pi s)\big)\Big)}{2\pi^3}, & 0 < t < s \leqslant 1. \end{cases}$$

The expression of this function is more complicated than the one obtained for $M = 0$. In any case, we can also obtain $\tilde{u}(t,s)$ as the continuous extension of $u(t,s)$, given by the

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expression:

$$\begin{cases} \dfrac{\operatorname{csch}(\pi)\Big(\sinh(\pi s)\big(\sin(\pi t)-\sinh(\pi-\pi t)\big)-\sin(\pi s)\big(\sinh(\pi t)-\sinh(\pi)\cos(\pi t)+\cosh(\pi)\sin(\pi t)\big)\Big)}{2\pi^3 s\,(1-s)}, & 0<s\leqslant t<1,\\[2ex] \dfrac{\operatorname{csch}(\pi)\Big(\sin(\pi t)\big(\sinh(\pi s)+\sinh(\pi)\cos(\pi s)-\cosh(\pi)\sin(\pi s)\big)-\sinh(\pi t)\big(\sin(\pi s)+\sinh(\pi-\pi s)\big)\Big)}{2\pi^3 s\,(1-s)}, & 0<t<s<1. \end{cases}$$

By studying this function, it can be seen that for each $t \in (0,1)$, the maximum is found at $s=0$. So, we have:

$$k_1(t) = \frac{\cos(\pi t) - \cosh(\pi t) + \tan\left(\frac{\pi}{2}\right)\Big(\sinh(\pi t) - \sinh(\pi t)\Big)}{2\,\pi^2}. \tag{6.4.19}$$

The expression of $k_2(t)$ is difficult to find, but we can see that the minimum of $k_2(t)$ is given by:

$$k_2\left(\frac{1}{2}\right) = \tilde{u}\left(\frac{1}{2},1\right) = -\frac{4}{\pi^3}.$$

Thus, let us consider this lower bound for $\tilde{u}(t,s)$. In Figure 6.4.3, we represent the function $\tilde{u}(t,s) = \dfrac{g(t,s)}{\phi(s)}$, bounded from below by $k_2\left(\dfrac{1}{2}\right)$ and from above by $k_1(t)$. In addition, in Figure 6.4.4, it appears the same representation considering the constant values $t_0 = \dfrac{9}{10}$ and $s_0 = \dfrac{9}{10}$, respectively.

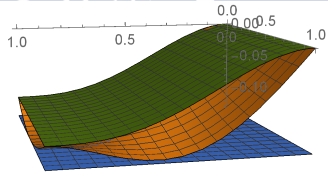

Figure 6.4.3: $\tilde{u}(t,s)$ (orange) bounded from below by $k_2(t)$ (blue) and from above by $k_1(t)$ (green).

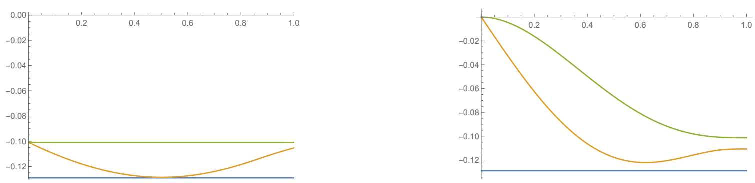

Figure 6.4.4: Figure 6.4.3 for $t_0 = \dfrac{9}{10}$ on the left and for $s_0 = \dfrac{9}{10}$ on the right.

As before, we can directly obtain:

$$K_1 = -k(1) = \frac{1}{\pi^2}, \quad \text{and} \quad K_2 = \frac{4}{\pi^3}.$$

Moreover, as in previous example, k_1 is an decreasing function on $[0,1]$, if we choose $I_1=[c,1]$, where $c\in(0,1)$, then $m_1=-k_1(c)$. If we consider $c=\dfrac{1}{3}$, as in the previous example, we have:

$$m_1=\frac{-1+2\,\cosh\left(\frac{\pi}{6}\right)\,\operatorname{sech}\left(\frac{\pi}{2}\right)+\sqrt{3}\,\tanh\left(\frac{\pi}{2}\right)}{4\,\pi^2}.$$

Moreover, (6.4.8) is still applicable for this case with $c=\dfrac{1}{3}$ and:

$$\int_{\frac{1}{3}}^{1}\left|k_1(s)\right|\phi(s)\,\mathrm{d}\,s=\frac{18\left(\sqrt{3}+\tanh\left(\frac{\pi}{6}\right)\right)+\pi\left(3-3\sqrt{3}\tanh\left(\frac{\pi}{2}\right)+6\cosh\left(\frac{\pi}{6}\right)\operatorname{sech}\left(\frac{\pi}{2}\right)\right)}{36\pi^5}$$
$$+\frac{2\pi\left(\sqrt{3}+\tanh\left(\frac{\pi}{2}\right)+2\sinh\left(\frac{\pi}{6}\right)\operatorname{sech}\left(\frac{\pi}{2}\right)\right)}{36\pi^5}.$$

So, we have:

$$\frac{p}{K_2\displaystyle\int_0^1\phi(s)\,\mathrm{d}\,s}=\frac{3\,\pi^3}{2}\,p, \tag{6.4.20}$$

$$\frac{K_2\,u}{K_1\displaystyle\int_{\frac{1}{3}}^{1}\left|k_1(s)\right|\phi(s)\,\mathrm{d}\,s}=\frac{4\,u}{\pi\displaystyle\int_{\frac{1}{3}}^{1}\left|k_1(s)\right|\phi(s)\,\mathrm{d}\,s}\approxeq 135.19\,u<136\,u,\quad u>0 \tag{6.4.21}$$

$$\frac{u}{m_1\displaystyle\int_{\frac{1}{3}}^{1}\phi(s)\,\mathrm{d}\,s}=\frac{162\pi^2u}{5\left(-1+\sqrt{3}\tanh\left(\frac{\pi}{2}\right)+2\cosh\left(\frac{\pi}{6}\right)\operatorname{sech}\left(\frac{\pi}{2}\right)\right)} \tag{6.4.22}$$
$$\approxeq 213.55\,u<214\,u,\quad u>0.$$

Moreover, we have for this case:

$$\frac{m_1}{K_2}=\frac{1}{16}\pi\left(-1+\sqrt{3}\tanh\left(\frac{\pi}{2}\right)+2\cosh\left(\frac{\pi}{6}\right)\operatorname{sech}\left(\frac{\pi}{2}\right)\right)\approxeq 0.29402>\frac{147}{500}. \tag{6.4.23}$$

To achieve the correspondent Theorems 6.3.4 and 6.3.5 for this problem, we use the approximations of the bounds which are also applicable for obtaining the results, their expressions are much easier to work with and we do not lose much information. Indeed, in many cases, we can avoid the strict inequalities. First, let us write (H_1) and (H_2) for this particular case.

(H_1) There exists $p>0$ such that:

$$f(t,u)\leqslant\frac{3\,\pi^3}{2}\,p,\quad\forall\,t\in[0,1],\ \forall\,u\in[-p,0].$$

(H_2) There exists $q > 0$ such that:

$$f(t,u) \geqslant -136\,u, \quad \forall\, t \in \left[\frac{1}{3}, 1\right],\ \forall\, u \in \left[-q, -\frac{147}{500}\,q\right].$$

Finally, let us show the results collected in Section 6.3. Using (6.4.20), (6.4.21), (6.4.22) and (6.4.23), we transform Theorem 6.3.4 for this case.

Theorem 6.4.3. *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r,$$

and assume that function f satisfies the following conditions:

(i) $f(t,u) \geqslant -214\,u$ *for all* $t \in \left[\frac{1}{3}, 1\right]$ *and* $u \in \left[-\frac{500}{147}\,r, -r\right]$,

(ii) $f(t,u) \leqslant \frac{3\,\pi^3}{2}\,q$ *for all* $t \in [0,1]$ *and* $u \in \left[-\frac{500}{147}\,q, 0\right]$, *being the inequality strict at* $u = q$,

(iii) $f(t,u) \geqslant -136\,u$ *for all* $t \in \left[\frac{1}{3}, 1\right]$ *and* $u \in \left[-p, -\frac{147}{500}\,p\right]$.

Then, problem (6.4.18) *has at least two negative solutions, u_1 and u_2, such that:*

$$p < \|u_1\|_{C(I)}, \qquad \min_{t \in \left[\frac{1}{3}, 1\right]} u_1(t) > -q > \min_{t \in \left[\frac{1}{3}, 1\right]} u_2(t), \qquad \max_{t \in \left[\frac{1}{3}, 1\right]} u_2(t) > -r. \tag{6.4.24}$$

As an example, consider the following continuous function:

$$f(t,u) = \begin{cases} \dfrac{6000}{49}\,t, & u \geqslant -\dfrac{49}{2000}, \\[2ex] -\dfrac{3\,t}{u}, & u \in \left(-\sqrt[7]{36}, -\dfrac{49}{2000}\right), \\[2ex] \dfrac{t\,u^2\sqrt{-u}}{2}, & u \leqslant -\sqrt[7]{36}. \end{cases} \tag{6.4.25}$$

Let us choose $p = \frac{1}{12}$, $q = \frac{66}{25}$ and $r = 121$. So, we have:

(i) For $u \leqslant -r$, $f(t,u) \geqslant -\dfrac{121\,\sqrt{121}\,u}{3 \cdot 2} = -\dfrac{1331\,u}{6} \geqslant -214\,u$ for every $t \in \left[\frac{1}{3}, 1\right]$,

(ii) For all $t \in [0,1]$ and $u \in \left[-\frac{440}{49}, 0\right] \subset [-9, 0]$ we have:

$$\begin{aligned} f(t,u) &\leqslant \max\left\{\max_{t\in[0,1]} f\,(t,0),\ \max_{t\in[0,1]} f\,(t,9)\right\} = \max\left\{\frac{6000}{49}, \frac{243}{2}\right\} \\ &= \frac{6000}{49} < \frac{3\,\pi^3}{2} \cdot \frac{66}{25}, \end{aligned}$$

being the inequality strict for $u < -\dfrac{49}{2000}$. In particular, for $u = \dfrac{66}{25}$.

(iii) $f(t,u) = -\dfrac{3\,t}{u^2} u \geqslant -144\,u > -136\,u$ for every $t \in \left[\dfrac{1}{3}, 1\right]$ and $u \in \left[-\dfrac{1}{12}, -\dfrac{49}{2000}\right]$.

So, we can ensure the existence of at least two negative solutions for problem (6.4.18), with f defined in (6.4.25), satisfying (6.4.24).

Now, we have the next result which ensures the existence of three solutions, which is a particular case of Theorem 6.3.5.

Theorem 6.4.4. *Let p, q and r be positive numbers satisfying the relation:*

$$0 < p < \frac{500}{147} q < q < \frac{500}{147} q \leqslant r.$$

Assume, moreover, that function f satisfies the following conditions:

(a) $f(t,u) \leqslant \dfrac{3\,\pi^3}{2}\, r$ *for all* $t \in [0,1]$ *and* $u \in [-r, 0]$,

(b) $f(t,u) < \dfrac{3\,\pi^3}{2}\, p$ *for all* $t \in [0,1]$ *and* $u \in \left[-\dfrac{500}{147} p, 0\right]$,

(c) $f(t,u) \geqslant -214\,u$ *for all* $t \in \left[\dfrac{1}{3}, 1\right]$ *and* $u \in \left[-\dfrac{500}{147} q, -q\right]$.

Then, problem (6.4.18) *has at least three solutions, u_1, u_2, u_3 such that $\|u_i\|_{C(I)} \leqslant r$ for $i = 1, 2, 3$ and:*

$$\min_{t \in \left[\frac{1}{3}, 1\right]} u_1(t) > -p > \min_{t \in \left[\frac{1}{3}, 1\right]} u_2(t), \quad \max_{t \in \left[\frac{1}{3}, 1\right]} u_2(t) > -q > \max_{t \in I_1} u_3(t). \tag{6.4.26}$$

Now, consider the continuous function:

$$f(t,u) = \begin{cases} 5\,\pi^3\,u^2, & u \geqslant -5, \\ \\ 125\,\pi^3, & u < -5. \end{cases} \tag{6.4.27}$$

In this case, we consider an autonomous function. Thus, the obtained bounds are applicable for all $t \in [0,1]$. Let us choose $r = \dfrac{250}{3}$, $p = \dfrac{147}{2000}$ and $q = \dfrac{7}{5}$, we have:

(a) $f(t,u) \leqslant f(t,5) = 125\,\pi^3 = \dfrac{3\,\pi^3}{2} \cdot \dfrac{250}{3}$ for all $u \in \left[-\dfrac{250}{3}, 0\right]$,

(b) $f(t,u) \leqslant f\left(t, \dfrac{1}{4}\right) = \dfrac{5\,\pi^3}{4} \cdot \dfrac{1}{4} < \dfrac{3\,\pi^3}{2} \cdot \dfrac{1}{4}$ for all $u \in \left[-\dfrac{1}{4}, 0\right]$,

(c) $f(t,u) \geqslant -5\,\pi^3 \dfrac{7}{5}\, u > -214\,u$ for all $u \in \left[-\dfrac{100}{21}, -\dfrac{7}{5}\right]$.

Thus, by using the previous result, we conclude that problem (6.4.18) has at least three solutions satisfying (6.4.26), for f defined in (6.4.27).

6.4.2 $(k, n-k)$ boundary conditions

This section is devoted to apply previous results for some particular cases with fixed $(k, n-k)$ boundary conditions. We distinguish the cases depending on the value of n.

- Second order: $n = 2$ and $k = 1$.

Let us study a second order operator with constant coefficients:

$$T_2[B, M]\, u(t) = u''(t) + B\, u'(t) + M\, u(t), \quad t \in [0, 1], \tag{6.4.28}$$

for a fixed $B \in \mathbb{R}$.

We consider the space of definition related to the boundary conditions $(1, 1)$. That is, X_1, with X_k defined in (1.1.8).

$T_2[B, 0]$ is always a disconjugate equation on $[0, 1]$ for all $B \in \mathbb{R}$ (it is a composition of two first order operators, see Theorem 1.1.7). In particular, the related Green's function of operator $T_2[B, 0]$, $g_B(t, s)$, satisfies property $\left(N_{g_1}\right)$ for all $B \in \mathbb{R}$.

We will use, in this case, the first option which appears in Remark 6.2.2. Thus, we will study the existence of positive solutions of the problem:

$$\begin{cases} u''(t) + B\, u'(t) + f\big(t, u(t)\big) = 0, & t \in [0, 1], \\ \\ u(0) = u(1) = 0. \end{cases} \tag{6.4.29}$$

Clearly, as in previous examples, $\eta = \gamma = 1$. Thus ϕ is given by (6.4.5).

First, consider $B = 0$. We have the expression of the related Green's function as follows (see [16, 21]):

$$-g_0(t, s) = \begin{cases} s\,(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ \\ (1-s)\, t, & 0 < t < s \leqslant 1, \end{cases}$$

thus,

$$\tilde{u}(t, s) = \begin{cases} \dfrac{1-t}{1-s}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \\ \dfrac{t}{s}, & 0 < t < s \leqslant 1. \end{cases}$$

From this expression, we directly obtain that:

$$\begin{cases} 1 - t \leqslant \tilde{u}(t, s) \leqslant 1, & 0 \leqslant s \leqslant t \leqslant 1, \\ \\ t \leqslant \tilde{u}(t, s) \leqslant 1, & 0 < t < s \leqslant 1. \end{cases}$$

Hence, $k_2(t) = 1 = K_2$ and:

$$k_1(t) = \min_{t \in [0,1]} \{1-t, t\} = \begin{cases} t, & 0 \leqslant t \leqslant \dfrac{1}{2}, \\ 1-t, & \dfrac{1}{2} < t \leqslant 1, \end{cases}$$

and $K_1 = k_1\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$.

In Figure 6.4.5, we represent the function $\tilde{u}(t,s) = -\dfrac{g(t,s)}{\phi(s)}$, bounded from above by $k_2(t)$ and from below by $k_1(t)$. Moreover, in Figure 6.4.6, it appears the same representation considering the constant values $t_0 = \dfrac{2}{3}$ and $s_0 = \dfrac{2}{3}$, respectively.

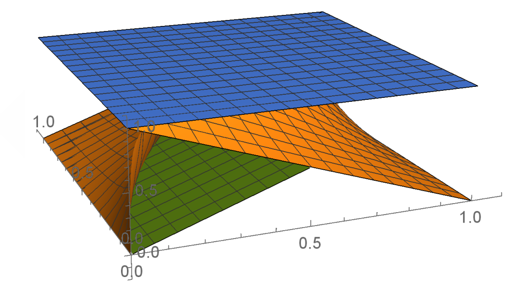

Figure 6.4.5: $\tilde{u}(t,s)$ (orange) bounded from above by $k_2(t)$ (blue) and from below by $k_1(t)$ (green).

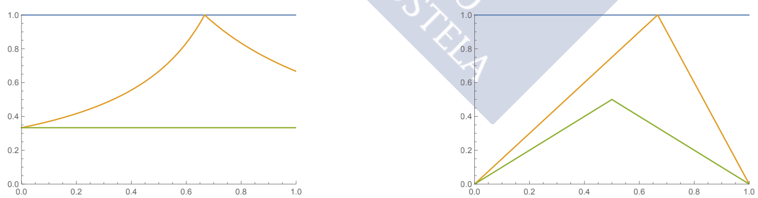

Figure 6.4.6: Figure 6.4.5 for $t_0 = \dfrac{2}{3}$ on the left and for $s_0 = \dfrac{2}{3}$ on the right.

Let us consider $[a_1, b_1] = \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$, then $m_1 = \dfrac{1}{4}$, and we obtain the following values:

$$\int_0^1 \phi(s)\,\mathrm{d}\,s = \frac{1}{6}, \quad \int_{1/4}^{3/4} \phi(s)\,\mathrm{d}\,s = \frac{11}{96}, \quad \int_{1/4}^{3/4} k_1(s)\,\phi(s)\,\mathrm{d}\,s = \frac{67}{1536}.$$

Thus, we are able to ensure the existence of at least two or three solutions for problem (6.4.29) by means of the previously obtained fixed point theorems. First, let us write the corresponding assumptions (H_1) and (H_2) for this situation.

(H_1) There exists $p > 0$ such that $f(t,u) \leqslant 6\,p$ for all $t \in [0,1]$ and $u \in [0,p]$.

(H_2) There exists $q > 0$ such that $f(t,u) \geqslant \dfrac{3072}{67} u$ for all $t \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$ and $u \in \left[\dfrac{q}{4}, q\right]$.

Finally, as a direct consequence of Theorems 6.3.1 and 6.3.3, we obtain the following results.

Theorem 6.4.5. *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r,$$

and assume that function f satisfies the following conditions:

(i) $f(t,u) \geqslant \dfrac{384}{11} u$ *for all* $t \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$ *and* $u \in [r, 4\,r]$, *being the inequality strict at* $u = r$,

(ii) $f(t,u) \leqslant 6\,q$ *for all* $t \in [0,1]$ *and* $u \in [0, 4\,q]$, *being the inequality strict at* $u = q$,

(iii) $f(t,u) > \dfrac{3072}{67} u$ *for all* $t \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$ *and* $u \in \left[\dfrac{p}{4}, p\right]$.

Then, for $B = 0$, problem (6.4.29) *has at least two positive solutions, u_1 and u_2, such that:*

$$p < \|u_1\|_{C(I)}, \quad \max_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_1(t) < q < \max_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_2(t) \quad \text{and} \quad \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_2(t) < r.$$

Theorem 6.4.6. *Let p, q and r be positive numbers satisfying the relation:*

$$0 < p < 4\,p < q < 4\,q \leqslant r.$$

Assume, moreover, that function f satisfies the following conditions:

(a) $f(t,u) \leqslant 6\,r$ *for all* $t \in [0,1]$ *and* $u \in [0,r]$,

(b) $f(t,u) < 6\,p$ *for all* $s \in [0,1]$ *and* $u \in [0, 4\,p]$,

(c) $f(t,u) \geqslant \dfrac{3072}{67} u$ *for all* $s \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$ *and* $u \in [q, 4\,q]$, *being the inequality strict at* $u = q$.

Then, for $B = 0$, problem (6.4.29) *has at least three positive solutions, u_1, u_2, u_3 such that $\|u_i\|_{C(I)} \leqslant r$ for $i = 1,2,3$ and:*

$$\max_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_1(t) < p < \max_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_2(t) \quad \text{and} \quad \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_2(t) < q < \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_3(t).$$

Now, let us see what happens for $B > 0$. In this case, the expression of the Green's function (see [21]) is:

$$-g_B(t,s) = \begin{cases} \dfrac{\left(e^{B\,s}-1\right)\left(e^{B\,(1-t)}-1\right)}{B\,(e^B-1)}, & 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{\left(e^{B}-e^{B\,s}\right)\left(1-e^{-B\,t}\right)}{B\,(e^B-1)}, & 0 < t < s \leqslant 1, \end{cases}$$

and, we have:

$$\tilde{u}(t,s) = \begin{cases} \dfrac{e^{B\,(1-t)}-1}{e^B-1}, & 0 = s \leqslant t \leqslant 1, \\[2ex] \dfrac{\left(e^{B\,s}-1\right)\left(e^{B\,(1-t)}-1\right)}{B\,(e^B-1)\,s\,(1-s)}, & 0 < s \leqslant t \leqslant 1, \\[2ex] \dfrac{\left(e^{B}-e^{B\,s}\right)\left(1-e^{-B\,t}\right)}{B\,(e^B-1)\,s\,(1-s)}, & 0 < t < s < 1, \\[2ex] \dfrac{e^B\,\left(1-e^{-B\,t}\right)}{e^B-1}, & 0 < t < s = 1. \end{cases}$$

For $s \in [0,1]$, let us consider:

$$f_1(s) = \frac{e^{B\,s}-1}{s\,(1-s)} \quad \text{and} \quad f_2(s) = \frac{e^B - e^{B\,s}}{s\,(1-s)}.$$

We have:

$$\frac{\partial}{\partial B} f_1'(s) = \frac{e^{B\,s}\big(1+B(1-s)\big)}{(1-s)^2} > 0,\ \forall B \geqslant 0,$$

and moreover, for $B = 0$, we have that $f_1'(s)_{|B=0} = 0$, thus $f_1'(s) \geqslant 0$ for all $B > 0$. Hence,

$$\max_{s\in[0,t]} f_1(s) = f_1(t) = \frac{e^{B\,t}-1}{t(1-t)} \ \text{ and } \ \min_{s\in[0,t]} f_1(s) = \lim_{s\to 0^+} f_1(s) = B.$$

Therefore, we construct the following functions:

$$k_{11}(t) = \frac{e^{B\,(1-t)}-1}{e^B-1} \quad \text{and} \quad k_{21}(t) = \frac{(e^B - e^{B\,t})\,(1-e^{-B\,t})}{t\,(1-t)\,B\,(e^B-1)}.$$

Due to the construction of k_{11} and k_{21}, for all $s \in [0,t]$ we have:

$$k_{11}(t) \leqslant \tilde{u}(t,s) \leqslant k_{21}(t).$$

On another hand, we have:

$$\frac{\partial}{\partial B} f_2''(s) = \frac{e^B \left(2 - 6\,(1-s)\,s\right) - e^{B\,s}\Big(2 + B\left(2 + B\,(1-s)\right)(1-s)\Big)\, s^3}{(1-s)^3 s^3}.$$

Let us see that the previous function is positive for all $B \in \left[0, \sqrt{6}\right]$. Consider the following positive functions:

$$\begin{aligned} f_{2_1}(s) &= e^B \left(2 - 6\,(1-s)\,s\right),\\ f_{2_2}(s) &= e^{B\,s}\Big(2 + B\left(2 + B\,(1-s)\right)(1-s)\Big)\, s^3. \end{aligned}$$

Clearly, $f_{2_1}''(s) = 12\, e^B > 0$ for all $B \geqslant 0$. Moreover,

$$f_{2_2}''(s) = e^{B\,s} s \left(8\, B^3 s\,(1-s)^2 + 6\left(1 + B\,(1-s)\right)^2 + B^2\left(B\,s\,(1-s) - 1\right)^2 + 6 - B^2\right),$$

which is positive for all $B \in \left[0, \sqrt{6}\right]$.

Hence, f_{2_1} and f_{2_2} can have at most two common points on $[0,1]$ for $B \in \left[0, \sqrt{6}\right]$. We have:

$$\begin{aligned} f_{2_1}(0) &= 2\, e^B > 0 = f_{2_2}(0),\\ f_{2_1}(1) &= 2\, e^B = f_{2_2}(1),\\ f_{2_1}'(1) &= 6\, e^B = f_{2_2}'(1),\\ f_{2_1}''(1) &= 12\, e^B = f_{2_2}''(1),\\ f_{2_1}^{(3)}(1) &= 0 < 12\, e^B + 2\, B^3 e^B = f_{2_2}^{(3)}(1). \end{aligned}$$

Thus, $f_{2_1}(s) > f_{2_2}(s)$ for all $s \in [0,1)$ and $B \in \left[0, \sqrt{6}\right]$, and we have:

$$\frac{\partial}{\partial B} f_2''(s) = f_{2_1}(s) - f_{2_2}(s) \geqslant 0, \quad s \in [0,1],\ B \in \left[0, \sqrt{6}\right].$$

As a consequence, since $f_2''(s) = 0$ for $B = 0$, then $f_2''(s) \geqslant 0$ for all $B \in \left[0, \sqrt{6}\right]$.

Moreover, $f_2'(1) = \dfrac{1}{2}(B-2)\, B\, e^B < 0$ for all $B < 2$. So, f_2 is a decreasing function in $[0,1]$ for $B \in [0,2]$ and we have:

$$\max_{s \in [t,1]} f_2(s) = f_2(t) = \frac{e^B - e^{B\,t}}{t(1-t)} \quad \text{and} \quad \min_{s \in [t,1]} f_2(s) = \lim_{s \to 1^-} f_2(s) = B\, e^B,$$

define, now, the following functions:

$$k_{12}(t) = \frac{e^B - e^{B\,(1-t)}}{e^B - 1} \quad \text{and} \quad k_{22}(t) = k_{21}(t).$$

Due to the construction of k_{12} and k_{22}, for all $s \in [t,1]$ we have:

$$k_{12}(t) \leqslant \tilde{u}(t,s) \leqslant k_{22}(t).$$

Thus, $k_2(t) = k_{21}(t) = \dfrac{(e^B - e^{B\,t})\,(1-e^{-B\,t})}{t\,(1-t)\,B\,(e^B-1)} \leqslant k_2(1) = 1 = K_2.$

Moreover, we have:

$$k_1(t) = \begin{cases} \dfrac{e^B - e^{B\,(1-t)}}{e^B-1}, & 0 \leqslant t \leqslant t_1 = 1 - \dfrac{\log\left(\frac{1+e^B}{2}\right)}{B}, \\[2ex] \dfrac{e^{B\,(1-t)}-1}{e^B-1}, & t_1 < t \leqslant 1. \end{cases}$$

We have $K_1 = \max\limits_{t\in[0,1]} k_1(t) = k_1(t_1) = \dfrac{1}{2}$.

If we choose:

$$I_1 = [a_1, b_1] = \left[1 - \frac{\log\left(\frac{1+3\,e^B}{4}\right)}{B}, 1 - \frac{\log\left(\frac{3+e^B}{4}\right)}{B}\right], \tag{6.4.30}$$

we obtain $m_1 = k_1(a_1) = k_1(b_1) = \dfrac{1}{4}$.

Remark 6.4.7. *For $B \in [0,2]$, both a_1 and b_1 are decreasing functions of B. Moreover,*

$$a_1 \in \left[1 - \frac{1}{2}\log\left(\frac{1}{4}\left(1+3e^2\right)\right), \frac{1}{4}\right] \subset \left[\frac{3}{25}, \frac{1}{4}\right],$$

and,

$$b_1 \in \left[1 - \frac{1}{2}\log\left(\frac{1}{4}\left(3+e^2\right)\right), \frac{3}{4}\right] \subset \left[\frac{13}{25}, \frac{3}{4}\right].$$

To attain the rest of the constants involved on the different results, we use approximations of the values of the integrals. To this end we need to fix the value of $B \in [0,2]$.

The bounds shown below can be obtained for all $B \in [0,2]$, in order to simplify the description of k_1, let us choose B such that $t_1 = \dfrac{1}{3}$, that is, $B = \log(2+\sqrt{5}) \in [0,2]$. In such a case, we have:

$$a_1 = \frac{\log(\sqrt{5}-1)}{\log\,(2+\sqrt{5})} \text{ and } b_1 = \frac{\log\left(1+\dfrac{3}{\sqrt{5}}\right)}{\log(2+\sqrt{5})}.$$

Moreover, in this case, we obtain:

$$\int_{a_1}^{b_1} k_1(s)\,\phi(s)\,\mathrm{d}\,s \approxeq 0.035872 > \frac{3587}{100000},$$

$$\int_{a_1}^{b_1} \phi(s)\,\mathrm{d}\,s \approxeq 0.095719 > \frac{957}{10000}.$$

As for $B = 0$, in Figure 6.4.7, we represent function $\tilde{u}(t,s)$, bounded from above by $k_2(t)$ and from below by $k_1(t)$ for $B = \log(2+\sqrt{5})$. Moreover, in Figure 6.4.8, we plot the same representation considering the constant values $t_0 = \dfrac{2}{3}$ and $s_0 = \dfrac{2}{3}$, respectively.

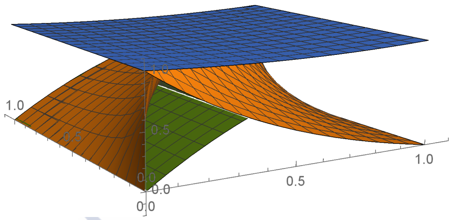

Figure 6.4.7: $\tilde{u}(t,s)$ (orange) bounded from above by $k_2(t)$ (blue) and from below by $k_1(t)$ (green).

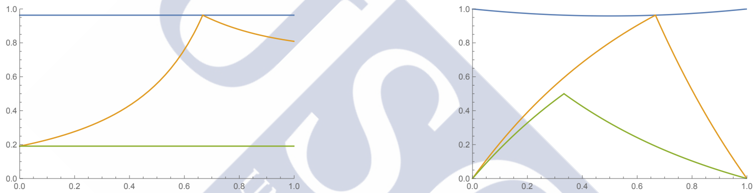

Figure 6.4.8: Figure 6.4.7 for $t_0 = \dfrac{2}{3}$ on the left and for $s_0 = \dfrac{2}{3}$ on the right.

Thus, as in the case where $B = 0$, we obtain the correspondent results of existence of solution for such a problem. First, let us write the related conditions (H_1) and (H_2). Let $I = [0,1]$ and I_1 be defined in (6.4.30) for $B = \log(2+\sqrt{5})$.

(H_1) There exists $p > 0$ such that $f(t,u) \leqslant 6\,p$ for all $t \in I$ and $u \in [0,p]$.

(H_2) There exists $q > 0$ such that $f(t,u) \geqslant \dfrac{200000}{3587}u > \dfrac{u}{\dfrac{1}{2}\displaystyle\int_{a_1}^{b_1} k_1(s)\,\phi(s)\,\mathrm{d}\,s}$ for all $t \in I_1$ and $u \in \left[\dfrac{q}{4}, q\right]$.

Finally, we can rewrite Theorems 6.3.1 and 6.3.3 as follows.

Theorem 6.4.8. *Let I_1 be defined in* (6.4.30) *for $B = \log(2+\sqrt{5})$ and suppose that there exist positive numbers p, q and r such that $0 < p < q < r$.*

Assume, moreover, that function f satisfies the following conditions:

(i) $f(t,u) \geqslant \dfrac{40000}{957}u > \dfrac{u}{\dfrac{1}{4}\displaystyle\int_{a_1}^{b_1} \phi(s)\,\mathrm{d}\,s}$ *for all $t \in I_1$ and $u \in [r, 4\,r]$,*

(ii) $f(t,u) \leqslant 6\,q$ *for all $t \in [0,1]$ and $u \in [0, 4\,q]$, being the inequality strict for $u = q$,*

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(iii) $f(t,u \geqslant \dfrac{200000}{3587} u$ *for all* $t \in I_1$ *and* $u \in \left[\dfrac{p}{4}, p\right]$.

Then, for $B = \log(2+\sqrt{5})$*, problem* (6.4.29) *has at least two positive solutions,* u_1 *and* u_2*, such that:*

$$p < \|u_1\|_{C(I)}, \ \max_{t \in I_1} u_1(t) < q < \max_{t \in I_1} u_2(t) \ \text{ and } \ \min_{t \in I_1} u_2(t) < r.$$

Theorem 6.4.9. *Let* I_1 *be defined in* (6.4.30) *for* $B = \log(2+\sqrt{5})$ *and suppose* p*,* q *and* r *are positive numbers such that:*

$$0 < p < 4\,p < q < 4\,q \leqslant r.$$

Assume, moreover, that the function f *satisfies the following conditions:*

(a) $f(t,u) \leqslant 6\,r$ *for all* $t \in [0,1]$ *and* $u \in [0,r]$*,*

(b) $f(t,u) < 6\,p$ *for all* $t \in [0,1]$ *and* $u \in [0,4\,p]$*,*

(c) $f(t,u) \geqslant \dfrac{40000}{957} u$ *for all* $s \in I_1$ *and* $u \in [q, 4\,q]$.

Then, for $B = \log(2+\sqrt{5})$*, problem* (6.4.29) *has at least three solutions,* u_1*,* u_2*,* u_3 *such that* $\|u_i\|_{C(I)} \leqslant r$ *for* $i = 1,2,3$ *and:*

$$\max_{t \in I_1} u_1(t) < p < \max_{t \in I_1} u_2(t) \ \text{ and } \ \min_{t \in I_1} u_2(t) < q < \min_{t \in I_1} u_3(t).$$

Finally, repeating the same arguments for $B < 0$, we obtain that for $B \in [-2, 0]$:

$$k_2(t) = \frac{\left(1 - e^{B\,t}\right)\,\left(e^{-B} - e^{-B\,t}\right)}{t\,(1-t)\,B\,(1-e^{-B})} \leqslant 1 = K_2,$$

$$k_1(t) = \begin{cases} \dfrac{e^{-B\,t} - 1}{e^{-B} - 1}, & 0 \leqslant t \leqslant t_2 = -\dfrac{\log\left(\frac{e^{-B}+1}{2}\right)}{B}, \\[3ex] \dfrac{e^{-B} - e^{-B\,t}}{e^{-B} - 1}, & t_2 < t \leqslant 1. \end{cases}$$

We have that $K_1 = \max\limits_{t \in [0,1]} k_1(t) = k_1(t_2) = \dfrac{1}{2}$.

In this case, if we choose

$$I_1 = [a_1, b_1] = \left[-\frac{\log(\frac{e^{-B}+3}{4})}{B}, -\frac{\log(\frac{3\,e^{-B}+1}{4})}{B}\right], \tag{6.4.31}$$

it is satisfied that $m_1 = k_1(a_1) = k_1(b_1) = \dfrac{1}{4}$.

Remark 6.4.10. *For $B \in [-2,0]$, both a_1 and b_1 are decreasing functions of B. Moreover,*

$$a_1 \in \left[\frac{1}{4}, \frac{1}{2}\log\left(\frac{1}{4}\left(3+e^2\right)\right)\right] \subset \left[\frac{1}{4}, \frac{12}{25}\right],$$

and

$$b_1 \in \left[\frac{3}{4}, \frac{1}{2}\log\left(\frac{1}{4}\left(1+3e^2\right)\right)\right] \subset \left[\frac{3}{4}, \frac{22}{25}\right].$$

Moreover, for the choice $B = \log(\sqrt{5}-2) \in [-2,0]$ and the correspondent interval I_1 defined in (6.4.31), we attain the same bounds as for $B = \log(2+\sqrt{5})$. Thus, (H_1) and (H_2) coincide in both cases and Theorems 6.4.8 and 6.4.9 remain applicable for $B = \log(\sqrt{5}-2)$.

It is important to recall that for all $B \in [-2,2]$ any suitable bounds can be obtained by using the expression of k_1 and k_2 without any additional difficulty.

If $B \notin [-2,2]$, the study is much more complicated. In any case, the approach can also be done. For instance let us choose $B = -2\,\pi$, we have:

$$-g(t,s) = \begin{cases} \dfrac{\left(1-e^{-2\,\pi\,s}\right)\left(e^{2\,\pi}-e^{2\pi\,t}\right)}{2\,\pi\,(e^{2\pi}-1)}, & 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{\left(e^{2\,\pi\,(1-s)}-1\right)\left(e^{2\,\pi\,t}-1\right)}{2\,\pi\,(e^{2\,\pi}-1)}, & 0 < t < s \leqslant 1, \end{cases}$$

and, as a consequence:

$$\tilde{u}(t,s) = \begin{cases} \dfrac{e^{2\,\pi}-e^{2\,\pi\,t}}{e^{2\,\pi}-1}, & 0 = s \leqslant t \leqslant 1, \\[2ex] \dfrac{\left(1-e^{-2\,\pi\,s}\right)\left(e^{2\,\pi}-e^{2\,\pi\,t}\right)}{2\,\pi\,(e^{2\,\pi}-1)\,s\,(1-s)}, & 0 < s \leqslant t \leqslant 1, \\[2ex] \dfrac{\left(e^{2\,\pi\,(1-s)}\right)\left(e^{2\,\pi\,t}-1\right)}{2\,\pi\,(e^{2\,\pi}-1)\,s\,(1-s)}, & 0 < t < s < 1, \\[2ex] \dfrac{e^{2\,\pi\,t}-1}{e^{2\,\pi}-1}, & 0 < t < s = 1. \end{cases}$$

As for $B \in [-2,2]$, we can see that for all $s \in [t,1]$, the following inequalities are fulfilled:

$$\frac{e^{2\,\pi\,t}-1}{e^{2\,\pi}-1} \leqslant \tilde{u}(t,s) \leqslant \frac{\left(e^{2\,\pi\,(1-t)}-1\right)\,\left(e^{2\,\pi\,t}-1\right)}{t\,(1-t)\,2\,\pi\,\left(e^{2\,\pi}-1\right)} \leqslant 1,\ \forall t \in [0,1].$$

Now, let us consider the following function:

$$h_1(s) = \frac{1-e^{-2\,\pi\,s}}{s\,(1-s)}.$$

We have:

$$h_1''(s) = \frac{2\left(1-e^{-2\,\pi\,s}\right)}{(1-s)^3 s^3}\left((1-s)^2-(1-s)\,s+s^2\right)+\frac{4\,\pi\,e^{-2\,\pi\,s}}{(1-s)^2 s^2}\left(1-(s-1)\,s\,\pi\right)\geqslant 0,$$

$$h_1'\left(\frac{451}{1000}\right)<0 \quad \text{and} \quad h_1'\left(\frac{452}{1000}\right)>0.$$

Thus, the minimum of h_1 is attained at $s_1 \in \left(\frac{451}{1000},\frac{452}{1000}\right)$ and:

$$h_1(s_1)=\frac{1-e^{-2\,\pi\,s_1}}{s_1\,(1-s_1)}\geqslant\frac{1-e^{-2\,\pi\,\frac{451}{1000}}}{\frac{452}{100}\cdot\frac{549}{1000}}=\frac{250000}{62037}\left(1-e^{\frac{-451}{500}\,\pi}\right).$$

Hence, if we consider:

$$\tilde{k}_1(t)=\frac{250000}{62037}\left(1-e^{\frac{-451}{500}\,\pi}\right)\frac{e^{2\,\pi}-e^{2\,\pi\,t}}{2\,\pi\,(e^{2\,\pi}-1)}.$$

we have that $\tilde{k}_1(t)\leqslant\tilde{u}(t,s)$ for all $t\in[0,1]$ and $s\in[0,t]$.

Moreover, for all $s\in[0,t]$:

$$\begin{aligned}\tilde{u}(t,s)&\leqslant\max\left\{h_1(0),h_1(t)\right\}\frac{e^{2\,\pi}-e^{2\,\pi\,t}}{2\,\pi(e^{2\,\pi}-1)}\\&=\max\left\{\frac{e^{2\,\pi}-e^{2\,\pi\,t}}{e^{2\,\pi}-1},\frac{\left(e^{2\,\pi\,(1-t)}-1\right)\,\left(e^{2\,\pi\,t}-1\right)}{t\,(1-t)\,2\,\pi\,\left(e^{2\,\pi}-1\right)}\right\}\leqslant 1.\end{aligned}$$

As a consequence, $K_2=1$ and:

$$k_1(t)=\begin{cases}\dfrac{e^{2\,\pi\,t}-1}{e^{2\,\pi}-1}, & 0\leqslant t\leqslant t_3,\\[2ex] \dfrac{250000}{62037}\left(1-e^{\frac{-451}{500}\,\pi}\right)\dfrac{e^{2\,\pi}-e^{2\,\pi\,t}}{2\,\pi\,(e^{2\,\pi}-1)}, & t_3<t\leqslant 1,\end{cases}$$

where $t_3=\dfrac{1}{2\,\pi}\log\left(\dfrac{125000\left(e^{\frac{549}{500}\,\pi}-e^{2\,\pi}\right)-62037\,\pi}{125000\left(e^{-\frac{451}{500}\,\pi}-1\right)-62037\,\pi}\right)\approxeq 0.844992.$

In this case, we conclude that:

$$K_1=\max_{t\in[0,1]}k_1(t)=k_1(t_3)\approxeq 0.37642>\frac{376}{1000}=\frac{47}{125}.$$

As in previous cases, in Figure 6.4.9, we represent the function $\tilde{u}(t,s)$, bounded from above by $K_2=1$ and from below by $k_1(t)$. Moreover, in Figure 6.4.10, we plot the same representation considering the constant values $t_0=\frac{21}{25}$ and $s_0=\frac{21}{25}$, respectively.

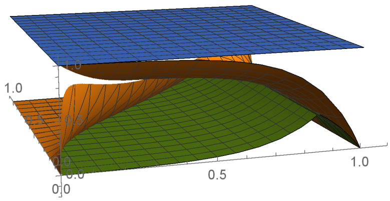

Figure 6.4.9: $\tilde{u}(t,s)$ (orange) bounded from above by $k_2(t)$ (blue) and from below by $k_1(t)$ (green).

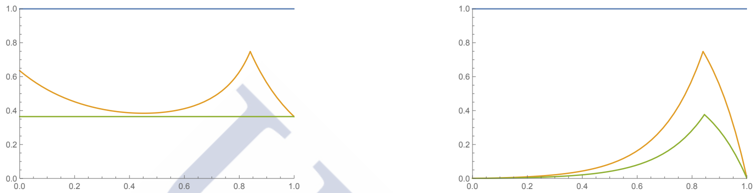

Figure 6.4.10: Figure 6.4.9 for $t_0 = \dfrac{21}{25}$ on the left and for $s_0 = \dfrac{21}{25}$ on the right.

If we choose:

$$I_1 = [a_1, b_1] = \left[\frac{\log\left(\frac{3+e^{2\,\pi}}{4}\right)}{2\,\pi}, 0.9151 \right] \subset [0.78, 0.9151], \tag{6.4.32}$$

we have that $m_1 = k_1(a_1) = \dfrac{1}{4}$ and:

$$\int_{a_1}^{b_1} \phi(s)\,\mathrm{d}\,s \approxeq 0.0172072 > \frac{43}{2500}, \quad \int_{a_1}^{b_1} k_1(s)\,\phi(s)\,\mathrm{d}\,s \approxeq 0.005393 > \frac{539}{100000}.$$

Thus, as in the previous cases, we deduce the correspondent results of existence of solution for this problem. Let $I = [0,1]$ and I_1 be defined in (6.4.32). First, let us write the related (H_1) and (H_2) as follows:

(H_1) There exists $p > 0$ such that $f(t,u) \leqslant 6\,p$ for all $t \in I$ and $u \in [0,p]$.

(H_2) There exists $q > 0$ such that $f(t,u) \geqslant \dfrac{12500000\,u}{25333} > \dfrac{u}{\dfrac{47}{125}\displaystyle\int_{a_1}^{b_1} k_1(s)\,\phi(s)\,\mathrm{d}\,s}$ for all $t \in I_1$ and $u \in \left[\dfrac{q}{4}, q\right]$.

So, we can rewrite Theorems 6.3.1 and 6.3.3 as follows.

Theorem 6.4.11. *Let I_1 be defined in* (6.4.32) *and suppose that there exist positive numbers p, q and r such that*

$$0 < p < q < r.$$

Assume, moreover, that function f satisfies the following conditions:

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(i) $f(t,u) \geqslant \dfrac{10000}{43} u > \dfrac{u}{\dfrac{1}{4}\displaystyle\int_{a_1}^{b_1} \phi(s)\, \mathrm{d}\, s}$ *for all* $t \in I_1$ *and* $u \in [r, 4\, r]$,

(ii) $f(t,u) \leqslant 6\, q$ *for all* $t \in [0,1]$ *and* $u \in [0, 4\, q]$, *being the inequality strict for* $u = q$

(iii) $f(t,u) \geqslant \dfrac{12500000}{25333} u$ *for all* $t \in I_1$ *and* $u \in \left[\dfrac{p}{4}, p\right]$.

Then, for $B = -2\,\pi$, *problem* (6.4.29) *has at least two positive solutions,* u_1 *and* u_2*, such that:*

$$p < \|u_1\|_{C(I)},\ \max_{t\in I_1} u_1(t) < q < \max_{t\in I_1} u_2(t) \ \ \textit{and}\ \ \min_{t\in I_1} u_2(t) < r.$$

Let us consider the following continuous function:

$$f(t,u) = \begin{cases} \left(\dfrac{1007}{88} + \dfrac{225}{88}\, t\right) \dfrac{50000000}{1190651} u, & u \leqslant \dfrac{1}{28}, \\[2ex] \left(\dfrac{1007}{88} + \dfrac{225}{88}\, t\right) \dfrac{3125000}{58341899\, u}, & \dfrac{1}{28} < u \leqslant 14, \\[2ex] \left(\dfrac{1007}{88} + \dfrac{225}{88}\, t\right) \dfrac{3125000}{58341899\, u} + \dfrac{10000}{43}(u-14)\, u, & u > 14. \end{cases} \tag{6.4.33}$$

It is easy to verify that this function f satisfies the hypotheses of Theorem 6.4.11.

(i) For the construction of f, (i) is trivially fulfilled for all $u \geqslant 15$.

(ii) If $q = \dfrac{7}{2}$, for all $t \in I$ and $u \in [0, 14]$, it is satisfied:

$$f(t,u) \leqslant f\left(t, \frac{1}{28}\right) = \left(\frac{1007}{88} + \frac{225}{88}\, t\right) \frac{50000000}{1190651} \cdot \frac{1}{28} \leqslant \frac{25000000}{1190651} < \frac{7}{2} \cdot 6.$$

(iii) If $p = \dfrac{1}{28}$, clearly $f(t,u) \geqslant \dfrac{47}{4} \cdot \dfrac{50000000}{1190651} u = \dfrac{12500000}{25333} u$, for all $u \leqslant p$ and $t \in I_1$ for all $t \in \left[\dfrac{3}{25}, 1\right]$, in particular for $t \in I_1$ and $u \in \left[\dfrac{1}{112}, \dfrac{1}{28}\right]$.

Remark 6.4.12. *Realise that, from Remarks 6.4.7 and 6.4.10, the conditions imposed to* f *are stronger than those imposed in Theorems 6.4.5 and 6.4.8, then these results can also be applied to this* f.

Thus, we can conclude that for $B \in \left\{ -2\,\pi,\ \log(\sqrt{5}-2),\ 0,\ \log(\sqrt{5}+2)\right\}$ problem (6.4.29), with f defined in (6.4.33), has at least two positive solutions, u_1 and u_2, such that:

$$\frac{1}{28} < \|u_1\|_{C(I)},\ \max_{t\in I_1} u_1(t) < \frac{7}{2} < \max_{t\in I_1} u_2(t) \ \ \text{and}\ \ \min_{t\in I_1} u_2(t) < 1.$$

Theorem 6.4.13. *Let I_1 be defined in* (6.4.32) *and suppose p, q and r are positive numbers such that:*

$$0 < p < 4\,p < q < 4\,q \leqslant r.$$

Assume, moreover, that function f satisfies the following conditions:

(a) $f(t,u) \leqslant 6\,r$ *for all* $t \in [0,1]$ *and* $u \in [0,r]$,

(b) $f(t,u) < 6\,p$ *for all* $s \in [0,1]$ *and* $u \in [0,4\,p]$,

(c) $f(t,u) \geqslant \dfrac{10000}{43} u$ *for all* $s \in I_1$ *and* $u \in [q,4\,q]$.

Then, for $B = -2\,\pi$, problem (6.4.29) *has at least three solutions, u_1, u_2, u_3 such that $\|u_i\| \leqslant r$ for $i = 1,2,3$ and:*

$$\max_{t\in I_1} u_1(t) < p < \max_{t\in I_1} u_2(t) \;\; \textit{and} \;\; \min_{t\in I_1} u_2(t) < q < \min_{t\in I_1} u_3(t).$$

Let us consider the following continuous function:

$$f(t,u) = \begin{cases} 12\,\left(\dfrac{31}{28}t + \dfrac{25}{28}\right)\,u^3, & u \leqslant 16, \\[2ex] 49152\,\left(\dfrac{31}{28}t + \dfrac{25}{28}\right), & u > 16. \end{cases} \tag{6.4.34}$$

Let us verify that f fulfils the hypotheses of Theorem 6.4.13.

(a) If $r = 16384$, then $f(t,u) \leqslant 98304 = 6 \cdot 16384$ for all $t \in I$ and $u \in [0,16384]$,

(b) If $p = \dfrac{1}{8}$, then $f(t,u) \leqslant 12 \cdot 2 \cdot \dfrac{1}{4} = 6 \cdot \dfrac{1}{2}$, for all $t \in I$ and $u \in \left[0, \dfrac{1}{2}\right]$,

(c) If $q = 4$, then $f(t,u) \geqslant 12 \cdot \dfrac{5}{4} \cdot 16\,u = 240\,u > \dfrac{10000}{43} u$ for all $t \in \left[\dfrac{3}{25}, 1\right]$, in particular for $t \in I_1$ and $u \in [4,16]$.

As in Remark 6.4.12, the imposed conditions on f are stronger than those in Theorems 6.4.5 and 6.4.9. So, we can conclude that for:

$$B \in \Big\{ -2\,\pi,\; \log(\sqrt{5}-2),\; 0,\; \log(\sqrt{5}+2) \Big\}$$

problem (6.4.29), with f defined in (6.4.34), has at least three solutions, u_1, u_2, u_3 such that $\|u_i\| \leqslant 16384$ for $i = 1,2,3$ and:

$$\max_{t\in I_1} u_1(t) < \frac{1}{8} < \max_{t\in I_1} u_2(t) \;\; \text{and} \;\; \min_{t\in I_1} u_2(t) < 4 < \min_{t\in I_1} u_3(t).$$

- Fourth order: $n = 4$ and $k = 2$.

In Chapter 4, we have proved that the related Green's function of operator:

$$T_4^0[M]\,u(t) = u^{(4)}(t) + M\,u(t)$$

on X_2, with $I = [0,1]$, satisfies property $\left(P_{g_1}\right)$ if, and only if, $M \in \left(-\left(\lambda_4^2\right)^4, \left(\lambda_4^1\right)^4\right]$, where $\lambda_4^1 \approxeq 5.55$ and $\lambda_4^2 \approxeq 4.73$ are the least positive solutions of (2.1.5) and (2.1.6), respectively.

In particular, it is fulfilled for $M = 0$. Thus, let us study the following problem:

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ u(1) = u'(1) = 0. \end{cases} \tag{6.4.35}$$

The related Green's function (see [21]) is given by:

$$g(t,s) = \begin{cases} \dfrac{s^2}{6}\,(1-t)^2\,(3\,t - s - 2\,s\,t), & 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{(1-s)^2}{6}\,t^2\,(3\,s - t - 2\,s\,t), & 0 < t < s \leqslant 1. \end{cases}$$

In this case, $\eta = \gamma = 2$, so $\phi(s) = s^2(1-s)^2$ and:

$$\tilde{u}(t,s) = \begin{cases} \dfrac{1}{6}\left(\dfrac{1-t}{1-s}\right)^2 (3\,t - s - 2\,s\,t), & 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{1}{6}\left(\dfrac{t}{s}\right)^2 (3\,s - t - 2\,s\,t), & 0 < t < s \leqslant 1. \end{cases}$$

For each $t \in [0,1]$, let us consider:

$$g_{1t}(s) = \tilde{u}(t,s) = \frac{1}{6}\left(\frac{1-t}{1-s}\right)^2 (3\,t - s - 2\,s\,t), \quad s \in [0,t].$$

Using the expression:

$$g_{1t}'(s) = \frac{(1-t)^2(4\,t - 2\,s\,t - s - 1)}{6\,(1-s)^3},$$

it is clear that $g_{1t}'(s) = 0$ if, and only if, either $t = 1$ or $s = \dfrac{4\,t-1}{1+2\,t}$.

If $t = 1$, then $g_{1t}(s) = 0$ for all $s \in [0,1]$.

It is immediate to verify that $\dfrac{4\,t-1}{1+2\,t} \in [0,1]$ for $t \in [0,1]$ if, and only if, $t \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right]$.

For $t \in \left[0, \dfrac{1}{4}\right)$, g_{1t} is a decreasing function in $[0,t]$, thus we have:

$$g_{1t}(0) = \frac{t}{2}(1-t)^2 \geqslant \tilde{u}(t,s) \geqslant \frac{t}{3}(1-t) = g_{1t}(t), \ \forall s \in [0,t].$$

For $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$, we have that ${g_1}_t''\left(\frac{4\,t-1}{1+2\,t}\right) = -\frac{(1+2\,t)^2}{48\,(1-t)} < 0$, and, furthermore, ${g_1}_t\left(\frac{4\,t-1}{1+2\,t}\right) = \frac{1-t}{24}\,(1+2\,t)^2$, hence for all $s \in [0,t]$, we have:

$$\frac{1-t}{24}\,(1+2\,t)^2 \geqslant \tilde{u}(t,s) \geqslant \min\left\{{g_1}_t(0), {g_1}_t(t)\right\} = \min\left\{\frac{t}{2}\,(1-t)^2, \frac{t}{3}\,(1-t)\right\},$$

or, which is the same:

$$\frac{1-t}{24}\,(1+2\,t)^2 \geqslant \tilde{u}(t,s) \geqslant \frac{t}{3}\,(1-t),\ \forall\, s \in [0,t],\ t \in \left[\frac{1}{4}, \frac{1}{3}\right),$$
$$\frac{1-t}{24}\,(1+2\,t)^2 \geqslant \tilde{u}(t,s) \geqslant \frac{t}{2}\,(1-t)^2,\ \forall\, s \in [0,t],\ t \in \left[\frac{1}{3}, \frac{1}{2}\right].$$

Finally, for $t \in \left(\frac{1}{2}, 1\right]$, ${g_1}_t$ is an increasing function in $[0,t]$, hence:

$${g_1}_t(t) = \frac{t}{3}\,(1-t) \geqslant \tilde{u}(t,s) \geqslant \frac{t}{2}\,(1-t)^2 = {g_1}_t(0),\ \forall s \in [0,t].$$

Thus, we construct the following functions:

$$k_{11}(s) = \begin{cases} \frac{t}{3}\,(1-t), & 0 \leqslant t \leqslant \frac{1}{3}, \\ \frac{t}{2}\,(1-t)^2, & \frac{1}{3} < t \leqslant 1, \end{cases}$$

and:

$$k_{21}(s) = \begin{cases} \frac{t}{2}\,(1-t)^2, & 0 \leqslant t < \frac{1}{4}, \\ \frac{1-t}{24}\,(1+2\,t)^2, & \frac{1}{4} \leqslant t \leqslant \frac{1}{2}, \\ \frac{t}{3}\,(1-t), & \frac{1}{2} < t \leqslant 1. \end{cases}$$

For their construction, we conclude that for all $t \in [0,1]$, then:

$$k_{11}(s) \leqslant \tilde{u}(t,s) \leqslant k_{21}(s),\ \forall s \in [0,t].$$

Repeating these arguments for $s \in [t,1]$, we obtain that for all $t \in [0,1]$:

$$k_{12}(s) \leqslant \tilde{u}(t,s) \leqslant k_{22}(s),\ \forall s \in [t,1],$$

where:

$$k_{12}(s) = \begin{cases} \frac{t^2}{2}\,(1-t), & 0 \leqslant t \leqslant \frac{2}{3}, \\ \frac{t}{3}\,(1-t), & \frac{2}{3} < t \leqslant 1, \end{cases}$$

and,

$$k_{22}(s) = \begin{cases} \dfrac{t}{3}\,(1-t), & 0 \leqslant t < \dfrac{1}{2}, \\[2ex] \dfrac{t}{24}\,(3-2\,t)^2, & \dfrac{1}{2} \leqslant t \leqslant \dfrac{3}{4}, \\[2ex] \dfrac{t^2}{2}\,(1-t), & \dfrac{3}{4} < t \leqslant 1. \end{cases}$$

So, we can define:

$$k_1(t) = \begin{cases} \dfrac{t^2}{2}\,(1-t), & 0 \leqslant t \leqslant \dfrac{1}{2}, \\[2ex] \dfrac{t}{2}\,(1-t)^2, & \dfrac{1}{2} < t \leqslant 1, \end{cases}$$

and,

$$k_2(t) = \begin{cases} \dfrac{t}{2}\,(1-t)^2, & 0 \leqslant t \leqslant \dfrac{1}{4}, \\[2ex] \dfrac{1-t}{24}\,(1+2\,t)^2, & \dfrac{1}{4} < t \leqslant \dfrac{1}{2}, \\[2ex] \dfrac{t}{24}\,(3-2\,t)^2, & \dfrac{1}{2} < t \leqslant \dfrac{3}{4}, \\[2ex] \dfrac{t^2}{2}\,(1-t), & \dfrac{3}{4} < t \leqslant 1. \end{cases}$$

Thus, we obtain that:

$$K_2 = \max_{t \in [0,1]} k_2(t) = k_2\left(\frac{1}{2}\right) = \frac{1}{12},$$

and,

$$K_1 = \max_{t \in [0,1]} k_1(t) = k_1\left(\frac{1}{2}\right) = \frac{1}{16}.$$

As in second order case, in Figure 6.4.11, we represent the function $\tilde{u}(t,s)$, bounded from above by $k_2(t)$ and from below by $k_1(t)$. Moreover, in Figure 6.4.12, we show the same representation considering the constant values $t_0 = \dfrac{3}{4}$ and $s_0 = \dfrac{3}{4}$, respectively.

Let us choose $I_1 = [a_1, b_1] = \left[\dfrac{1}{3}, \dfrac{2}{3}\right]$. In this case we have:

$$m_1 = \min_{t \in I_1} k_1(t) = k_1\left(\frac{1}{3}\right) = \frac{1}{27},$$

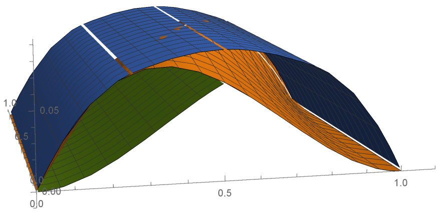

Figure 6.4.11: $\tilde{u}(t,s)$ (orange) bounded from above by $k_2(t)$ (blue) and from below by $k_1(t)$ (green).

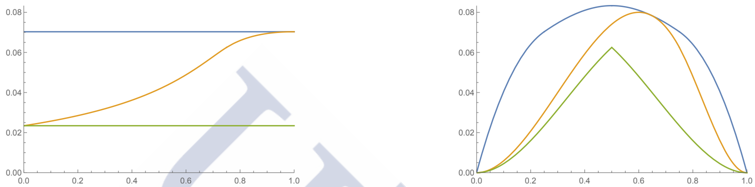

Figure 6.4.12: Figure 6.4.11 for $t_0 = \dfrac{3}{4}$ on the left and for $s_0 = \dfrac{3}{4}$ on the right.

and,

$$\int_0^1 \phi(s)\,\mathrm{d}\,s = \frac{1}{30}, \quad \int_{\frac{1}{3}}^{\frac{2}{3}} \phi(s)\,\mathrm{d}\,s = \frac{47}{2430}, \quad \int_{\frac{1}{3}}^{\frac{2}{3}} k_1(s)\,\phi(s)\,\mathrm{d}\,s = \frac{462461}{470292480}.$$

Hence, as in the second order case, we can obtain the correspondent (H_1) and (H_2), as follows:

(H_1) There exists $p > 0$ such that $f(t,u) \leqslant 360\,p$ for all $t \in [0,1]$ and $u \in [0,p]$.

(H_2) There exists $q > 0$ such that $f(t,u) \geqslant \dfrac{627056640\,u}{426461}$ for every $t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right]$ and $u \in \left[\dfrac{4}{9}q, q\right]$.

Finally, we can rewrite Theorems 6.3.1 and 6.3.3 as follows.

Theorem 6.4.14. *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r,$$

and suppose that function f satisfies the following conditions:

(i) $f(t,u) \geqslant \dfrac{65610}{47} u$ *for all* $t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right]$ *and* $u \in \left[r, \dfrac{9}{4} r\right]$, *being the inequality strict for* $u = r$,

(ii) $f(t,u) < 360\,q$ *for all* $t \in [0,1]$ *and* $u \in \left[0, \dfrac{9}{4} q\right]$, *being the inequality strict for* $u = q$,

(iii) $f(t,u) > \dfrac{627056640}{426461} u$ *for all* $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$ *and* $u \in \left[\frac{4}{9}p, p\right]$.

Then problem (6.4.35) *has at least two positive solutions,* u_1 *and* u_2*, such that:*

$$p < \|u_1\|_{C(I)}, \max_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} u_1(t) < q < \max_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} u_2(t) \text{ and } \min_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} u_2(t) < r.$$

Let us choose a particular function f that the satisfies the conditions imposed in previous result.

$$f(t,u) = \begin{cases} 1296\,t, & u \leqslant \dfrac{1}{36}, \\[2ex] \dfrac{t}{u^2}, & \dfrac{1}{36} < u \leqslant \dfrac{33}{4}, \\[2ex] \dfrac{64\,t}{35937}\left(u - \dfrac{29}{4}\right)^5 u, & u > \dfrac{33}{4}. \end{cases}$$

Let us choose $p = \dfrac{1}{16}$, $q = \dfrac{11}{3}$ and $r = 27$, we have:

(i) For all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $u \in \left[27, \frac{243}{4}\right]$, we have:

$$f(t,u) = \frac{64\,t}{35937}\left(u - \frac{29}{4}\right)^5 u \geqslant \frac{64}{3 \cdot 35937}\left(\frac{79}{4}\right)^5 u = \frac{3077056399\,u}{1724976} > \frac{65610\,u}{47}.$$

(ii) For all $t \in [0,1]$ and $u \in \left[0, \frac{33}{4}\right]$, we have $f(t,u) \leqslant \dfrac{1}{(1/36)^2} = 1296 < \dfrac{11}{3} \cdot 360.$

(iii) For all $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $u \in \left[\frac{1}{36}, \frac{1}{16}\right]$, we have:

$$f(t,u) = \frac{t}{u^3}\,u \geqslant \frac{1/3}{(1/16)^3}\,u = \frac{4096}{3}\,u > \frac{627056640\,u}{462461}.$$

Hence, for this function the problem (6.4.35) has at least two positive solutions, u_1 and u_2, such that $\dfrac{1}{16} < \|u_1\|_{C(I)}$, $\max_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} u_1(t) < \dfrac{11}{3} < \max_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} u_2(t)$ and $\min_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} u_2(t) < 27$.

Now, Theorem 6.3.3 reads as follows:

Theorem 6.4.15. *Suppose* p*,* q *and* r *are positive numbers for which the following inequalities are fulfilled:*

$$0 < p < \frac{9}{4}p < q < \frac{9}{4}q \leqslant r.$$

Assume, moreover, that the function f *satisfies the following conditions:*

(a) $f(t,u) \leqslant 360\,r$ *for all* $t \in [0,1]$ *and* $u \in [0,r]$,

(b) $f(t,u) < 360\, p$ *for all* $s \in [0,1]$ *and* $u \in \left[0, \dfrac{9}{4}\, p\right]$,

(c) $f(t,u) \geqslant \dfrac{65610}{47} u$ *for all* $t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right]$ *and* $u \in \left[q, \dfrac{9}{4} q\right]$, *being the inequality strict for* $u = q$.

Then problem (6.4.35) *has at least three solutions,* u_1, u_2, u_3 *such that* $\|u_i\| \leqslant r$ *for* $i = 1, 2, 3$ *and:*

$$\max_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_1(t) < p < \max_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_2(t) \ \ \text{and} \ \ \min_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_2(t) < q < \min_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_3(t).$$

Let us see that the following function satisfies the imposed hypotheses:

$$f(t,u) = \begin{cases} (2 + 3\, t)\, u^2, & u \leqslant \dfrac{1}{2}, \\[2ex] \left(u - \dfrac{1}{2}\right) u^4 + (2 + 3\, t)\, u^2, & \dfrac{1}{2} < u \leqslant 14, \\[2ex] 519008 + 588\, t, & u > 14. \end{cases}$$

Let us choose $p = \dfrac{2}{9}$, $q = \dfrac{56}{9}$ and $r = 1444$, we can check all the hypotheses of Theorem 6.4.15.

(a) For all $t \in [0,1]$ and $u \in [0, 1444]$, it is fulfilled:

$$f(t,u) \leqslant 519008 + 588 = 519596 < 360 \cdot 1444 = 519840\, r.$$

(b) For all $s \in [0,1]$ and $u \in \left[0, \dfrac{1}{2}\right]$, then $f(t,u) = (2 + 3\, t)\, u^2 \leqslant \dfrac{5}{4} < 360 \cdot \dfrac{1}{2} = 180.$

(c) For all $s \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right]$ and $u \in \left[\dfrac{56}{9}, 14\right]$, we have:

$$\begin{aligned} f(t,u) &= \left(u - \frac{1}{2}\right) u^4 + (2 + 3\, t)\, u^2 = \left(\left(u - \frac{1}{2}\right) u^3 + (2 + 3\, t)\, u \right) u \\ &\geqslant \left(\left(\frac{56}{9} - \frac{1}{2}\right) \left(\frac{56}{9}\right)^3 + 3\, \frac{56}{9} \right) = \frac{9166696\, u}{6561} > \frac{65610\, u}{47}. \end{aligned}$$

Hence, for such a f the problem (6.4.35) has at least three positive solutions, u_1, u_2 and u_3, such that

$$\max_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_1(t) < \frac{1}{2} < \max_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_2(t) \ \ \text{and} \ \ \min_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_2(t) < \frac{56}{9} < \min_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} u_3(t).$$

6.4.3 Simply supported beam boundary conditions

In Chapter 5, we have proved that the Green's function related to the fourth order operator:

$$T_4^0[M]\,u(t) = u^{(4)}(t) + M\,u(t), \quad t \in [0,1], \tag{6.4.36}$$

in $X_{\{0,2\}}^{\{0,2\}}$, $g_M^0(t,s)$, satisfies property $\left(P_{g_1}\right)$ if, and only if,

$$M \in (-\lambda_1(0), -\lambda_2(0)] \approxeq \left(-\pi^4, 5.55^4\right],$$

moreover, it satisfies property $\left(N_{g_1}\right)$ if, and only if,

$$M \in [-\lambda_3(0), -\lambda_1(0)) \approxeq \left[-3.927^4, -\pi^4\right).$$

In particular, property $\left(P_{g_1}\right)$ is fulfilled for $M = 0$ and property $\left(N_{g_1}\right)$ for $M = -100$. Now, let us study these two particular cases.

- $T_4^0[0] = \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}$

We are interested into prove the existence of one or multiple positive solutions of the problem:

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) = u''(1) = 0. \end{cases} \tag{6.4.37}$$

We have:

$$\phi(s) = s\,(1-s). \tag{6.4.38}$$

Now, let us calculate the correspondent k_1 and k_2.

We have to calculate the related Green's function. By means of the *Mathematica* program developed in [21], we obtain:

$$g_0^0(t,s) = \begin{cases} \dfrac{1}{6}\,s\,(1-t)\left((2-t)\,t - s^2\right), & 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{1}{6}\,t\,(1-s)\left((2-s)\,s - t^2\right), & 0 < t < s \leqslant 1. \end{cases}$$

Thus, clearly, for this case:

$$\tilde{u}(t,s) = \begin{cases} \dfrac{1}{6}\cdot\dfrac{1-t}{1-s}\left((2-t)\,t - s^2\right), & 0 \leqslant s \leqslant t < 1,\ t \neq 0, \\[2ex] \dfrac{1}{6}\cdot\dfrac{t}{s}\left((2-s)\,s - t^2\right), & 0 < t < s \leqslant 1. \end{cases}$$

So, we have:

$$\frac{\partial \tilde{u}(t,s)}{\partial s} = \begin{cases} \dfrac{1-t}{6}\cdot\dfrac{(s-t)\,(s+t-2)}{(1-s)^2}, & 0 \leqslant s \leqslant t < 1,\ t \neq 0, \\[2ex] \dfrac{t}{6}\cdot\dfrac{(t-s)\,(t+s)}{s^2}, & 0 < t < s \leqslant 1. \end{cases}$$

Hence, $\tilde{u}(t,s)$ is non-decreasing as a function of s for $s \in [0,t]$ and it is non-increasing for $s \in [t,1]$.

Therefore, we obtain:

$$k_2(t) = \tilde{u}(t,t) = \frac{t\,(1-t)}{3},$$

and, for all $t \in [0,1]$:

$$k_1(t) = \min\left\{\tilde{u}(t,0), \tilde{u}(t,1)\right\} = \min\left\{\frac{1}{6}\,t\,(1-t)\,(2-t),\ \frac{1}{6}\,t\,(1-t^2)\right\},$$

or, which is the same:

$$k_1(t) = \begin{cases} \dfrac{1}{6}\,t\,(1-t^2), & 0 \leqslant t \leqslant \dfrac{1}{2}, \\[2ex] \dfrac{1}{6}\,t\,(1-t)\,(2-t), & \dfrac{1}{2} < t \leqslant 1. \end{cases}$$

As in previous examples, in Figure 6.4.13, we represent the function $\tilde{u}(t,s) = \dfrac{g(t,s)}{\phi(s)}$, bounded from above by $k_2(t)$ and from below by $k_1(t)$. Moreover, in Figure 6.4.14, we show the same representation considering the constant values $t_0 = \dfrac{1}{3}$ and $s_0 = \dfrac{1}{3}$, respectively.

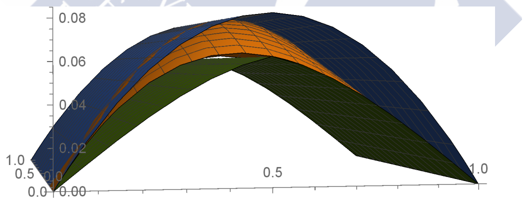

Figure 6.4.13: $\tilde{u}(t,s)$ (orange) bounded from above by $k_2(t)$ (blue) and from below by $k_1(t)$ (green).

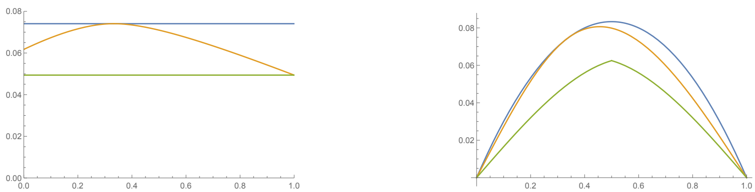

Figure 6.4.14: Figure 6.4.13 for $t_0 = \dfrac{1}{3}$ on the left and for $s_0 = \dfrac{1}{3}$ on the right.

Directly, we obtain:

$$K_1 = \|k_1\|_{C(I)} = k_1\left(\frac{1}{2}\right) = \frac{1}{16} \quad \text{and} \quad K_2 = \|k_2\|_{C(I)} = k_2\left(\frac{1}{2}\right) = \frac{1}{12}.$$

Moreover, since k_1 is a symmetric function with respect to $t = \dfrac{1}{2}$, we can choose:

$$I_1 = [a_1, b_1] = \left[\frac{1}{2} - c, \frac{1}{2} + c\right], \quad c \in \left(0, \frac{1}{2}\right).$$

Clearly,

$$m_1 = k_1\left(\frac{1}{2} - c\right) = \frac{1}{48}\,\left(1 - 4c^2\right)\,(3 - 2c).$$

Now, let us obtain the bounds which appear in Theorems 6.2.1, 6.3.1 and 6.3.3, for the different values of $c \in \left(0, \dfrac{1}{2}\right)$.

ϕ is defined in (6.4.38), so:

$$\int_0^1 \phi(s)\,\mathrm{d}\,s = \frac{1}{6}, \quad \int_{\frac{1}{2}-c}^{\frac{1}{2}+c} \phi(s)\,\mathrm{d}\,s = \frac{1}{6}\,c\,\left(3 - 4\,c^2\right), \tag{6.4.39}$$

and,

$$\int_{\frac{1}{2}-c}^{\frac{1}{2}+c} k_1(s)\,\phi(s)\,\mathrm{d}\,s = \frac{c\,(45 - 15\,c - 120\,c^2 + 60\,c^3 + 144\,c^4 - 80\,c^5)}{1444}.$$

Thus, we obtain:

$$\frac{p}{K_2 \displaystyle\int_0^1 \phi(s)\,\mathrm{d}\,s} = 72\,p, \tag{6.4.40}$$

$$\frac{K_2\,u}{K_1 \displaystyle\int_{\frac{1}{2}-c}^{\frac{1}{2}+c} k_1(s)\,\phi(s)\,\mathrm{d}\,s} = \frac{1083\,u}{c\,\left(45 - 15\,c - 120\,c^2 + 60\,c^3 + 144\,c^4 - 80\,c^5\right)}, \tag{6.4.41}$$

$$\frac{u}{m_1 \displaystyle\int_{\frac{1}{2}-c}^{\frac{1}{2}+c} \phi(s)\,\mathrm{d}\,s} = \frac{288\,u}{c\,(3 - 4\,c^2)\,(3 - 2\,c)\,(1 - 4\,c^2)}. \tag{6.4.42}$$

Studying the behaviour of (6.4.41) and (6.4.42) for $c \in \left(0, \dfrac{1}{2}\right)$, we can easily see that (6.4.41) is decreasing with respect to c and (6.4.42) attains its minimum on the interval $\left[\dfrac{1}{5}, \dfrac{3}{10}\right]$.

Our interest is to make these bounds as lower as possible, but we are not able to minimise them together. We choose $c = \dfrac{3}{10}$, then $I_1 = \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ and we obtain the bounds (6.4.41) and (6.4.42) for this particular case:

$$\frac{K_2\,u}{K_1 \displaystyle\int_{\frac{1}{5}}^{\frac{4}{5}} k_1(s)\,\phi(s)\,\mathrm{d}\,s} = \frac{1600000\,u}{8073}, \tag{6.4.43}$$

$$\frac{u}{m_1 \displaystyle\int_{\frac{1}{5}}^{\frac{4}{5}} \phi(s)\,\mathrm{d}\,s} = \frac{15625\,u}{66}. \tag{6.4.44}$$

Thus, we can rewrite (H_1) and (H_2) for this case as follows.

(H_1) There exists $p > 0$ such that:

$$f(t,u) \leqslant 72\,p, \quad \forall\, t \in [0,1],\ \forall\, u \in [0,p].$$

(H_2) There exists $q > 0$ such that:

$$f(t,u) \geqslant \frac{1600000\,u}{8073}, \quad \forall\, t \in \left[\frac{1}{5}, \frac{4}{5}\right],\ \forall\, u \in \left[\frac{48}{125}\,q, q\right].$$

Using (6.4.40), (6.4.43) and (6.4.44), we rewrite Theorem 6.3.1 for this case.

Theorem 6.4.16. *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r.$$

Assume, moreover, that function f satisfies the following conditions:

(i) $f(t,u) \geqslant \dfrac{15625\,u}{66}$ *for all* $t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ *and* $u \in \left[r, \dfrac{125}{48}\,r\right]$*, being the inequality strict at* $u = r$*,*

(ii) $f(t,u) \leqslant 72\,q$ *for all* $t \in [0,1]$ *and* $u \in \left[0, \dfrac{125}{48}\,q\right]$*, being the inequality strict at* $u = q$*,*

(iii) $f(t,u) > \dfrac{1600000\,u}{8073}$ *for all* $t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ *and* $u \in \left[\dfrac{48}{125}\,p, p\right]$.

Then, the problem (6.4.37) *has at least two positive solution, u_1 and u_2, such that:*

$$p < \|u_1\|_{C(I)}, \quad \max_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_1(t) < q < \max_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_2(t), \quad \min_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_2(t) < r.$$

Now, consider the following continuous function:

$$f(t,u) = \begin{cases} \dfrac{11390625}{512}\,t\,(1-t)\,u, & u \leqslant \dfrac{8}{225}, \\[2ex] \dfrac{t\,(1-t)}{u^2}, & u \in \left(\dfrac{8}{225}, 4\right), \\[2ex] t\,(1-t)\left(\dfrac{56607479}{15250000}u^3 - \dfrac{3621925531}{244000000}u^2\right), & u \geqslant 4. \end{cases} \tag{6.4.45}$$

Let us choose $p = \dfrac{5}{54}$, $q = 3$ and $r = \dfrac{111}{5}$. So, we have:

(i) For $u \leqslant r$, $f(t,u) \geqslant \dfrac{5\,f\left(\frac{1}{5},\frac{111}{5}\right)\,u}{111} = \dfrac{9149203310439\,u}{38125000000} \geqslant \dfrac{15625\,u}{66}$ for every $t \in \left[\dfrac{1}{5},\dfrac{4}{5}\right]$.

(ii) For all $t \in [0,1]$ and $u \in \left[0,\dfrac{125}{16}\right]$, we have:

$$f(t,u) \leqslant \max\left\{\max_{t\in[0,1]} f\left(t,\frac{8}{225}\right), \max_{t\in[0,1]} f\left(t,\frac{125}{16}\right)\right\} = \max\left\{\frac{50625}{256},216\right\}$$
$$= 216 = 3\cdot 72,$$

being the inequality strict for $u < \dfrac{125}{16}$. In particular, the inequality is strict for $u = 3$.

(iii) $f(t,u) = \dfrac{t\,(1-t)}{u^3}u \geqslant \dfrac{4}{25}\cdot\dfrac{157464}{125}u > \dfrac{1600000\,u}{8073}$ for every $t \in \left[\dfrac{1}{5},\dfrac{4}{5}\right]$ and $u \in \left[\dfrac{8}{225},\dfrac{5}{54}\right]$.

So, we can ensure the existence of at least two positive solutions for problem (6.4.37) with f defined in (6.4.45).

Finally, we have the next result which makes sure the existence of three solutions by using (6.4.40) and (6.4.44).

Theorem 6.4.17. *Let p, q and r be positive numbers satisfying the relation:*

$$0 < p < \frac{125}{8}p < q < \frac{125}{48}q \leqslant r.$$

Assume, moreover, that the function f satisfies the following conditions:

(a) $f(t,u) \leqslant 72\,r$ *for all* $t \in [0,1]$ *and* $u \in [0,r]$,

(b) $f(t,u) < 72\,p$ *for all* $t \in [0,1]$ *and* $u \in \left[0,\dfrac{125}{48}\,p\right]$,

(c) $f(t,u) \geqslant \dfrac{15625\,u}{66}$ *for all* $t \in \left[\dfrac{1}{5},\dfrac{4}{5}\right]$ *and* $u \in \left[q,\dfrac{125}{48}q\right]$, *being the inequality strict for* $u = q$.

Then, the problem (6.4.37) *has at least three solutions, u_1, u_2, u_3 such that: $\|u_i\|_{C(I)} \leqslant r$ for $i = 1,2,3$ and:*

$$\max_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_1(t) < p < \max_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_2(t), \qquad \min_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_2(t) < q < \min_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_3(t).$$

Consider the continuous function:

$$f(t,u)=\begin{cases}1400\,t\,(1-t)\,u^2, & u\geqslant \dfrac{7}{2},\\[2ex] 17150\,t\,(1-t), & u>\dfrac{7}{2}.\end{cases}\tag{6.4.46}$$

Let us choose $r=60$, $p=\dfrac{48}{625}$ and $q=\dfrac{5}{4}$, we have:

(a) $f(t,u)\leqslant \dfrac{17150}{4}<60\cdot 72$ for all $t\in[0,1]$ and $u\in[0,60]$,

(b) $f(t,u)\leqslant \dfrac{1400}{4}\cdot\dfrac{1}{5^2}=14<72\cdot\dfrac{1}{5}$ for all $t\in[0,1]$ and $u\in\left[0,\dfrac{1}{5}\right]$,

(c) $f(t,u)\geqslant \dfrac{1400\cdot 4}{5^2}\cdot\dfrac{5}{4}u=280\,u>\dfrac{15625\,u}{66}$ for all $t\in\left[\dfrac{1}{5},\dfrac{4}{5}\right]$ and $u\in\left[\dfrac{5}{4},\dfrac{625}{192}\right]$.

Thus, by using the previous result, we conclude that problem (6.4.37) has at least three solutions for f defined in (6.4.46).

- $T_4^0[-100]=\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}-100$

In this case, as for problem (6.4.18) we took the second option in Remark 6.2.2, we want to prove the existence of negative solutions of the problem:

$$\begin{cases}u^{(4)}(t)-100\,u(t)=f\big(t,u(t)\big),\ t\in[0,1],\\ u(0)=u''(0)=0,\\ u(1)=u''(1)=0.\end{cases}\tag{6.4.47}$$

Clearly, ϕ is the same as the one given in (6.4.38).

By means the *Mathematica* program, defined in [21], we obtain $g^0_{-100}(t,s)$, the related Green's function:

$$\begin{cases}\dfrac{\operatorname{csch}\left(\sqrt{10}\right)\sinh\left(\sqrt{10}s\right)\sinh\left(\sqrt{10}(t-1)\right)-\csc\left(\sqrt{10}\right)\sin\left(\sqrt{10}s\right)\sin\left(\sqrt{10}(t-1)\right)}{20\sqrt{10}}, & 0\leqslant s\leqslant t\leqslant 1,\\[4ex] \dfrac{\csc\left(\sqrt{10}\right)\left(\cos\left(\sqrt{10}(s+t-1)\right)-\cos\left(\sqrt{10}(-s+t+1)\right)\right)}{40\sqrt{10}} & \\ \quad+\dfrac{\operatorname{csch}\left(\sqrt{10}\right)\left(\cosh\left(\sqrt{10}(s+t-1)\right)-\cosh\left(\sqrt{10}(-s+t+1)\right)\right)}{40\sqrt{10}}, & 0<t<s\leqslant 1.\end{cases}$$

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Despite the difficulty of the obtained expression, we can obtain $\tilde{u}(t,s)$ as the continuous extension of $u(t,s)$, given by the expression:

$$\begin{cases} \dfrac{\operatorname{csch}\left(\sqrt{10}\right)\sinh\left(\sqrt{10}s\right)\sinh\left(\sqrt{10}(t-1)\right)-\csc\left(\sqrt{10}\right)\sin\left(\sqrt{10}s\right)\sin\left(\sqrt{10}(t-1)\right)}{20\,s\,(1-s)\sqrt{10}}, & 0<s\leqslant t<1, \\[2ex] \dfrac{\csc\left(\sqrt{10}\right)\left(\cos\left(\sqrt{10}(s+t-1)\right)-\cos\left(\sqrt{10}(-s+t+1)\right)\right)}{40\,s\,(1-s)\sqrt{10}} \\ \quad +\dfrac{\operatorname{csch}\left(\sqrt{10}\right)\left(\cosh\left(\sqrt{10}(s+t-1)\right)-\cosh\left(\sqrt{10}(-s+t+1)\right)\right)}{40\,s\,(1-s)\sqrt{10}}, & 0<t<s<1. \end{cases}$$

to $(0,1)\times[0,1]$.

Studying this function, it can be seen that for each $t\in(0,1)$, the maximum is found either at $s=0$, for $t\in\left[0,\dfrac{1}{2}\right]$, or at $s=1$, for $t\in\left(\dfrac{1}{2},1\right]$. So, we consider:

$$k_1(t)=\begin{cases} \dfrac{1}{20}\left(\operatorname{csch}\left(\sqrt{10}\right)\sinh\left(\sqrt{10}(t-1)\right)-\csc\left(\sqrt{10}\right)\sin\left(\sqrt{10}(t-1)\right)\right), & 0\leqslant t\leqslant\dfrac{1}{2}, \\[2ex] \dfrac{1}{20}\left(\csc\left(\sqrt{10}\right)\sin\left(\sqrt{10}t\right)-\operatorname{csch}\left(\sqrt{10}\right)\sinh\left(\sqrt{10}t\right)\right), & \dfrac{1}{2}<t\leqslant 1. \end{cases} \tag{6.4.48}$$

Again, the expression of $k_2(t)$ is quite difficult to find. However, we can see that the minimum of $k_2(t)$ is given by:

$$k_2\left(\frac{1}{2}\right)=\tilde{u}\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\tan\left(\sqrt{\frac{5}{2}}\right)-\tanh\left(\sqrt{\frac{5}{2}}\right)}{10\sqrt{10}}.$$

So, we consider this inferior bound for $\tilde{u}(t,s)$. In Figure 6.4.15, we represent the function $\tilde{u}(t,s)=\dfrac{g(t,s)}{\phi(s)}$, bounded from below by $k_2\left(\dfrac{1}{2}\right)$ and from above by $k_1(t)$. Moreover, in Figure 6.4.16, the same functions are represented for the constant values $t_0=\dfrac{1}{2}$ and $s_0=\dfrac{1}{2}$, respectively.

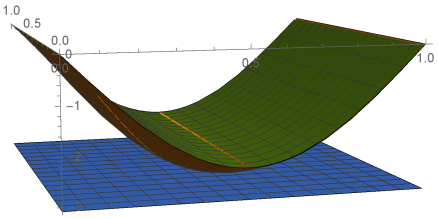

Figure 6.4.15: $\tilde{u}(t,s)$ (orange) bounded from below by $k_2(t)$ (blue) and from above by $k_1(t)$ (green).

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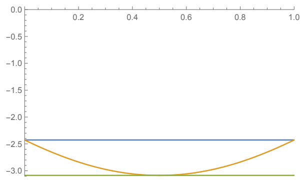

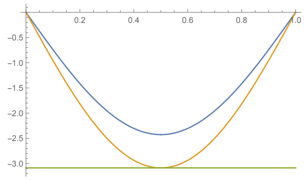

Figure 6.4.16: Figure 6.4.15 for $t_0 = \dfrac{1}{2}$ on the left and for $s_0 = \dfrac{1}{2}$ on the right.

In this case, $k_1(t)$ is a symmetric function with respect to $t = \dfrac{1}{2}$. And, moreover, it attains its minimum value at this point. Thus, we obtain:

$$K_1 = -k\left(\frac{1}{2}\right) = \frac{\operatorname{sech}\left(\sqrt{\frac{5}{2}}\right) - \sec\left(\sqrt{\frac{5}{2}}\right)}{40}.$$

On the other hand, since we are considering the lower bound of $\tilde{u}(t,s)$ as a constant function, we have:

$$K_2 = \frac{\tanh\left(\sqrt{\frac{5}{2}}\right) - \tan\left(\sqrt{\frac{5}{2}}\right)}{10\sqrt{10}}.$$

Moreover, as in previous example, we choose $I_1 = \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ which is a symmetric interval with respect to $t = \dfrac{1}{2}$.

Therefore, we obtain the minimum value of $-k_1$ in $\left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ as follows:

$$m_1 = k_1\left(\frac{1}{5}\right) = \frac{\left(\sin\left(4\sqrt{\frac{2}{5}}\right)\csc\left(\sqrt{10}\right) - \sinh\left(4\sqrt{\frac{2}{5}}\right)\operatorname{csch}\left(\sqrt{10}\right)\right)}{20}.$$

Clearly (6.4.39) remains true for this case with $c = \dfrac{3}{10}$ and:

$$\begin{aligned}\int_{\frac{1}{5}}^{\frac{4}{5}} \left|k_1(s)\right| \phi(s)\,\mathrm{d}\,s =& \frac{9\sqrt{10}\csc\left(\sqrt{\frac{5}{2}}\right)\left(4\cos\left(4\sqrt{\frac{2}{5}}\right)\sec\left(\sqrt{\frac{5}{2}}\right) - 5\right) - 5\sqrt{10}\operatorname{csch}\left(\sqrt{\frac{5}{2}}\right)}{20000}\\ &+ \frac{3\sin\left(4\sqrt{\frac{2}{5}}\right)\csc\left(\sqrt{10}\right) + 3\sinh\left(4\sqrt{\frac{2}{5}}\right)\operatorname{csch}\left(\sqrt{10}\right)}{500}\\ &+ \frac{9\sqrt{10}\cos\left(4\sqrt{\frac{2}{5}}\right)\csc\left(\sqrt{\frac{5}{2}}\right)\sec\left(\sqrt{\frac{5}{2}}\right)}{5000}.\end{aligned}$$

So, we have:

$$\frac{p}{K_2 \displaystyle\int_0^1 \phi(s)\,\mathrm{d}\,s} = \frac{60\sqrt{10}\,p}{\tanh\left(\sqrt{\frac{5}{2}}\right) - \tan\left(\sqrt{\frac{5}{2}}\right)} \approxeq 1.944\,p > \frac{97}{50}\,p, \tag{6.4.49}$$

$$\frac{K_2\,u}{K_1 \displaystyle\int_{\frac{1}{3}}^1 k_1(s)\,\phi(s)\,\mathrm{d}\,s} \approxeq 4.558\,u < \frac{114}{25}\,u, \tag{6.4.50}$$

$$\begin{aligned}\frac{u}{m_1 \displaystyle\int_{\frac{1}{5}}^{\frac{4}{5}} \phi(s)\,\mathrm{d}\,s} &= \frac{5000}{33\left(\sin\left(4\sqrt{\frac{2}{5}}\right)\left(-\csc\left(\sqrt{10}\right)\right) + \sinh\left(4\sqrt{\frac{2}{5}}\right)\operatorname{csch}\left(\sqrt{10}\right)\right)} \\ &\approxeq 5.354\,u < \frac{134}{25}\,u.\end{aligned} \tag{6.4.51}$$

Moreover, we have for this case:

$$\frac{m_1}{K_2} = \frac{\sqrt{\frac{5}{2}}\left(\sin\left(4\sqrt{\frac{2}{5}}\right)\csc\left(\sqrt{10}\right) - \sinh\left(4\sqrt{\frac{2}{5}}\right)\operatorname{csch}\left(\sqrt{10}\right)\right)}{\tan\left(\sqrt{\frac{5}{2}}\right) - \tanh\left(\sqrt{\frac{5}{2}}\right)} \approxeq 0.485 < \frac{23}{50}. \tag{6.4.52}$$

As in Problem (6.4.4), to obtain the correspondent of previous results for this case, we use the approximations of the bounds which are also applicable for obtaining the results and their expressions are much easier to work with. First, let us write (H_1) and (H_2).

(H_1) There exists $p > 0$ such that:

$$f(t,u) \leqslant \frac{97}{50}\,p, \quad \forall\, t \in [0,1],\ \forall\, u \in [-p, 0].$$

(H_2) There exists $q > 0$ such that:

$$f(t,u) \geqslant -\frac{114}{25}\,u, \quad \forall\, t \in \left[\frac{1}{5}, \frac{4}{5}\right],\ \forall\, u \in \left[-q, -\frac{23}{50}\,q\right].$$

Finally, we use (6.4.49), (6.4.50), (6.4.51) and (6.4.52) to write the results collected in Section 6.3.

Theorem 6.4.18. *Suppose that there exist positive numbers p, q and r such that:*

$$0 < p < q < r,$$

and assume that function f satisfies the following conditions:

(i) $f(t,u) \geqslant -\frac{134}{25}\,u$ *for all* $t \in \left[\frac{1}{5}, \frac{4}{5}\right]$ *and* $u \in \left[-\frac{50}{23}\,r, -r\right]$,

(ii) $f(t,u) \leqslant \dfrac{97}{50}\, q$ *for all* $t \in [0,1]$ *and* $u \in \left[-\dfrac{50}{23}\, q, 0\right]$,

(iii) $f(t,u) \geqslant -\dfrac{114}{25}\, u$ *for all* $t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ *and* $u \in \left[-p, -\dfrac{23}{50}\, p\right]$.

Then, the problem (6.4.47) *has at least two positive solution,* u_1 *and* u_2*, such that:*

$$p < \|u_1\|_{C(I)}, \quad \min_{t \in \left[\frac{1}{5},\frac{4}{5}\right]} u_1(t) > -q > \min_{t \in \left[\frac{1}{5},\frac{4}{5}\right]} u_2(t), \quad \max_{t \in \left[\frac{1}{5},\frac{4}{5}\right]} u_2(t) > -r.$$

Now, consider the following continuous function:

$$f(t,u) = \begin{cases} 25\, t\,(1-t), & u \geqslant -\dfrac{23}{125}\,, \\[2ex] \dfrac{23\, t\,(t-1)}{5\, u}, & u \in \left(-\dfrac{23}{3}, -\dfrac{23}{125}\right), \\[2ex] \dfrac{27}{2645} t\,(1-t) u^2, & u \leqslant -\dfrac{23}{3}. \end{cases} \tag{6.4.53}$$

Let us choose $p = \dfrac{2}{5}$ and $q = \dfrac{625}{194}$, we have:

(i) Since $\displaystyle\lim_{u \to -\infty} \frac{f(t,u)}{u} = -\infty$, we have ensured the existence of $r > 0$ such that this property is fulfilled.

(ii) For all $t \in [0,1]$ and $u \in \left[-\dfrac{15625}{2231}, 0\right]$ we have:

$$f(t,u) \leqslant f\,(t,0) = 25\, t\,(1-t) \leqslant \frac{25}{4} = \frac{97}{50} \cdot \frac{625}{194}.$$

(iii) $f(t,u) = \dfrac{t\,(1-t)}{u^2}(-u) \geqslant \dfrac{23}{5} \cdot \dfrac{\frac{4}{25}}{\left(\frac{2}{5}\right)^2}(-u) = -\dfrac{23}{5}\, u > -\dfrac{114}{25}\, u$ for every t in $\left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ and $u \in \left[-\dfrac{2}{5}, -\dfrac{23}{125}\right]$.

So, we can ensure the existence of at least two positive solutions for problem (6.4.49) with f defined in (6.4.53).

Now, we have the next result which warrants the existence of three solutions.

Theorem 6.4.19. *Let* p*,* q *and* r *be positive numbers satisfying the relation:*

$$0 < p < \frac{50}{23} p < q < \frac{50}{23} q \leqslant r.$$

Assume, moreover, that the function f *satisfies the following conditions:*

(a) $f(t,u) \leqslant \dfrac{97}{50}\, r$ *for all* $t \in [0,1]$ *and* $u \in [-r,0]$,

(b) $f(t,u) \leqslant \dfrac{97}{50}\, p$ *for all* $t \in [0,1]$ *and* $u \in \left[-\dfrac{50}{23}\, p, 0\right]$,

(c) $f(t,u) \geqslant -\dfrac{134}{25}\, u$ *for all* $t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ *and* $u \in \left[-\dfrac{50}{23} q, -q\right]$.

Then, the problem (6.4.47) *has at least three solutions,* u_1*,* u_2*,* u_3 *such that* $\|u_i\|_{C(I)} \leqslant r$ *for* $i = 1,2,3$ *and:*

$$\min_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_1(t) > -p > \min_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_2(t), \quad \max_{t\in I_1} u_2(t) > -q > \max_{t\in\left[\frac{1}{5},\frac{4}{5}\right]} u_3(t).$$

To finish this chapter, consider the continuous function:

$$f(t,u) = \begin{cases} 15\, t\,(1-t)\, u^2, & u \geqslant -\dfrac{125}{23}, \\[2ex] \dfrac{234375\, t\,(1-t)}{529}, & u < -\dfrac{125}{23}. \end{cases} \tag{6.4.54}$$

Let us choose $p = \dfrac{23}{100}$ and $q = \dfrac{5}{2}$, we have:

(a) Since $f(t,s)$ is a bounded function in $\mathbb{R}$, it is clear that it must exists $r > 0$ such that this property is fulfilled.

(b) $f(t,u) \leqslant f\left(t, \dfrac{1}{2}\right) \leqslant \dfrac{15}{8} \cdot \dfrac{1}{2} \geqslant \dfrac{97}{50} \cdot \dfrac{1}{2}$ for all $t \in [0,1]$ and $u \in \left[0, \dfrac{1}{2}\right]$.

(c) $f(t,u) \geqslant \dfrac{15 \cdot 4}{25} \cdot (-u)\,(-u) \geqslant -6\, u > -\dfrac{134\, u}{25}$ for every $t \in \left[\dfrac{1}{5}, \dfrac{4}{5}\right]$ and every $u \in \left[-\dfrac{125}{23}, -\dfrac{5}{2}\right]$.

Thus, by using the previous result, we conclude that problem (6.4.47) has at least three solutions for f defined in (6.4.54).

Chapter 7

Existence results for non-linear problems via variational methods

Contents

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This chapter is devoted to develop a new technique on the study of non-linear differential problems. In this case the non-linearities appear in all the non-null coefficients. Thus, it is not possible to use the linear part of the considered problem to obtain the related inverse integral operator by means of the related Green's function. In this case, instead of looking for the inverse operator and fixed points of it, we prove that the solutions of the considered problems coincide with critical points of associated functionals.

In this chapter we study two different problems by means of variational methods, both include the non-linearities given by the p-Laplacian functional, which is defined below.

Definition 7.0.1. *Let $p \in (1, +\infty)$, we consider the p-Laplacian functional as:*

$$\varphi_p \colon \mathbb{R} \to \mathbb{R},$$

defined by:

$$\varphi_p(t) = \begin{cases} t|t|^{p-2}, & t \neq 0, \\ \\ 0, & t = 0. \end{cases}$$

Despite both considered problems involve the p-Laplacian functional, they are different in the sense that in the first part of the chapter we study a continuous problem with periodic boundary conditions and in the second part we prove the existence of solution for a discrete problem involving the p-Laplacian, with different additional considerations.

Problems involving the p-Laplacian have been widely studied in the literature. For instance, in [90] it is considered the following second order impulsive problem coupled with the Dirichlet boundary conditions:

$$\begin{cases} \Big[\varphi_p\big(u'(t)\big)\Big]' + f\big(t, u(t), u'(t)\big) = 0, \\ \Delta\, u'(t_i) = I_i\big(u(t_i)\big), \quad i = 1, 2, \ldots, \ell, \\ u(0) = u(T) = 0, \end{cases}$$

where $p \geqslant 2$, $0 = t_0 < t_1 < \cdots < t_\ell < t_{\ell+1} = T$ and:

$$\Delta\, u'(t_i) = \lim_{t \to {t_i}^+} \Phi(u'(t)) - \lim_{t \to {t_i}^-} \Phi(u'(t)).$$

In [38], it is obtained the existence of anti-periodic solutions for the following $2\, n^{\text{th}}$-order problem:

$$\begin{cases} \Big[\varphi_p\left(u^{(n)}(t)\right)\Big]^{(n)} + f\big(u(t)\big)\, u'(t) + g\big(t, u(t)\big) = e(t), \\ & t \in \mathbb{R}, \\ u(t + \pi) = -u(t), \end{cases}$$

where $p > 1$, $n > 2$ and imposing several conditions on the continuous functions f, g and e.

Furthermore, in [70] Liu studies a fourth order problem involving the p-Laplacian with deviating terms:

$$\Big[q(t)\, \varphi_p\big(u''(t)\big)\Big]'' = f\Big(t, u(t), u\big(\tau_1(t)\big), \ldots, u\big(\tau_m(t)\big)\Big),\; t \in \mathbb{R}.$$

In Section 7.2, we focus on discrete non-linear problems. The theory of non-linear difference equations is broadly used in the study of discrete models in different fields of science. Recently, the problems for difference equations are treated by topological and variational methods. Topological methods for higher order difference equations using Green's functions and fixed point theorems are used in [2], [5]. The variational methods coupled with critical point theory have been extensively applied to the resolvability of problems for difference equations during the last decade. We refer the reader to [1], [56], [89] and references therein. A survey on applications of critical point theory to existence results for difference equations is given in [25]. Periodic and homoclinic orbits for $2\, n^{\text{th}}$ order difference equations are studied in [35] using linking theorem and in [37] by mountain-pass and symmetric mountain-pass theorems.

The second section is divided in two parts. The first part is based on the mountain-pass theorem of Brezis and Nirenberg [14]. Following the steps of [26], we obtain the existence of a non-trivial homoclinic solution of equation (7.2.1), i.e. a non-zero solution u, such that:

$$\lim_{|k| \to +\infty} |u(k)| = 0.$$

In the second part, we obtain the existence of at least three solutions for the difference equation with $p_i = q$ for all $i = 1, \ldots, n$ and the Dirichlet boundary conditions, by generalizing a result given in [36] for the problem:

$$-\Delta\Big[\varphi_p\Big(\Delta u(k-1)\Big)\Big] = \lambda\, f\big(k, u(k)\big),\; k \in [1, T], \quad u(0) = u(T+1) = 0,$$

where $[1,T]=\{1,2,\ldots,T\}$ for a fixed integer $T>0$.

Such a result is obtained by applying [36, Theorem 2.1], which is a modification of the theorem of D. Averna and G. Bonanno (see [6]), to our boundary value problem.

The study of p-Laplacian difference equations has been developed in the literature. In addition to the previously mentioned ([26], [36]), we refer to [58], where the following problem is studied:

$$\Delta\Big[\varphi_p\Big(\Delta u(k-1)\Big)\Big]+a(t)\,f\big(k,u(k)\big)=0,\ k\in[1,T+1],\quad \Delta u(0)=u(T+2)=0,$$

where $a(t)$ is a is a positive function. Moreover, in [94], the existence of three positive solutions of this problems is studied.

Recently, in [45], Dimitrov has proved the existence of at least three solutions of the problem:

$$\begin{aligned}\Delta^2\Big[\varphi_p\Big(\Delta^2 u(k-2)\Big)\Big]+\alpha\,\varphi_p\big(u(k)\big)&=\lambda f\big(k,u(k)\big),\quad k\in[1,T],\\ u(0)=\Delta u(-1)&=\Delta^2 u(T)=0,\\ \Delta\Big[\varphi_p\Big(\Delta^2 u\,(T-1)\,\Big)\Big]&=\mu g\big(u(T+1)\big),\end{aligned}$$

where α, λ and μ are real parameters, f and g are continuous.

Moreover, we refer to [59, 74, 89], where the existence of homoclinic solution for different discrete second order problems is proved. Finally, in [69] there is studied the existence of periodic solutions for a higher order difference equation involving the p-Laplacian.

This chapter is structured in two different sections. One of them is devoted to prove the existence of solutions for a non-linear differential boundary value problem organised as follows: firstly, we construct the variational approach of the considered problem, then the existence results and, after that, it is shown how the results are modified for an impulsive problem. Finally, some particular cases are presented. In Section 7.2, they are considered two problems involving a discrete differential equation, at first we focus on proving the existence of homoclinic non-trivial solutions and, we finish the chapter by studying a discrete boundary value problem with boundary conditions which correspond to the discretisation of a particular case of the $(k,n-k)$ boundary conditions.

7.1 Periodic $2\,n^{\mathrm{th}}$-order non-linear p-Laplacian differential equations

This section is devoted to the study of a $2\,n^{\mathrm{th}}$-order non-linear differential equation involving the p-Laplacian, with periodic boundary conditions. Our aim is to study the existence of solutions for the following problem:

$$\begin{aligned}\Big[\varphi_p\Big(u^{(n)}(t)\Big)\Big]^{(n)}+\sum_{i=1}^{n-1}(-1)^i a_i\Big[\varphi_p\Big(u^{(n-i)}(t)\Big)\Big]^{(n-i)}&\\ +(-1)^n\Big(f\big(t,u(t)\big)-h\big(t,u(t)\big)\Big)&=0,\ t\in[0,T],\end{aligned}\tag{7.1.1}$$

coupled with the periodic boundary conditions:

$$u(T)-u(0)=\cdots=u^{(2\,n-1)}(T)-u^{(2\,n-1)}(0)=0, \tag{7.1.2}$$

where $T \geqslant 0$ and $a_i \geq 0$ for $i=1,\ldots,n-1$.

Remark 7.1.1. *In this chapter, for the sake of simplifying the notation, we consider the interval* $[0,T]$ *instead of* $[a,b]$.

The results here presented are published in [83] and generalise the ones obtained in [49] for the particular second order case.

This section is devoted to the study the existence of solutions for the boundary value problem (7.1.1)-(7.1.2). The approach is done by means of variational calculus. Thus, in next section we describe some of the tools which allow us to obtain the results.

7.1.1 Variational approach and previous results

At first, we need to introduce a Banach space to work in. Let us define:

$$W_p := \Big\{u \in W^{n,p}\,(0,T)\ \big|\ u(T)-u(0)=\cdots=u^{(n-1)}(T)-u^{(n-1)}(0)=0\Big\}, \tag{7.1.3}$$

where $W^{n,p}\,(0,T)$ is the Sobolev space defined as follows:

$$W^{n,p}\,(0,T)=\Big\{u\in L^p(0,T)\,\Big|\,u^{(i)}\in L^p(0,T),\ i=1,\ldots,n\Big\}.$$

Its associated norm is:

$$\|u\|_p=\left(\sum_{i=0}^{n}\int_0^T \left|u^{(i)}(t)\right|^p \mathrm{d}\,t\right)^{1/p}<\infty,\qquad \forall u\in W^{n,p}.$$

Now, let us define the classical solutions for the problem (7.1.1)-(7.1.2).

Definition 7.1.2. *A function* $u \in C^n\big([0,T]\big)$ *is said to be a classical solution of problem* (7.1.1)-(7.1.2) *if* $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n\big([0,T]\big)$ *and it satisfies equation* (7.1.1) *for* $t\in(0,T)$ *coupled with periodic boundary conditions* (7.1.2).

Remark 7.1.3. *Realise that, from the derivative chain rule, it is clear that the assumption* $\varphi_p\big(u^{(n)}(\cdot)\big) \in C^n\big([0,T]\big)$ *implies that* $u^{(n)} \in C^n\big([0,T]\big)$. *Moreover, if* $u^{(n)}$ *is not of constant sign on* $[0,T]$, *in order to have* $\varphi_p\big(u^{(n)}(\cdot)\big) \in C^n\big([0,T]\big)$, *we should ask for* $\varphi_p \in C^n(\mathbb{R})$.

So, in particular, $u\in C^{2\,n}\big([0,T]\big)$.

However, we need to study the regularity of the p*-Laplacian to ensure that a function which satisfies* $u\in C^{2\,n}\big([0,T]\big)$ *also satisfies* $\varphi_p\big(u^{(n)}(\cdot)\big)\in C^n\big([0,T]\big)$.

For instance, let us consider $n=2$ *and* $p=3$, *for* $u\in C^4\big([0,T]\big)$ *we have:*

$$\Big[\varphi_3\big(u''(t)\big)\Big]'=\varphi_3'\big(u''(t)\big)\,u^{(3)}(t)=|u''(t)|\;u^{(3)}(t),$$

hence $\varphi_3\big(u''(\cdot)\big)\in C^1\big([0,T]\big)$, *but* $\varphi_3\big(u''(\cdot)\big)\notin C^2\big([0,T]\big)$ *if* u'' *is not of constant sign on* $[0,T]$ *even if* $u^{(3)}\in C^1\big([0,T]\big)$.

Now, we introduce a concept which has already been mentioned in Chapter 5, for a weaker concept of solution of (7.1.1)-(7.1.2).

Definition 7.1.4. *The function $u \in W_p$ is said to be a weak solution of* (7.1.1)-(7.1.2) *if for every $v \in W_p$ it is satisfied the following equality:*

$$\begin{aligned}\int_0^T \varphi_p\left(u^{(n)}(t)\right) v^{(n)}(t)\,\mathrm{d}\,t + \sum_{i=1}^{n-1} a_i \int_0^T \varphi\left(u^{(n-i)}(t)\right)\, v^{(n-i)}(t)\,\mathrm{d}\,t \\ + \int_0^T \Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big)\, v(t)\,\mathrm{d}\,t = 0.\end{aligned} \tag{7.1.4}$$

Our aim is to ensure the existence of multiple weak solutions of (7.1.1)-(7.1.2). That is, we look for $u \in W_p$ such that (7.1.4) is satisfied. Then, we ensure that, in some cases, the weak solution is also a classical solution.

Now, we introduce a condition that f and h must satisfy. Let us consider the following functions:

$$F(t,u) = \int_0^u f(t,s)\,\mathrm{d}\,s, \quad H(t,u) = \int_0^u h(t,s)\,\mathrm{d}\,s.$$

(FH) We say that f and h satisfy condition (FH) if they are continuous functions on $\mathbb{R}^2$ and there exist positive constants b_1, b_2, c_1, c_2 and $q > r > 1$ such that for all $u \in \mathbb{R}$ the following inequalities are fulfilled:

$$b_1\,|u|^q \leqslant F(t,u) \leqslant b_2\,|u|^q, \quad c_1\,|u|^r \leqslant H(t,u) \leqslant c_2\,|u|^r. \tag{7.1.5}$$

Remark 7.1.5. *Realise that we can write $b_i = \dfrac{\tilde{b}_i}{q}$ and $c_i = \dfrac{\tilde{c}_i}{r}$ for $i = 1, 2$.*

In the sequel, we explain several previous results which will allow us to obtain the existence results for the solutions of (7.1.1)-(7.1.2).

First, we introduce a different norm in W_p:

$$\|u\| = \left(\int_0^T |u^{(n)}(t)|^p\,\mathrm{d}\,t + \int_0^T |u(t)|^p\,\mathrm{d}\,t\right)^{1/p}.$$

Now, we show a result which proves that $\|\cdot\|$ and $\|\cdot\|_p$ are equivalent norms in W_p.

Lemma 7.1.6. *If $u \in W_p$, then:*

$$\int_0^T \left|u^{(i)}(t)\right|^p \mathrm{d}\,t \leqslant T^{p\,(n-i)} \int_0^T \left|u^{(n)}(t)\right|^p \mathrm{d}\,t, \quad i = 1, \ldots, n-1.$$

Proof. First, as it is proved in [73, pp. 8-9], we have that $u \in W^{1,p}(0,T)$ and $\displaystyle\int_0^T u(t)\,\mathrm{d}\,t = 0$, then:

$$\int_0^T |u(t)|^p\,\mathrm{d}\,t \leqslant T^p \int_0^T |u'(t)|^p\,\mathrm{d}\,t.$$

Now, since $u \in W_p$:

$$\int_0^T u^{(j)}(t)\,\mathrm{d}\,t = u^{(j-1)}(T) - u^{(j-1)}(0) = 0,\; j = 1,\dots,n-1.$$

We have:

$$\int_0^T \left|u^{(j)}(t)\right|^p \mathrm{d}\,t \leqslant T^p \int_0^T \left|u^{(j+1)}(t)\right|^p \mathrm{d}\,t, \quad j = 1,\dots,n-1. \tag{7.1.6}$$

Applying an induction argument, we prove the result. For $i = n-1$, the result follows from (7.1.6).

For $i \in \{2,\dots,n-1\}$, from (7.1.6), we have:

$$\int_0^T \left|u^{(i-1)}(t)\right|^p \mathrm{d}\,t \leqslant T^p \int_0^T \left|u^{(i)}(t)\right|^p \mathrm{d}\,t, \quad i = 1,\dots,n-1,$$

and the result follows by the induction hypothesis. $\square$

Using this result, the equivalence between the norms $\|\cdot\|$ and $\|\cdot\|_p$ is obvious:

$$\|u\| \leqslant \|u\|_p \leqslant \left(1 + \sum_{i=1}^{n-1} T^{p\,(n-i)}\right)^{1/p} \|u\| := k_1\|u\|.$$

Considering again [73, pp. 8-9], we have the following result.

Lemma 7.1.7. *For $u \in W_p$, the following inequality is fulfilled:*

$$\|u\|_{C([0,T])} = \max_{t\in[0,T]} |u(t)| \leqslant \left(T^{-\frac{1}{p}} + T^{\frac{p-1}{p}}\right) \|u\|_p := k_2\|u\|_p.$$

Now, let us consider the following space:

$$W = \left\{u \in W_p \;\middle|\; \int_0^T u(t)\,\mathrm{d}\,t = 0\right\},$$

coupled with the norm:

$$\|u\|_W = \left(\int_0^T \left|u^{(n)}(t)\right|^p \mathrm{d}\,t\right)^{1/p}.$$

Taking into account that $\displaystyle\int_0^T u(t)\,\mathrm{d}\,t = 0$ for $u \in W$, we have an analogous result to Lemma 7.1.6 to show that $\|\cdot\|_p$ and $\|\cdot\|_W$ are equivalent in W. The result is the following.

Lemma 7.1.8. *If $u \in W$, then*

$$\int_0^T \left|u^{(i)}(t)\right|^p \mathrm{d}\,t \leqslant T^{p\,(n-i)} \int_0^T \left|u^{(n)}(t)\right|^p \mathrm{d}\,t, \quad i = 0,\dots,n-1.$$

The equivalence between $\|\cdot\|_p$ and $\|\cdot\|_W$ in W follows directly from Lemma 7.1.8:

$$\|u\|_W \leqslant \|u\|_p \leqslant \left(1+\sum_{i=0}^{n-1} T^{p\,(n-i)}\right)^{1/p} \|u\|_W := k_3\|u\|_W .$$

Now, we introduce the variational approach of problem (7.1.1)-(7.1.2).
Let us consider the function:

$$\Phi_p(t) = \frac{|t|^p}{p}. \tag{7.1.7}$$

It is clear that $\Phi_p'(t) = \varphi_p(t)$.
Define a functional $K\colon W_p \to \mathbb{R}$, as follows:

$$K(u) := K_1(u) + K_2(u), \tag{7.1.8}$$

where:

$$K_1(u) = \int_0^T \left[\Phi_p\Big(u^{(n)}(t)\Big) + \sum_{i=1}^{n-1} a_i \Phi_p\Big(u^{(n-i)}(t)\Big)\right] \mathrm{d}\,t, \tag{7.1.9}$$

$$K_2(u) = \int_0^T \Big(F\big(t,u(t)\big) - H\big(t,u(t)\big)\Big)\mathrm{d}\,t. \tag{7.1.10}$$

The functional K is Gateaux differentiable and for all $u, v \in W_p$, we have:

$$\begin{aligned}\langle K'(u), v\rangle &= \lim_{h\to 0} \frac{K(u+h\,v) - K(u)}{h} \\ &= \int_0^T \varphi_p\left(u^{(n)}(t)\right) v^{(n)}(t)\,\mathrm{d}\,t + \sum_{i=1}^{n-1} a_i \int_0^T \varphi_p\left(u^{(n-i)}(t)\right) v^{(n-i)}(t)\,\mathrm{d}\,t \\ &\quad + \int_0^T \Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big)\, v(t)\,\mathrm{d}\,t.\end{aligned}$$

Hence the critical points of K are exactly the weak solutions of (7.1.1)-(7.1.2).

Thus, our procedure to prove the existence of weak solutions for problem (7.1.1)-(7.1.2) is to find critical points of the functional K on W_p.

In the sequel, let us introduce some concepts and results which ensure, under some different hypotheses, the existence of one or multiple critical points of I on W_p, see for instance [50, 73].

Definition 7.1.9. *Let X be a normed space, we define the dual space X^* as the set of all the continuous linear forms:*

$$X^* = \Big\{\varphi\colon X \to \mathbb{R} \;\Big|\; \varphi \textit{ is continuous and linear} \Big\}.$$

Now, we introduce the concept of weak convergence.

Definition 7.1.10. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements in a normed linear space X. We say that $\{u_n\}_{n=1}^{\infty}$ converges weakly to $x \in X$ ($x_n \rightharpoonup x$) if, and only if,*

$$\lim_{n\to+\infty} f(x_n) = f(x), \ \forall f \in X^*.$$

Definition 7.1.11. *Let $E : X \to \mathbb{R}$ be a functional and $\mathcal{M} \subset X$. Then, E is said to be weakly sequentially continuous at $u_0 \in \mathcal{M}$ relative to $\mathcal{M}$ if for any sequence $\{u_n\}_{n=1}^{\infty}$ such that $u_n \rightharpoonup u_0$ we have:*

$$E(u_0) = \lim_{n\to+\infty} E(u_n).$$

Analogously, E is said to be weakly sequentially lower semi-continuous at $u_0 \in \mathcal{M}$ relative to $\mathcal{M}$ if for any sequence $\{u_n\}_{n=1}^{\infty}$ such that $u_n \rightharpoonup u_0$ we have:

$$E(u_0) \leqslant \liminf_{n\to+\infty} E(u_n).$$

We say that E is weakly sequentially continuous (lower semi-continuous) in $\mathcal{M}$ if it is weakly sequentially continuous (lower semi-continuous) at every point $u \in \mathcal{M}$ relative to $\mathcal{M}$.

Definition 7.1.12. *Let $X^{**} = (X^*)^*$ be the dual space of X^*. We define the canonical embedding from X into X^{**} as follows:*

$$\begin{aligned} Q\colon X &\longrightarrow X^{**} \\ x &\longmapsto Q(x)\colon X^* \longrightarrow \mathbb{R} \\ &\qquad\qquad\qquad \varphi \longmapsto \varphi(x). \end{aligned}$$

Definition 7.1.13. *A Banach space X is said to be reflexive if Q is surjective.*

Definition 7.1.14. *A Banach space X is said to be uniformly convex if for every $\varepsilon > 0$, there is $\delta > 0$ such that:*

$$x,\, y \in X,\ \|x\| = \|y\| = 1,\ \|x - y\| \geqslant \varepsilon \Rightarrow 1 - \left\| \frac{x+y}{2} \right\| \geqslant \delta.$$

Now, we have a result which makes a connection between these last two concepts, see for instance [97, Chapter V, 2].

Theorem 7.1.15. *Every uniformly convex space is reflexive.*

Remark 7.1.16. *W_p is a uniformly convex space, thus it is reflexive.*

The following result gives a property which ensures that a functional is weakly sequentially lower semi-continuous.

Proposition 7.1.17 ([50, Proposition 6.2.10])**.** *Let E be a convex and continuous functional defined in a convex set, $\mathcal{M} \subset X$. Then E is weakly sequentially lower semi-continuous.*

Remark 7.1.18. *K_1 is a convex and continuous functional, since Φ_p also is. Hence, K_1 is weakly sequentially lower semi-continuous on W_p.*

Now, let us see that K_2 is also a weakly sequentially lower semi-continuous functional.

Definition 7.1.19. *An embedding of X into Y is compact if a ball in X is relatively compact in Y, we denote it by $X \hookrightarrow\hookrightarrow Y$.*

Remark 7.1.20. $W_p \hookrightarrow\hookrightarrow C\big([0,T]\big)$

This remark coupled with property (FH) implies that K_2 is a weakly sequentially continuous functional. Thus, in particular, K is a weakly sequentially lower semi-continuous functional.

Now, we introduce the classical minimisation argument, which is used to prove the existence of weak solution for (7.1.1)-(7.1.2).

Theorem 7.1.21 ([73, Theorem 1.1])**.** *Let $E\colon X \to \mathbb{R}$ be a weakly sequentially lower semi-continuous functional on a reflexive Banach space X and let E have a bounded minimizing sequence. Then E has a minimum on X, i.e., there exists $u_0 \in X$ such that:*

$$E(u_0) = \inf_{u \in X} E(u).$$

Moreover, if E is differentiable, then u_0 is a critical point of E.

Before introducing a result which will give the multiplicity result, let us introduce another property on the functional which will be useful.

Definition 7.1.22. *Let E be a C^1-functional on X. We say that E satisfies the Palais-Smale condition $\big((PS)$-condition$\big)$ if every sequence $\{u_m\}_{m=1}^{\infty} \subset X$ such that:*

- $\Big\{E(u_m)\Big\}_{m=1}^{\infty}$ *is bounded,*
- $\lim\limits_{m \to +\infty} E'(u_m) = 0$ *in X^*,*

has a convergent subsequence.

Clark's theorem, see [40], gives the existence of a sequence of critical points for an even functional. This result or some of its modifications have been used to obtain multiplicity results in many cases (see for instance [91] and [100]). We present a result, which is a generalisation of Clark's theorem given in [71].

Theorem 7.1.23. *Let X be a Banach space and $E \in C^1(X,\mathbb{R})$. Assume that E satisfies the (PS)-condition, it is even, bounded from below and $E(0) = 0$.*

If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} E < 0$, where $S_\rho = \{u \in X,\ \|u\|_X = \rho\}$, then at least one of the following conclusions holds:

1. *There exists a sequence of critical points $\{u_k\}$ satisfying $E(u_k) < 0$ for all k and:*

$$\lim_{k \to \infty} \|u_k\|_X = 0.$$

2. *There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u such that $\|u\|_X = \alpha$ and $E(u) = 0$.*

Remark 7.1.24. *Realise that Theorem 7.1.23 implies the existence of many pairs of critical points $(u_m, -u_m)$, $u_m \neq 0$ such that $E(u_m) \leqslant 0$, $\lim_{m\to+\infty} E(u_m) = 0$ and:*

$$\lim_{m\to+\infty} \|u_m\|_X = 0.$$

7.1.2 Main results

This section is devoted to obtain the existence results for problem (7.1.1)-(7.1.2).

First, we introduce some previous lemmas which are used in the proof of the main results.

Lemma 7.1.25 ([67, Lemma 4.2])**.** *If $p \geq 2$, then:*

$$|y|^p \geqslant |x|^p + p\,|x|^{p-2}\,x\,(y-x) + \frac{|y-x|^p}{2^{p-1}-1}, \quad \forall\, x,\, y \in \mathbb{R}.$$

If $1 < p < 2$, then:

$$|y|^p \geqslant |x|^p + p\,|x|^{p-2}\,x\,(y-x) + C(p)\,\frac{|y-x|^p}{(|x|+|y|)^{2-p}}, \quad \forall\, x,\, y \in \mathbb{R},$$

where $C(p) = \dfrac{3}{16}\,p\,(p-1) > 0$.

Remark 7.1.26. *Realise that, from Lemma 7.1.25, we obtain, directly, the following assertions:*

If $p \geqslant 2$, then:

$$|x|^{p-2}\,x\,(x-y) \geqslant \frac{1}{p}\left(|x|^p - |y|^p + \frac{|y-x|^p}{2^{p-1}-1}\right), \quad \forall x, y \in \mathbb{R},$$

and, in particular:

$$|x|^{p-2}\,x\,(x-y) + |y|^{p-2}\,y\,(y-x) \geqslant \frac{2}{p}\left(\frac{|y-x|^p}{2^{p-1}-1}\right), \quad \forall x, y \in \mathbb{R}.$$

If $1 < p < 2$, then:

$$|x|^{p-2}\,x\,(x-y) \geqslant \frac{1}{p}\left(|x|^p - |y|^p + C(p)\,\frac{|y-x|^p}{(|x|+|y|)^{2-p}}\right), \quad \forall x, y \in \mathbb{R},$$

and, in particular:

$$|x|^{p-2}\,x\,(x-y) + |y|^{p-2}\,y\,(y-x) \geqslant \frac{2}{p}\,C(p)\,\frac{|y-x|^p}{(|x|+|y|)^{2-p}}, \quad \forall x, y \in \mathbb{R}.$$

Using this result, we can prove the following one.

Lemma 7.1.27. *Let $p>1$, $a_i \geqslant 0$ for $i=1,\ldots,n-1$. If f and h satisfy condition (FH), then functional $K\colon W_p \to \mathbb{R}$, defined in* (7.1.8)*, satisfies the Palais-Smale condition in W_p.*

Proof. Let $\{u_m\}$ be a Palais-Smale sequence, i.e., $K(u_m)$ is bounded in $\mathbb{R}$ and $K'(u_m)\to 0$ in W_p^*, where W_p^* is the dual space of W_p.

First, let us see that $\{u_m\}$ is a bounded sequence in W_p.

Consider the function $g(t)=\dfrac{1}{q}\,\tilde b_1 t^q - \dfrac{1}{r}\,\tilde c_2\, t^r$ for $t>0$.

We can prove that g achieves its minimum on $(0,+\infty)$ at $d_1 := \left(\dfrac{\tilde c_2}{\tilde b_1}\right)^{\frac{1}{q-r}}$. Then:

$$g(t) \geqslant g(d_1) = \frac{r-q}{r\,q}\left(\frac{\tilde c_2^{\,q}}{\tilde b_1^{\,r}}\right)^{\frac{1}{q-r}} =: d_2, \quad t>0. \tag{7.1.11}$$

Let us write $u_m=\tilde u_m+\bar u_m$, where $\tilde u_m\in W$ and $\bar u_m\in\mathbb{R}$. Using condition (FH), we can bound K from below on W_p as follows:

$$\begin{aligned} K(u_m) &\geqslant \int_0^T \Phi_p\left(u_m^{(n)}(t)\right)\mathrm{d}\,t + \int_0^T \left(\frac{1}{q}\,\tilde b_1\,|u_m(t)|^q - \frac{1}{r}\,\tilde c_2\,|u_m(t)|^r\right)\mathrm{d}\,t \\ &\geqslant \frac{1}{p}\,\|u_m\|_W^p + T\,d_2 = \frac{1}{p}\,\|\tilde u_m\|_W^p + T\,d_2. \end{aligned}$$

Then, since $K(u_m)$ is bounded, $\{\tilde u_m\}$ is a bounded sequence in W. Using the fact that $\|\cdot\|_W$ and $\|\cdot\|$ are equivalent norms in W, we conclude that $\{\tilde u_m\}$ is also a bounded sequence in W_p.

Now, analogously to [49], it can be proved that $\{\bar u_m\}$ is bounded in $\mathbb{R}$ using Lemma 7.1.7. For convenience of the reader, we describe this proof.

Suppose, to arrive a contradiction, that:

$$\lim_{m\to+\infty} |\bar u_m| = +\infty.$$

Since $\{\tilde u_m\}$ is a bounded sequence in W, by Lemma 7.1.7, there is $N>0$ such that $\|\tilde u_m\|_{C([0,T])}\leqslant N$. So, for $t\in[0,T]$, we have:

$$|u_m(t)| \geqslant |\bar u_m| - \left|\tilde u_m(t)\right| \geqslant |\bar u_m| - N.$$

Thus, $\displaystyle\lim_{m\to+\infty}\|u_m(t)\| = +\infty$ uniformly in $[0,T]$. That is, for any $R>0$, there exists $M=M(R)$ such that for any $m>M$, we have that $|u_m(t)|\geqslant R$, for every $t\in[0,T]$.

It can be proved that g is an increasing function in $(d_1,+\infty)$. Then, taking $R\geqslant d$ and $m\geqslant M(R)$, using property (FH), we have:

$$\begin{aligned} K(u_m) &\geqslant \int_0^T \left(\frac{1}{q}\,\tilde b_1\,|u_m(t)|^q - \frac{1}{r}\,\tilde c_2\,|u_m(t)|^r\right)\mathrm{d}\,t \geqslant \int_0^T \left(\frac{1}{q}\,\tilde b_1\,R^q - \frac{1}{r}\,\tilde c_2\,R^r\right)\mathrm{d}\,t \\ &= T\left(\frac{1}{q}\,\tilde b_1\,R^q - \frac{1}{r}\,\tilde c_2\,R^r\right). \end{aligned}$$

However,

$$\lim_{R\to+\infty}\left(\frac{1}{q}\,\tilde{b}_1\,R^q-\frac{1}{r}\,\tilde{c}_2\,R^r\right)=+\infty,$$

which is a contradiction with the fact that $I(u_m)$ is bounded. Hence, we can affirm that $\{u_m\}$ is a bounded sequence in W_p.

Passing to a subsequence, if it is necessary, we may assume the existence of $u\in W_p$ such that $u_m\rightharpoonup u$ weakly in W_p and $u_m\to u$ in $C\big([0,T]\big)$.

Since $K'(u_m)\to 0$ in W_p^*, taking into account the weak convergence in W_p, we have:

$$\begin{aligned}
0 &= \lim_{m\to+\infty}\langle K'(u_m)-K'(u),u_m-u\rangle\\
&= \lim_{m\to+\infty}\int_0^T\left[\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right]\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\mathrm{d}\,t\\
&\quad+\lim_{m\to+\infty}\sum_{i=1}^{n-1}a_i\int_0^T\left[\varphi_p\left(u_m^{(n-i)}(t)\right)-\varphi_p\left(u^{(n-i)}(t)\right)\right]\left(u_m^{(n-i)}(t)-u^{(n-i)}(t)\right)\mathrm{d}\,t\\
&\quad+\lim_{m\to+\infty}\int_0^T\Big(f\big(t,u_m(t)\big)-f\big(t,u(t)\big)\Big)\Big(u_m(t)-u(t)\Big)\mathrm{d}\,t\\
&\quad-\lim_{m\to+\infty}\int_0^T\Big(h\big(t,u_m(t)\big)-h\big(t,u(t)\big)\Big)\Big(u_m(t)-u(t)\Big)\mathrm{d}\,t.
\end{aligned}\tag{7.1.12}$$

Since $u_m\to u$ in $C\big([0,T]\big)$, we have that:

$$\begin{aligned}
&\lim_{m\to+\infty}\int_0^T\Big(f\big(t,u_m(t)\big)-f\big(t,u(t)\big)\Big)\Big(u_m(t)-u(t)\Big)\mathrm{d}\,t=0,\\
&\lim_{m\to+\infty}\int_0^T\Big(h\big(t,u_m(t)\big)-h\big(t,u(t)\big)\Big)\Big(u_m(t)-u(t)\Big)\mathrm{d}\,t=0.
\end{aligned}\tag{7.1.13}$$

Now, using Lemma 7.1.25 and Remark 7.1.26, we have for each $i=1,\dots,n-1$:

1. for $p\geqslant 2$,

$$\begin{aligned}
&\left[\varphi_p\left(u_m^{(n-i)}(t)\right)-\varphi_p\left(u^{(n-i)}(t)\right)\right]\left(u_m^{(n-i)}(t)-u^{(n-i)}(t)\right)\\
&=\varphi_p\left(u_m^{(n-i)}(t)\right)\left(u_m^{(n-i)}(t)-u^{(n-i)}(t)\right)+\varphi_p\left(u^{(n-i)}(t)\right)\left(u^{(n-i)}(t)-u_m^{(n-i)}(t)\right)\\
&\geqslant\frac{2\left|u_m^{(n-i)}(t)-u^{(n-i)}(t)\right|^p}{p\;(2^{p-1}-1)}\geqslant 0,
\end{aligned}$$

2. $1<p<2$,

$$\begin{aligned}
&\left[\varphi_p\left(u_m^{(n-i)}(t)\right)-\varphi_p\left(u^{(n-i)}(t)\right)\right]\left(u_m^{(n-i)}(t)-u^{(n-i)}(t)\right)\\
&=\varphi_p\left(u_m^{(n-i)}(t)\right)\left(u_m^{(n-i)}(t)-u^{(n-i)}(t)\right)+\varphi_p\left(u^{(n-i)}(t)\right)\left(u^{(n-i)}(t)-u_m^{(n-i)}(t)\right)\\
&\geqslant\frac{2\,C(p)\;\left|u_m^{(n-i)}(t)-u^{(n-i)}(t)\right|^p}{p\;\left(\left|u_m^{(n)}(t)\right|+|u^{(n)}(t)|\right)^{2-p}}\geqslant 0.
\end{aligned}$$

This coupled with (7.1.12) and (7.1.13), implies that:

$$\begin{aligned}
0 = & \lim_{m\to+\infty}\int_0^T \left[\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right]\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\mathrm{d}\,t\\
& +\lim_{m\to+\infty}\sum_{i=1}^{n-1}a_i\int_0^T\left[\varphi_p\left(u_m^{(n-i)}(t)\right)-\varphi_p\left(u^{(n-i)}(t)\right)\right]\left(u_m^{(n-i)}(t)-u^{(n-i)}(t)\right)\mathrm{d}\,t\\
\geqslant & \lim_{m\to+\infty}\int_0^T \left[\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right]\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\mathrm{d}\,t.
\end{aligned}$$

If $p\geqslant 2$, using again Lemma 7.1.25, we conclude that:

$$\begin{aligned}
0 \geqslant & \lim_{m\to+\infty}\int_0^T \left[\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right]\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\mathrm{d}\,t,\\
\geqslant & \frac{2}{p\,(2^{p-1}-1)}\lim_{m\to+\infty}\int_0^T\left|u_m^{(n)}(t)-u^{(n)}(t)\right|^p\mathrm{d}\,t\geqslant 0.
\end{aligned}$$

Thus, taking into account that $u_m\to u$ in $C\big([0,T]\big)$, we have:

$$0=\lim_{m\to+\infty}\int_0^T\left(\left|u_m^{(n)}(t)-u^{(n)}(t)\right|^p+|u_m(t)-u(t)|^p\right)\mathrm{d}\,t=\lim_{m\to+\infty}\|u_m-u\|^p.$$

So, $u_m\to u$ in W_p.
If $1<p<2$, we have, using Hölder inequality:

$$\begin{aligned}
0 \geqslant & \lim_{m\to+\infty}\int_0^T \left[\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right]\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\mathrm{d}\,t,\\
\geqslant & \lim_{m\to+\infty}\left\{\int_0^T\left|u_m^{(n)}(t)\right|^p\mathrm{d}\,t+\int_0^T\left|u^{(n)}(t)\right|^p\mathrm{d}\,t\right.\\
& -\left(\int_0^T\left|u_m^{(n)}(t)\right|^p\mathrm{d}\,t\right)^{\frac{p-1}{p}}\left(\int_0^T\left|u^{(n)}(t)\right|^p\mathrm{d}\,t\right)^{\frac{1}{p}}\\
& \left.-\left(\int_0^T\left|u_m^{(n)}(t)\right|^p\mathrm{d}\,t\right)^{\frac{1}{p}}\left(\int_0^T\left|u^{(n)}(t)\right|^p\mathrm{d}\,t\right)^{\frac{p-1}{p}}\right\}\\
= & \lim_{m\to+\infty}\Big(\|u_m\|_W-\|u\|_W\Big)\left(\|u_m\|_W^{p-1}-\|u\|_W^{p-1}\right)\geqslant 0.
\end{aligned}$$

So, considering that $u_m\to u$ in $C\big([0,T]\big)$, we conclude:

$$\lim_{m\to+\infty}\|u_m\|=\|u\|.$$

Finally, from the weak convergence in W_p, since W_p is uniformly convex, we have that $u_m\to u$ in W_p and the result is proved. □

In the sequel, we show a result which ensures the existence of at least a weak solution of (7.1.1)-(7.1.2).

Theorem 7.1.28. *Let $p > 1$, f and h satisfy (FH), $a_i \geqslant 0$ for $i = 1, \dots, n-1$. Then* (7.1.1)-(7.1.2) *has at least a weak solution.*

If, in addition, we have that $p > r$ and f and h are odd functions, then (7.1.1)-(7.1.2) *has infinitely many pairs of weak solutions $(u_m, -u_m)$, such that $u_m \neq 0$ and:*

$$\lim_{m\to+\infty} \max_{t\in[0,T]} |u_m(t)| = \lim_{m\to+\infty} \|u_m\|_{C([0,T])} = 0.$$

Proof. As we have said, K, defined in (7.1.8), is a weakly sequentially lower semi-continuous functional on the reflexive Banach space W_p.

Using the arguments of Lemma 7.1.27, we can prove that

$$K(u) \geqslant \frac{1}{p}\, \|u\|_W^p + T\, d_2,$$

so, $\inf_{u\in W_p} K(u) > -\infty$.

Moreover, if we choose a minimizing sequence, i.e. $\{u_m\} \in W_p$, such that

$$\lim_{m\to+\infty} K(u_m) = \inf_{u\in W_p} K(u),$$

we can prove that it is bounded in W_p following the same steps as in Lemma 7.1.27 again.

Then, from Theorem 7.1.21, the existence of a critical point of K is proved. This critical point corresponds to a weak solution of problem (7.1.1)-(7.1.2).

The multiplicity result follows from Theorem 7.1.23, using similar arguments as in [49]. For the convenience of the reader, we recall some of these steps.

From Lemma 7.1.27, K satisfies the (PS)-condition.

K is bounded from below in W_p and $K(0) = 0$. Since f and h are odd functions then K is even.

Let $k \in \mathbb{N}$ be arbitrary and X^k be a k-dimensional subspace of W_p, with basis elements:

$$\{\Theta_1, \dots, \Theta_k\} \subset W \subset W_p.$$

Realise that the separability of W_p allows this construction.

The set $S_\rho^k := \left\{ u = \alpha_1\, \Theta_1 + \cdots + \alpha_k\, \Theta_k \;\middle|\; \sum_{j=1}^k |\alpha_j|^p = \rho^k \right\} \subset X^k$ is homeomorphic to the unit sphere $S^{k-1} \subset \mathbb{R}^k$.

For $u = \sum_{j=1}^k \alpha_j\, \Theta_j$, the expression:

$$\|u\|_{X^k} = \left(\sum_{j=1}^k |\alpha_j|^p \right)^{1/p},$$

defines a norm in X^k.

From Lemma 7.1.8, we can use the fact that norms $\|\cdot\|_{X^k}$, $\|\cdot\|_W$, $\|\cdot\|_{L^q}$ and $\|\cdot\|_{L^r}$ are equivalent.

As in [49], using condition (FH) and Lemma 7.1.6, we obtain that for any $u \in S_\rho^k$:

$$\begin{aligned} K(u) &= \int_0^T \left[\Phi_p\left(u^{(n)}(t)\right) + \sum_{i=1}^{n-1} a_i \Phi_p\left(u^{(n-i)}(t)\right)\right] \mathrm{d}\,t + \int_0^T \Big(F\big(t,u(t)\big) - H\big(t,u(t)\big)\Big)\,\mathrm{d}\,t \\ &\leqslant \left(1 + \sum_{i=1}^{n-1} T^{p\,i} a_i\right) \frac{1}{p}\|u\|_W^p + b_2 \|u\|_{L^q}^q - c_1\,\|u\|_{L^r}^r. \end{aligned}$$

From the equivalence of the norms, there exists three positive constants, e_1, e_2 and e_3, such that:

$$K(u) \leqslant \|u\|_W^r \left(e_1\|u\|_W^{p-r} + e_2\|u\|_W^{q-r} - e_3\right).$$

Since $1 < r < p$ and $r < q$ and taking into account the equivalence of the norms $\|\cdot\|_W$ and $\|\cdot\|_{X^k}$, there exists $\rho > 0$ such that:

$$\sup_{u \in S_\rho^k} K(u) < 0.$$

Hence, by applying Theorem 7.1.23, the result is proved. □

Now, we are interested into see when this weak solutions are also classical ones. In order to do that, let us denote:

$$i_0 = \min\Big\{ i \in \{1, \dots, n-1\} \;\mid\; a_i \neq 0 \Big\}. \tag{7.1.14}$$

So, we rewrite (7.1.1) as follows:

$$\begin{aligned} \left[\varphi_p\left(u^{(n)}(t)\right)\right]^{(n)} + \sum_{i=i_0}^{n-1} (-1)^i a_i \left[\varphi_p\left(u^{(n-i)}(t)\right)\right]^{(n-i)} & \\ + (-1)^n \Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big) = 0,\; t \in [0,T]. & \end{aligned} \tag{7.1.15}$$

We have a preliminary result to deduce the regularity of φ_p.

Lemma 7.1.29. *Let us consider $\alpha = \lceil p \rceil - 2$, where $\lceil\cdot\rceil$ means the ceiling function, then $\varphi_p \in C^\alpha(\mathbb{R})$. Moreover, if $p \in \mathbb{N}$ is even, then $\varphi_p \in C^\infty(\mathbb{R})$.*

Proof. The proof follows from the derivatives of φ_p, which are given by:

$$\begin{cases} \varphi_p^{(2k-1)}(t) = (p-1)\,\dots\,(p-2\,k)\,\Phi_{p-2\,k}(t) & t \neq 0,\; k \in \mathbb{Z}, \\ \varphi_p^{(2k)}(t) = (p-1)\,\dots\,(p-2\,k)\,\varphi_{p-2\,k}(t) & t \neq 0,\; k \in \mathbb{Z}. \end{cases}$$

So, if $p = 2\,k$, with $k \in \mathbb{N}$, then $\varphi_p \in C^\infty(\mathbb{R})$.

If $0 < p - 2\,k \leqslant 1$ or, equivalently, $\lceil p \rceil = 2\,k+1$, then $2\,k - 1 = \alpha$ and $\varphi_p \in C^\alpha(\mathbb{R})$.

Moreover, if $1 < p - 2\,k < 2$ or, which is the same, $\lceil p \rceil = 2\,k + 2$, then $2\,k = \alpha$ and $\varphi_p \in C^\alpha(\mathbb{R})$. □

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Remark 7.1.30. *From Lemma 7.1.29, we obtain, in particular, the following assertions.*

- *If* $p \in (1,2)$*, then* $\varphi_p \in C(\mathbb{R})$.
- *If* $p = 2$*, then* $\varphi_2 \in C^\infty(\mathbb{R})$.
- *If* $p \in (2,3]$*, then* $\varphi_p \in C^1(\mathbb{R})$.

As a general conclusion, from previous Remark, we have that if $p \in (n-1, n]$, then $\varphi_p \in C^{n-2}(\mathbb{R})$. Moreover, if $p = n = 2\,k$, where $k \in \mathbb{N}$, then $\varphi_{2\,k} \in C^\infty(\mathbb{R})$.

Lemma 7.1.31. *If* $i \leqslant \min\left\{\alpha, \dfrac{n}{2}\right\}$*, with* α *given in Lemma 7.1.29, then the following equality is fulfilled for all* u*,* $v \in W_p$*:*

$$\int_0^T \varphi_p\left(u^{(i)}(t)\right) v^{(i)}(t)\,\mathrm{d}\,t = (-1)^i \int_0^T \left[\varphi_p\left(u^{(i)}(t)\right)\right]^{(i)} v(t)\,\mathrm{d}\,t.$$

Proof. Since $u \in W_p$, using Lemma 7.1.29, integrating by parts we have:

$$\begin{aligned}\int_0^T \varphi_p\left(u^{(i)}(t)\right) v^{(i)}(t)\,\mathrm{d}\,t = &\sum_{j=1}^{i} (-1)^{j-1}\left[\varphi_p\left(u^{(i)}(t)\right)\right]^{(j-1)} v^{(i-j)}(t)\Bigg]_0^T \\ &+ (-1)^i \int_0^T \left[\varphi_p\left(u^{(i)}(t)\right)\right]^{(i)} v(t)\,\mathrm{d}\,t.\end{aligned}$$

The result follows from the boundary conditions of u and v, taking into account that $i \leqslant \dfrac{n}{2}$. □

Lemma 7.1.32. *If* $u \in C^\beta\big([0,T]\big)$ *and* $\varphi_p\left(u^{(\beta)}(\cdot)\right) \in C^\beta\big([0,T]\big)$*, then for* $k = 1, \dots, \beta-1$*:*

$$\begin{aligned}\left[\varphi_p\left(u^{(\beta)}(t)\right)\right]^{(k)} = &\, g_k\left(u^{(\beta)}(t), \dots, u^{(\beta+k-1)}(t)\right) \\ &+ (p-1)(p-2)\,\Phi_{p-2}\left(u^{(\beta)}(t)\right) u^{(\beta+k)}(t),\end{aligned} \tag{7.1.16}$$

where $g_k\left(u^{(\beta)}(\cdot), \dots, u^{(\beta+k-1)}(\cdot)\right) \in C^{\beta-k}\big([0,T]\big)$.

Proof. First, realise that from Remark 7.1.3 we know that $u \in C^{2\,\beta}(I)$.

Let us prove the result by induction. For $k = 1$:

$$\left[\varphi_p\left(u^{(\beta)}(t)\right)\right]' = (p-1)(p-2)\,\Phi_{p-2}\left(u^{(\beta)}(t)\right) u^{(\beta+1)}(t),$$

so, $g_1 \equiv 0$.

Now, assume that (7.1.16) is true for some $k \in \{1, \dots, n-2\}$, for $k+1$:

$$\begin{aligned}\left[\varphi_p\left(u^{(\beta)}(t)\right)\right]^{(k+1)} =& \frac{\mathrm{d}}{\mathrm{d}\,t} g_k\left(u^{(\beta)}(t), \dots, u^{(\beta+k-1)}(t)\right)\\ &+ (p-1)(p-2)\,\varphi_{p-2}\left(u^{(\beta)}(t)\right) u^{(\beta+1)}(t)\, u^{(\beta+k)}(t)\\ &+ (p-1)(p-2)\,\Phi_{p-2}\left(u^{(\beta)}(t)\right) u^{(\beta+k+1)}(t).\end{aligned}$$

Hence, the result is true considering:

$$\begin{aligned}g_{k+1}(u^{(\beta)}(t), \cdots, u^{(\beta+k)}(t)) =& \frac{\mathrm{d}}{\mathrm{d}\,t} g_k\left(u^{(\beta)}(t), \dots, u^{(\beta+k-1)}(t)\right)\\ &+ (p-1)(p-2)\varphi_{p-2}\left(u^{(\beta)}(t)\right) u^{(\beta+1)}(t)\, u^{(\beta+k)}(t).\end{aligned}$$

□

Now, we can show a result which ensures that a weak solution is, indeed, a classical solution.

Theorem 7.1.33. *Let i_0, defined in* (7.1.14)*, be such that $i_0 \geqslant \left\lfloor \frac{n}{2} \right\rfloor$ and $\lceil p \rceil \geqslant n - i_0 + 2$ or $p = 2\,k$, where $k \in \mathbb{N}$, then every weak solution of* (7.1.15) *is also a classical solution.*

Proof. Taking into account that $u \in W_p$ is a weak solution, from Lemma 7.1.31, we have for all $v \in W_p$:

$$\begin{aligned}\int_0^T \varphi_p\left(u^{(n)}(t)\right) v^{(n)}(t)\,\mathrm{d}\,t =& \sum_{i=i_0}^{n-1} (-1)^{n-i+1} a_i \int_0^T \left[\varphi_p\left(u^{(n-i)}(t)\right)\right]^{(n-i)} v(t)\,\mathrm{d}\,t\\ &- \int_0^T \Big(f\big(t, u(t)\big) - h\big(t, u(t)\big)\Big) v(t)\,\mathrm{d}\,t.\end{aligned} \tag{7.1.17}$$

So, in distributional sense:

$$\left[\varphi_p\left(u^{(n)}(t)\right)\right]^{(n)} = \sum_{i=i_0}^{n-1} (-1)^{i+1} a_i \left[\varphi_p\left(u^{(n-i)}(t)\right)\right]^{(n-i)} + (-1)^{n+1}\Big(f\big(t, u(t)\big) - h\big(t, u(t)\big)\Big).$$

Hence, following a basic theorem of calculus of variations [73, page 6: Fundammental Lemma], taking into account that f and h are continuous functions, we conclude that equation (7.1.15) is satisfied for all $t \in (0, T)$.

We only have to check the boundary conditions, since $u \in W_p$, then:

$$u(T) - u(0) = \cdots = u^{(n-1)}(T) - u^{(n-1)}(0) = 0.$$

Now, integrating in (7.1.17) by parts, taking into account that (7.1.15) is fulfilled, we obtain that for all $v \in W_p$:

$$\left[\sum_{j=1}^{n} (-1)^{j-1} \left[\varphi_p\left(u^{(n)}(t)\right)\right]^{(j-1)} v^{(n-j)}(t)\right]_0^T = 0.$$

From the arbitrariness of $v \in W_p$, we obtain:

$$\varphi_p\left(u^{(n)}(T)\right)-\varphi_p\left(u^{(n)}(0)\right)=\cdots=\left[\varphi_p\left(u^{(n)}(T)\right)\right]^{(n-1)}-\left[\varphi_p\left(u^{(n)}(0)\right)\right]^{(n-1)}=0,$$

and, taking into account that φ_p is an injective function, $u^{(n)}(T)-u^{(n)}(0)=0$.

Now, using Lemma 7.1.32, we obtain the rest of boundary conditions. For $k=1$, we have:

$$\begin{aligned}0=&\left[\varphi_p\left(u^{(n)}(T)\right)\right]'-\left[\varphi_p\left(u^{(n)}(0)\right)\right]'\\=&(p-1)\,(p-2)\Big(\Phi_{p-2}\left(u^{(n)}(T)\right)u^{(n+1)}(T)-\Phi_{p-2}\left(u^{(n)}(0)\right)u^{(n+1)}(0)\Big).\end{aligned}$$

Since $u^{(n)}(T)-u^{(n)}(0)=0$, we conclude that $u^{(n+1)}(T)-u^{(n+1)}(0)=0$.

Now, assume, as induction hypothesis, that for $k=1,\dots,n-2$:

$$u^{(n)}(T)-u^{(n)}(0)=\cdots=u^{(n+k)}(T)-u^{(n+k)}(0)=0.$$

From Lemma 7.1.32, for $k+1$:

$$\begin{aligned}0=&\left[\varphi_p\left(u^{(n)}(T)\right)\right]^{(k+1)}-\left[\varphi_p\left(u^{(n)}(0)\right)\right]^{(k+1)}\\=&(p-1)(p-2)\Big(\Phi_{p-2}\left(u^{(n)}(T)\right)u^{(n+k+1)}(T)-\Phi_{p-2}\left(u^{(n)}(0)\right)u^{(n+k+1)}(0)\Big),\end{aligned}$$

which implies that $u^{(n+k+1)}(T)-u^{(n+k+1)}(0)=0$.

Thus, the result is proved. □

As a direct consequence of Theorems 7.1.28 and 7.1.33, the following result is obtained.

Theorem 7.1.34. *Let $p>1$, f and h satisfying (FH), $a_i=0$ for $i=1,\dots,i_0-1$, $a_i\geqslant 0$ for $i=i_0,\dots,n-1$, $i_0\geqslant\left\lfloor\frac{n}{2}\right\rfloor$ and $\lceil p\rceil\geqslant n-i_0+2$ or $p=2\,k$, where $k\in\mathbb{Z}$. Then problem* (7.1.1)-(7.1.2) *has at least a classical solution.*

If, in addition, we have that $p>r$ and f and h are odd functions, then (7.1.1)-(7.1.2) *has infinitely many pairs of classical solutions $(u_m,-u_m)$, such that $u_m\neq 0$ and*

$$\lim_{m\to+\infty}\max_{t\in[0,T]}|u_m(t)|=\lim_{m\to+\infty}\|u_m\|_{C([0,T])}=0.$$

7.1.3 Impulsive problem

Denote $0 = t_0 < t_1 < \cdots < t_\ell < t_{\ell+1} = T$ and set $J_j = (t_j, t_{j+1})$ for $j = 0, \ldots, \ell$. We can study the following impulsive problem with analogous arguments:

$$\begin{aligned}
&\left[\varphi_p\left(u^{(n)}(t)\right)\right]^{(n)} + \sum_{i=1}^{n-1}(-1)^i a_i\left[\varphi_p\left(u^{(n-i)}(t)\right)\right]^{(n-i)} \\
&\qquad\qquad +(-1)^n\Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big) = 0, \quad t \in \bigcup_{j=0}^{\ell} J_j, \\
&\qquad u(T) - u(0) = \cdots = u^{(2\,n-1)}(T) - u^{(2\,n-1)}(0) = 0, \\
&\qquad\qquad\qquad \Delta\left[\varphi_p\left(u^{(n)}(t_j)\right)\right] = (-1)^{n+1} g_j\big(u(t_j)\big),
\end{aligned} \tag{7.1.18}$$

where $T \geqslant 0$, $a_i \geqslant 0$ for $i = 1, \ldots, n-1$:

$$\Delta\left[\varphi_p\left(u^{(n)}(t_j)\right)\right] := \varphi_p\left(u^{(n)}\left(t_j^+\right)\right) - \varphi_p\left(u^{(n)}\left(t_j^-\right)\right),$$

$u^{(n)}\left(t_j^\pm\right) = \lim_{t\to t_j^\pm} u^{(n)}(t)$ and $g_j : \mathbb{R} \to \mathbb{R}$ are continuous functions.

As in the non-impulsive case, we say that $u \in C^{n-1}\big([0,T]\big)$is a classical solution of problem (7.1.18) coupled with (7.1.2) if $u \in C^n\big(J_j\big)$ and $\varphi_p\left(u^{(n)}(\cdot)\right) \in C^n\big(J_j\big)$ for every $j = 0, \ldots, l$ and it satisfies (7.1.18) and the periodic boundary conditions (7.1.2).

Analogously, we say that $u \in W_p$ is a weak solution of (7.1.18) coupled with the boundary conditions (7.1.2) if for every $v \in W_p$, it is satisfied the following equation:

$$\begin{aligned}
&\int_0^T \varphi_p\left(u^{(n)}(t)\right) v^{(n)}(t)\,\mathrm{d}\,t + \sum_{i=1}^{n-1} a_i \int_0^T \varphi\left(u^{(n-i)}(t)\right) v^{(n-i)}(t)\,\mathrm{d}\,t \\
&\qquad + \int_0^T \Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big) v(t)\,\mathrm{d}\,t + \sum_{j=1}^{\ell} g_j\big(u(t_j)\big)\, v(t_j) = 0.
\end{aligned}$$

Thus, by analysing the critical points of the associated functional:

$$\tilde{K}(u) = K(u) + \sum_{j=1}^{\ell} G_j\big(u(t_j)\big),$$

where I has been defined in (7.1.8) and $G_j(t) = \displaystyle\int_0^t g_j(s)\,\mathrm{d}\,s$ for $j = 1, \ldots, \ell$, we obtain the analogous results to Theorem 7.1.28 and 7.1.34.

Theorem 7.1.35. *Let $p > 1$, f and h satisfying (FH), $a_i \geqslant 0$ for $i = 1, \ldots, n-1$, $g_j : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying for all $t \in \mathbb{R}$ and $j = 1, \ldots, \ell$:*

$$\int_0^t g_j(s)\,\mathrm{d}\,s \geqslant c,$$

for a given constant $c \in \mathbb{R}$. Then (7.1.18) *has at least a weak solution.*

If, in addition, we have that $p > r$ and f, h and g_j are odd functions and:

$$\int_0^t g_j(s)\,\mathrm{d}\,s \leqslant 0, \forall j = 1, \ldots, \ell.$$

Then (7.1.18) *has infinitely many pairs of weak solutions $(u_m, -u_m)$, $u_m \neq 0$, with:*

$$\lim_{m\to+\infty} \max_{t\in[0,T]} |u_m(t)| = \lim_{m\to+\infty} \|u_m\|_{C([0,T])} = 0.$$

Theorem 7.1.36. *Let $p > 1$, f and h satisfying (FH), $a_i \geqslant 0$ for $i = 1, \ldots, n-1$, $i_0 \geqslant \left\lfloor \dfrac{n}{2} \right\rfloor$ and $\lceil p \rceil \geqslant n - i_0 + 2$ or $p = 2\,k$, where $k \in \mathbb{Z}$, $g_j : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying for all $t \in \mathbb{R}$ and $j = 1, \ldots, \ell$:*

$$\int_0^t g_j(s)\,\mathrm{d}\,s \geqslant c,$$

for a given constant $c \in \mathbb{R}$. Then (7.1.18) *has at least a solution.*

If, in addition, we have that $p > r$ and f, h and g_j are odd functions and:

$$\int_0^t g_j(s)\,\mathrm{d}\,s \leqslant 0, \forall j = 1, \ldots, \ell.$$

Then (7.1.18) *has infinitely many pairs of solutions $(u_m, -u_m)$, $u_m \neq 0$, with:*

$$\lim_{m\to+\infty} \max_{t\in[0,T]} |u_m(t)| = \lim_{m\to+\infty} \|u_m\|_{C([0,T])} = 0.$$

Remark 7.1.37. *Proving these results for the impulsive problem is analogous to the ones proved before. The inferior bound in the integrals of g_j are imposed to warrant that the functional $\tilde{K}$ is bounded from below with a minimizing sequence.*

The odd character of g is required to let the functional $\tilde{K}$ be even.

Finally, the negative character of the integrals of g_j is used to prove the geometric hypothesis in the second part of the result, which involves Theorem 7.1.23.

7.1.4 Particular cases

This section is devoted to show the result for different examples.

First, realise that, as a particular case of functions satisfying condition (FH), we can consider $f\big(t, u(t)\big) = b(t)\,\varphi_q(t)$ and $h\big(t, u(t)\big) = c(t)\,\varphi_r(t)$, where b and c are positive continuous T-periodic functions.

As we have already said, for $n = 1$ the result has been obtained in [49].

Now, let us consider $n = 2$, we have:

$$\Big[\varphi_p\big(u''(t)\big)\Big]'' - a_1\Big[\varphi_p\big(u'(t)\big)\Big]' + \Big(f\big(t, u(t)\big) - h\big(t, u(t)\big)\Big) = 0, \quad t \in [0, T] \tag{7.1.19}$$

coupled with the boundary conditions:

$$u(T) - u(0) = u'(T) - u'(0) = u''(T) - u''(0) = u^{(3)}(T) - u^{(3)}(0) = 0. \tag{7.1.20}$$

If f and h satisfy condition (FH), then we can apply Theorem 7.1.28 to ensure the existence of at least a weak solution. If in addition, $p > r$, we have that there exist many pairs of weak solutions $(u_m, -u_m)$, $u_m \neq 0$, with:

$$\lim_{m\to+\infty} \max_{t\in[0,T]} |u_m(t)| = \lim_{m\to+\infty} \|u_m\|_{C([0,T])} = 0.$$

Moreover, if $a_1 = 0$ or $p \geqslant 2$, from Theorem 7.1.34, we can affirm that the obtained weak solutions are also classical solutions.

Finally, let us consider $n = 3$:

$$\begin{aligned}\left[\varphi_p\left(u^{(3)}(t)\right)\right]^{(3)} - a_1\Big(\varphi_p\big(u''(t)\big)\Big)'' + a_2\Big[\varphi_p\big(u'(t)\big)\Big]'\\ -\Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big) = 0, \quad t\in[0,T]\end{aligned} \tag{7.1.21}$$

coupled with the boundary conditions:

$$\begin{aligned}u(T)-u(0) = u'(T)-u'(0) = u''(T)-u''(0) = 0,\\ u^{(3)}(T)-u^{(3)}(0) = u^{(4)}(T)-u^{(4)}(0) = u^{(5)}(T)-u^{(5)}(0) = 0.\end{aligned} \tag{7.1.22}$$

If f and h satisfy condition (FH), then we can apply Theorem 7.1.28 to ensure the existence of at least a weak solution. If in addition, $p > r$, we have that there exist many pairs of weak solutions $(u_m, -u_m)$, $u_m \neq 0$, with:

$$\lim_{m\to+\infty} \max_{t\in[0,T]} |u_m(t)| = \lim_{m\to+\infty} \|u_m\|_{C([0,T])} = 0.$$

Moreover, if $a_1 = a_2 = 0$ or $a_1 = 0$ and $p \geqslant 2$, from Theorem 7.1.34, we can affirm that the obtained weak solutions are also classical solutions.

7.2 Non-linear $2\,n^{\text{th}}$-order p-Laplacian difference equations

All the previously studied problems, the linear problems, described in Chapters 1 to 5, and the non-linear ones, considered in Chapter 6 and in the first part of this chapter, have in common that they involve differential equations defined in a real interval. Here, we study a discrete problem, that is, we consider a difference equation. Indeed, we study the discretisation of a similar problem to the one studied in Section 7.1.

To be concise, consider the $2\,n^{\text{th}}$-order difference equation:

$$\begin{aligned}\Delta^n\Big[\varphi_{p_n}\Big(\Delta^n u(k-n)\Big)\Big] + \sum_{i=1}^{n-1}(-1)^i a_i\,\Delta^{n-i}\Big[\varphi_{p_{n-i}}\Big(\Delta^{n-i}u(k-i)\Big)\Big]\\ +(-1)^n\Big(V(k)\,\varphi_q\big(u(k)\big) - \lambda f\big(k,u(k)\big)\Big) = 0,\end{aligned} \tag{7.2.1}$$

where:

- φ_p has been introduced in Definition 7.0.1 for every $p > 1$,

- $V\colon \mathbb{Z} \to \mathbb{R}$ is a T-periodic positive function for T a fixed integer,
- $f\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a given function with growth conditions which will be specified below,
- $a_i \geqslant 0$ are fixed real numbers for all $i = 1, \dots, n-1$,
- $p_i \geqslant q \geqslant 1$ for $i = 1, \dots, n$,
- and the forward difference operators are given by:

$$\Delta u(k) = u(k+1) - u(k), \tag{7.2.2}$$

$$\Delta^i u(k) = \Delta^{i-1} u(k+1) - \Delta^{i-1} u(k), \text{ if } i \geqslant 2. \tag{7.2.3}$$

This section is devoted to prove the existence of one or multiple solutions for equation (7.2.1) by imposing different conditions on them. The results here presented can be seen in [84].

The partial cases where $p_1 = p_2 = 2$ are known as stationary extended Fisher-Kolmogorov equation (see Peletier and Troy [79], [91] and references therein).

This section is structured in two parts - Subsections 7.2.1 and 7.2.2. We introduce the considered problems at the beginning of these parts. Then, we construct the related variational formulation. After some preliminaries, we prove the existence of solutions. Finally, we present some examples of the obtained results.

7.2.1 Homoclinic solutions

This section is focused on the study of the existence of homoclinic solutions for the following problem:

$$\begin{aligned} \Delta^n \left[\varphi_{p_n}\Big(\Delta^n u(k-n)\Big)\right] + \sum_{i=1}^{n-1} (-1)^i a_i \, \Delta^{n-i} \left[\varphi_{p_{n-i}}\Big(\Delta^{n-i} u\big(k-(n-i)\big)\Big)\right] & \\ +(-1)^n \Big(V(k)\, \varphi_q\big(u(k)\big) - \lambda f\big(k, u(k)\big)\Big) = 0, & \quad (7.2.4) \\ \lim_{|k| \to +\infty} |u(k)| = 0, & \end{aligned}$$

where a_i, p_i, q, Δ^i and V have been previously introduced.

Denote:

$$0 < V_0 = \min\Big\{V(0), \dots, V(T-1)\Big\} \text{ and } V_1 = \max\Big\{V(0), \dots, V(T-1)\Big\}.$$

Let us define the potential function $F\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$F(k,t) = \int_0^t f(k,s) \, \mathrm{d}\, s. \tag{7.2.5}$$

Moreover, $f\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(F_1) For each $k \in \mathbb{Z}$, $f(k,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, for each $t \in \mathbb{R}$, $f(\cdot,t)\colon \mathbb{Z} \to \mathbb{R}$ is a T-periodic function.

(F_2) The potential function, $F(k,t)$, satisfies the Ambrosetti-Rabinowitz's type condition. There exists $\mu \in \mathbb{R}$, such that $\mu > p_i \geqslant q > 1$, for $i = 1, \dots, n$ and:

$$\mu\, F(k,t) \leqslant t\, f(k,t),\ \forall\, k \in \mathbb{Z},\ t \neq 0.$$

(F_3) There exists $s > 0$ such that $F(k,t) > 0$, $\forall k \in \mathbb{Z}$, $\forall t \geqslant s > 0$.

(F_4) Uniformly, for $k \in \mathbb{Z}$, $\displaystyle\lim_{|t|\to 0} \frac{f(k,t)}{|t|^{q-1}} = 0$.

Consider the functional Φ_p, previously introduced in (7.1.7).

Let $\ell^q = \left\{ (u(k))_{k\in\mathbb{Z}} \;\middle|\; \sum_{k\in\mathbb{Z}} |u(k)|^q < \infty \right\}$ be the Banach space with the norm:

$$|u|_q^q = \sum_{k\in\mathbb{Z}} |u(k)|^q,$$

and the functional:

$$\begin{aligned} J\colon \ell^q &\longrightarrow \mathbb{R} \\ u &\longmapsto J(u) = \Phi(u) - \lambda \sum_{k\in\mathbb{Z}} F\big(k, u(k)\big), \end{aligned}$$

where:

$$\Phi(u) = \sum_{k\in\mathbb{Z}} \left(\mathit{\Phi}_{p_n}\Big(\Delta^n\, u(k-n)\Big) + \sum_{i=1}^{n-1} a_i\, \mathit{\Phi}_{p_{n-i}}\Big(\Delta^{n-i}\, u\big(k-(n-i)\big)\Big) + V(k)\, \mathit{\Phi}_q\big(u(k)\big) \right).$$

We have the following result.

Lemma 7.2.1. *Let function f satisfy the assumptions (F_1) and (F_4). Then the functional $J\colon \ell^q \to \mathbb{R}$ is well-defined and C^1. Moreover, its critical points are solutions of the problem* (7.2.4).

Proof. Let us see first that the functional J is well-defined. In order to do that, we use the following elementary inequality:

$$(x+y)^p \leqslant 2^{p-1}\,(x^p + y^p), \tag{7.2.6}$$

which is fulfilled for every non-negative x, y and $p > 1$.

Applying (7.2.6), let us prove by induction that:

$$\sum_{k\in\mathbb{Z}} \mathit{\Phi}_p\Big(\Delta^j u(k-j)\Big) \leqslant \frac{2^{j\,p}}{p} \sum_{k\in\mathbb{Z}} |u(k)|^p, \tag{7.2.7}$$

for $j = 1, \dots, n-1$ and every $p > 1$.

First, consider $j=1$, directly from (7.2.6), we have:

$$\sum_{k\in\mathbb{Z}} \Phi_p\Big(\Delta u(k-1)\Big) \leqslant \frac{2^{p-1}}{p} \sum_{k\in\mathbb{Z}} \Big(|u(k)|^p + |u(k-1)|^p \Big) = \frac{2^p}{p} \sum_{k\in\mathbb{Z}} |u(k)|^p .$$

Now assume, as induction hypothesis, that (7.2.7) is fulfilled for j. So, for $j+1$, from (7.2.6) again, we have:

$$\sum_{k\in\mathbb{Z}} \phi_p\Big(\Delta^{j+1} u\big(k-(j+1)\big)\Big) \leqslant \frac{2^p}{p} \sum_{k\in\mathbb{Z}} \Big|\Delta^j u(k-j)\Big|^p \leqslant \frac{2^{(j+1)\,p}}{p} \sum_{k\in\mathbb{Z}} |u(k)|^p.$$

Thus, (7.2.7) is satisfied.

Now, since $p_i \geqslant q$, it is well-known that $\ell^q \subset \ell^{p_i}$ for $i=1,\dots,n$. Then, for $u\in\ell^q$, we conclude that:

$$\sum_{k\in\mathbb{Z}} \phi_{p_i}\Big(\Delta^i u(k-i)\Big) \leqslant \frac{2^{i\,p_i}}{p_i} \sum_{k\in\mathbb{Z}} |u(k)|^{p_i} < +\infty.$$

Moreover:

$$\sum_{k\in\mathbb{Z}} V(k)\,|u(k)|^q \leqslant V_1 \sum_{k\in\mathbb{Z}} |u(k)|^q < +\infty.$$

Finally, for all $\delta\in(0,1)$, there exists $N>0$ sufficiently large such that $|u(k)|^q<\delta<1$ if $|k|>N$. Moreover, under assumption (F_4), we have that there exists $\delta\in(0,1)$ such that $F\big(k,u(k)\big) < |u(k)|^q < \delta < 1, |k|>N$. Thus, $\sum_{k\in\mathbb{Z}} F\big(k,u(k)\big) < +\infty$ and J is a well-defined functional in ℓ^q.

For all $v\in\ell^q$, we have:

$$\begin{aligned}\langle J'(u),v\rangle &= \lim_{h\to 0}\frac{J(u+h\,v)-J(u)}{h}\\ &= \sum_{k\in\mathbb{Z}} \varphi_{p_n}\Big(\Delta^n u(k-n)\Big)\,\Delta^n v(k-n)\\ &\quad+ \sum_{i=1}^{n-1} a_i \sum_{k\in\mathbb{Z}} \varphi_{p_{n-i}}\Big(\Delta^{n-i} u\big(k-(n-i)\big)\Big)\Delta^{n-i} v\big(k-(n-i)\big)\\ &\quad+ \sum_{k\in\mathbb{Z}} V(k)\,\varphi_q\big(u(k)\big)v(k) - \lambda\sum_{k\in\mathbb{Z}} f\big(k,u(k)\big)v(k).\end{aligned} \tag{7.2.8}$$

Let us see that for all $j=1,\dots,n$ and every fixed $p>1$, we have:

$$\sum_{k\in\mathbb{Z}} \varphi_p\Big(\Delta^j u(k-j)\Big)\,\Delta^j v(k-j) = (-1)^j \sum_{k\in\mathbb{Z}} \Delta^j\Big[\varphi_p\Big(\Delta^j u(k-j)\Big)\Big] v(k). \tag{7.2.9}$$

For $j=1$, by direct calculations, we obtain:

$$\sum_{k\in\mathbb{Z}} \varphi_p\Big(\Delta u(k-1)\Big)\Delta v(k-1) = -\sum_{k\in\mathbb{Z}} \Delta\Big[\varphi_p\Big(\Delta u(k-1)\Big)\Big] v(k).$$

Assume, as induction hypothesis, that (7.2.9) is fulfilled for j, then for $j+1$ we have:

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\varphi_p\Big(\Delta^{j+1}u\big(k-(j+1)\big)\Big)\,\Delta^{j+1}v\big(k-(j+1)\big)=&-\sum_{k\in\mathbb{Z}}\varphi_p\Big[\Delta\Big(\Delta^{j+1}u\big(k-(j+1)\big)\Big)\Big]\Delta^{j}v(k-j)\\ =&-\sum_{k\in\mathbb{Z}}(-1)^j\Delta^j\Big[\varphi_p\,\Big(\Delta^{j+1}u(k-j)\Big)\Big]v(k)\\ &+\sum_{k\in\mathbb{Z}}(-1)^j\Delta^j\Big[\varphi_p\,\Big(\Delta^{j+1}u(k-(j+1))\Big)\,v(k)\Big]\\ =&(-1)^{j+1}\sum_{k\in\mathbb{Z}}\Delta^{j+1}\Big[\varphi_p\,\Big(\Delta^{j+1}u\big(k-(j+1)\big)\Big)\Big]v(k).\end{aligned}$$

Hence for all $v\in\ell^q$:

$$\begin{aligned}\langle J'(u),v\rangle=&(-1)^n\sum_{k\in\mathbb{Z}}\Delta^n\Big[\varphi_{p_n}\Big(\Delta^n u(k-n)\Big)\Big]v(k)\\ &+\sum_{k\in\mathbb{Z}}\sum_{i=1}^{n-1}(-1)^{n-i}a_i\,\Delta^{n-i}\Big[\varphi_{p_{n-i}}\Big(\Delta^{n-i}u\big(k-(n-i)\big)\Big)\Big]v(k)\qquad(7.2.10)\\ &+(-1)^n\sum_{k\in\mathbb{Z}}\Big(V(k)\,\varphi_q\big(u(k)\big)-\lambda f\big(k,u(k)\big)\Big)v(k).\end{aligned}$$

So, we can obtain the partial derivatives as follows:

$$\begin{aligned}\frac{\partial J(u)}{\partial u(k)}=&(-1)^n\left(\Delta^n\Big[\varphi_{p_n}\Big(\Delta^n u(k-n)\Big)\Big]+\sum_{i=1}^{n-1}(-1)^i a_i\Delta^{n-i}\Big[\varphi_{p_{n-i}}\Big(\Delta^{n-i}u\big(k-(n-i)\big)\Big)\Big]\right)\\ &+(-1)^n\Big(V(k)\,\varphi_q\big(u(k)\big)-\lambda f\big(k,u(k)\big)\Big),\end{aligned}$$

which are continuous functions.

Following the arguments of Iannizotto and Tersian [59, Propositions 5, 6 and 7], one can prove that the functional J is continuously differentiable and the critical points of J are the solutions of (7.2.4). □

Now, let us introduce the mountain-pass theorem of Brezis and Nirenberg [14], which we use to obtain the homoclinic solutions of (7.2.4).

Let X be a Banach space with norm $\|\cdot\|$ and $E\colon X\to\mathbb{R}$ be a C^1-functional.

As a modification of Definition 7.1.22, we say that E satisfies the $(PS)_c$-condition if every sequence $(x_k)\subset X$ such that

$$E(x_k)\to c,\quad E'(x_k)\to 0,\qquad(7.2.11)$$

has a convergent subsequence in X for $c\in\mathbb{R}$. Let us denote as a $(PS)_c$-sequence, every sequence $(x_k)\subset X$ that satisfies (7.2.11).

Remark 7.2.2. *Realise that if $(PS)_c$-condition is satisfied for any $c\in\mathbb{R}$, then the Palais-Smale condition introduced in Definition 7.1.22 is satisfied by E.*

Theorem 7.2.3 ([14, Mountain-pass Theorem, Brezis and Nirenberg])**.** *Let X be a Banach space with norm $\|\cdot\|$, $E\in C^1(X,\mathbb{R})$ and suppose that there exist $r>0$, $\alpha>0$ and $e\in X$ such that $\|e\|>r$ and:*

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1. *$E(x) \geqslant \alpha$ if $\|x\| = r$,*

2. *$E(e) < 0$.*

Let $c = \inf_{\gamma\in\Gamma} \left\{ \max\limits_{t\in[0,1]} E\big(\gamma(t)\big) \right\} \geqslant \alpha$, where:

$$\Gamma = \Big\{ \gamma \in C\big([0,1], X\big) \mid \gamma(0) = 0,\ \gamma(1) = e \Big\}.$$

Then, there exists a $(PS)_c$-sequence for E.

Moreover, if E satisfies the $(PS)_c$-condition, then c is a critical value of E, that is, there exists $u_0 \in X$ such that $E(u_0) = c$ and $E'(u_0) = 0$.

Let us consider the following norm in ℓ^q:

$$\|u\|_q := \left(\frac{1}{q} \sum_{k\in\mathbb{Z}} V(k)\ |u(k)|^q \right)^{1/q}.$$

From the assumptions on V, we have that it is an equivalent norm to $|\cdot|_q$, since we have:

$$\frac{V_0}{q}\, |u|_q^q \leq \|u\|_q^q \leq \frac{V_1}{q}\, |u|_q^q.$$

Now, we have the following result, which can be proved in the same way as [26, Lemma 2.3]:

Lemma 7.2.4. *Suppose that assumptions $(F_1) - (F_4)$ are satisfied. Then, there exist $\rho > 0$, $\alpha > 0$ and $e \in \ell^q$ such that $\|e\| > \rho$ and:*

1. *$J(u) \geqslant \alpha$ if $\|u\| = \rho$.*

2. *$J(e) < 0$.*

Proof. Taking into account (F_4), the following assertion is satisfied:

$$\exists\, \delta \in (0,1) \text{ such that } F(k,t) \leqslant \frac{V_0}{2\,q\,\lambda}\, |t|^q\,, \text{ if } |t| \leqslant \delta. \tag{7.2.12}$$

Let $\rho = \left(\dfrac{V_0}{q} \right)^{1/q} \delta$, if $\|u\|_q = \rho$, then:

$$\frac{V_0}{q}\, \delta^q = \rho^q = \|u\|_q^q = \frac{1}{q} \sum_{k\in\mathbb{Z}} V(k)\, |u(k)| \geqslant \frac{V_0}{q}\, |u(k)|^q\,,\ \forall k \in \mathbb{Z}.$$

So, for every $k \in \mathbb{Z}$, it is fulfilled that $|u(k)| < \delta$. Thence, we can apply inequality (7.2.12) to conclude:

$$\sum_{k\in\mathbb{Z}} F\big(k, u(k)\big) \leqslant \frac{V_0}{2q\lambda} \sum_{k\in\mathbb{Z}} |u(k)|^q \leqslant \frac{1}{2\lambda\, q} \sum_{k\in\mathbb{Z}} V(k)\ |u(k)|^q = \frac{1}{2\lambda}\, \|u\|_q^q\,.$$

Thus,

$$J(u) = \Phi(u) - \lambda \sum_{k\in\mathbb{Z}} F\big(k, u(k)\big) \geqslant \|u\|_q^q - \frac{1}{2}\,\|u\|_q^q = \frac{1}{2}\,\|u\|_q^q = \frac{\rho^q}{2} = \alpha > 0.$$

Integrating, from assumptions $(F_2) - (F_3)$, there exist $c_1, c_2 > 0$ such that:

$$F(k,t) \geqslant c_1\, t^{\mu} - c_2$$

for all $t > 0$ and $k \in \mathbb{Z}$, see [81, Remark 2.13].

Now, choose $v \in \ell^q$ such that:

$$v(k) = \begin{cases} b > 0, & k = 0, \\ \\ 0, & k \neq 0, \end{cases}$$

Then, we have for $\kappa > 0$,:

$$J(\kappa\, v) = \Phi(\kappa\, v) - \lambda \sum_{k\in\mathbb{Z}} F\big(k, v(k)\big) = \Phi(\kappa\, v) - \lambda\, F(0, \kappa\, b).$$

Let us see that the following equality is fulfilled for $i = 1, \dots n$ and $p > 1$:

$$\sum_{k\in\mathbb{Z}} \mathit{\Phi}_p\Big(\Delta^i \kappa\, v(k-i)\Big) = \frac{2^i}{p}\kappa^p b^p. \tag{7.2.13}$$

First, consider $i = 1$:

$$\sum_{k\in\mathbb{Z}} \mathit{\Phi}_p\Big(\Delta \kappa\, v(k-1)\Big) = 2\mathit{\Phi}_p(\kappa\, b) = \frac{2}{p}\kappa^p b^p.$$

Now, assume, as induction hypothesis, that (7.2.13) is fulfilled. For $i+1$, taking into account the definition of v, we have:

$$\sum_{k\in\mathbb{Z}} \mathit{\Phi}_p\Big(\Delta^{i+1} \kappa\, v\big(k-(i+1)\big)\Big) = \sum_{k\in\mathbb{Z}} \mathit{\Phi}_p\Big(\Delta^{i} \kappa\, v(k-i)\Big) + \sum_{k\in\mathbb{Z}} \mathit{\Phi}_p\Big(\Delta^{i} \kappa\, v\big(k-(i+1)\big)\Big) = \frac{2^{i+1}}{p}\kappa^p b^p.$$

Then:

$$J(\kappa\, v) \leqslant \sum_{i=1}^{n} a_{n-i}\frac{2^i}{p_i}\kappa^{p_i} b^{p_i} + V(0)\frac{\kappa^q b^q}{q} - \lambda\,(c_1 \kappa^{\mu} b^{\mu} - c_2),$$

with $a_0 = 1$.

Since $\mu > p_j \geqslant q$ for $j = 1, \dots, n$, there exists $M_1 > 0$ such that if $\kappa \geqslant M_1$, then $J(\kappa v) < 0$.

Moreover, $\|\kappa\, v\|_q^q = \dfrac{V(0)}{q}\,(\kappa\, b)^q \geqslant \dfrac{V_0}{q}\,(\kappa\, b)^q$. So, there exists $M_2 > 0$, such that if $\kappa \geqslant M_2$, then $\kappa\, b > \delta$ and $\|\kappa\, v\|_q > \rho$.

Then, let us consider $\kappa_1 \geqslant \max\Big\{M_1, M_2\Big\}$ and considering $e = \kappa_1\, v$ the result is proved. $\square$

In the sequel, we obtain a result which ensures the existence of a bounded $(PS)_c$-sequence for the functional J.

Lemma 7.2.5. *Assume that $(F_1)-(F_4)$ are satisfied. Then, there exists $c>0$ and a ℓ^q-bounded $(PS)_c$-sequence for J.*

Proof. From Lemma 7.2.4 and Theorem 7.2.3 there exists $(u_m)\subset \ell^q$ a $(PS)_c$-sequence for J, i.e, (7.2.11) is satisfied for $I=J$, where:

$$c=\inf_{\gamma\in\Gamma}\left\{\max_{t\in[0,1]} J\big(\gamma(t)\big)\right\},\quad \Gamma=\Big\{\gamma\in C\big([0,1],\ell^q\big)\ |gamma(0)=0,\ \gamma(1)=e\Big\},$$

where $e\in\ell^q$ has been introduced in Lemma 7.2.4.

Now, we have to prove that the sequence (u_m) is bounded in ℓ^q. We have:

$$\begin{aligned}\langle J'(u_m),u_m\rangle =&\sum_{k\in\mathbb{Z}}\left(\Big|\Delta^n u_m(k-n)\Big|^{p_n}+\sum_{i=1}^{n-1}a_i\Big|\Delta^{n-i}u_m\big(k-(n-i)\big)\Big|^{p_{n-i}}\right)\\&+\sum_{k\in\mathbb{Z}}\Big(V(k)\big|u_m(k)\big|^q-\lambda f\big(k,u_m(k)\big)\,u_m(k)\Big).\end{aligned}$$

Now, using (F_2) and taking into account that $\mu>p_i\geqslant q>1$ for $i=1,\dots,n$, using the same arguments as in [26, Lemma 2.4] we conclude that:

$$\begin{aligned}\mu J(u_m)-\langle J'(u_m),u_m\rangle =&\left(\frac{\mu}{p_n}-1\right)\sum_{k\in\mathbb{Z}}\Big|\Delta^n u_m(k-n)\Big|^{p_n}\\&+\sum_{i=1}^{n-1}a_i\left(\frac{\mu}{p_{n-i}}-1\right)\sum_{k\in\mathbb{Z}}\Big|\Delta^{n-i}u_m\big(k-(n-i)\big)\Big|^{p_{n-i}}\\&+\sum_{k\in\mathbb{Z}}\left[\left(\frac{\mu}{q}-1\right)V(k)\big|u_m(k)\big|^q+\lambda\Big(f\big(k,u_m(k)\big)u_m(k)-\mu F\big(k,u_m(k)\big)\Big)\right]\\&\geqslant\left(\frac{\mu}{q}-1\right)q\,\|u_m\|_q^q=(\mu-q)\,\|u_m\|_q^q\,.\end{aligned}$$

Thus, (u_m) is a bounded sequence in ℓ^q. □

Now, we can prove the main result of this part.

Theorem 7.2.6. *Suppose that $a_i>0$ for $i=1,\dots,n-1$, the function $V:\mathbb{Z}\to\mathbb{R}$ is positive and T-periodic and assumptions $(F_1)-(F_4)$ are fulfilled. Then, for $\lambda>0$, problem* (7.2.4) *has a non-trivial homoclinic solution $u\in\ell^q$, which is a critical point of the functional $J\colon\ell^q\to\mathbb{R}$.*

Proof. The proof is analogous to the proof of [26, Theorem 1.1]. For convenience of the reader we recall the different steps.

From Lemma 7.2.5, we have that the obtained $(PS)_c$-sequence (u_m) is bounded in ℓ^q, so:

$$\lim_{|k|\to+\infty}|u_m(k)|=0.$$

Hence, $|u_m(k)|$ attains its maximum at a value $k_m \in \mathbb{Z}$.

Let us denote as j_m, the unique integer such that $j_m\,T \leqslant k_m < (j_m+1)\,T$ and let us define the function:

$$w_m(k) := u_m(k + j_m\,T),$$

this function attains its maximum at $i_m = k_m - j_m T \in [0, T-1]$.

Taking into account the periodicity of V and hypothesis (F_1), we have:

$$\|w_m\|_q = \|u_m\|_q\,, \quad \text{and} \;\; J(w_m) = J(u_m).$$

Moreover, since (u_m) is bounded in ℓ^q it also is (w_m). Then, there exists $w \in \ell^q$ such that $w_m \rightharpoonup w$ weakly, i.e., $\langle w_m, v\rangle \to \langle w, v\rangle$ for all $v \in \ell^q$.

For each $k \in \mathbb{Z}$, let us choose $v_k \in \ell^q$ such that $v_k(j) = \delta_k^j$ for all $j \in \mathbb{Z}$, where:

$$\delta_i^j = \begin{cases} 1, & i = j, \\ \\ 0, & i \neq j. \end{cases}$$

Then, we have:

$$\lim_{m\to+\infty} w_m(k) = \lim_{m\to+\infty} \langle w_m, v_k\rangle = \langle w, v_k\rangle = w(k), \quad \forall k \in \mathbb{Z}.$$

Furthermore, for every $v \in \ell^q$, we have:

$$\begin{aligned} \Big|\langle J'(w_m), v\rangle\Big| \quad &= \quad \Big|\big\langle J'(u_m), v\big(\,\cdot + j_m\,T\big)\big\rangle\Big| \leqslant \big\|J'(u_m)\big\|\,\big\|v\,(\cdot + j_m T)\,\big\|_q, \\ &= \quad \big\|J'(u_m)\big\|\,\|v\|_q \to 0. \end{aligned}$$

Then:

$$\lim_{m\to+\infty} J'(w_m) = 0.$$

So, for every $v \in \ell^q$:

$$\begin{aligned} &\lim_{m\to+\infty}\left(\sum_{k\in\mathbb{Z}} \varphi_{p_n}\Big(\Delta^n w_m(k-n)\Big)\,\Delta^n v(k-n)\right. \\ &+\sum_{i=1}^{n-1} a_i \sum_{k\in\mathbb{Z}} \varphi_{p_{n-i}}\Big(\Delta^{n-i} w_m\big(k-(n-i)\big)\Big)\Delta^{n-i} v\big(k-(n-i)\big) \qquad (7.2.14) \\ &\left.+\sum_{k\in\mathbb{Z}} V(k)\,\varphi_q\big(w_m(k)\big)v(k) - \lambda\sum_{k\in\mathbb{Z}} f\big(k, w_m(k)\big)\,v(k)\right) = 0,\ \forall k \in \mathbb{Z}. \end{aligned}$$

Let us denote ℓ_0^q the set of all functions $v \in \ell^q$ with compact support, that is, for all $v \in \ell_0^q$ there exist c, $d \in \mathbb{Z}$, with $c < d$, such that $v(k) = 0$ if $k \in \mathbb{Z}\backslash[c, d]$.

Realise that ℓ_0^q is dense in ℓ^q, because for any $v \in \ell^q$, let us choose $v_k \in \ell_0^q$ such that:

$$v_k(j) = \begin{cases} v(j), & |j| \leqslant k, \\ \\ v(j) = 0, & |j| > k, \end{cases}$$

and, clearly $\lim_{k\to\infty} \|v - v_k\|_q = 0$.

If we consider $v \in \ell_0^q$ in (7.2.14), since we have finite sums and $f(k,\cdot)$ is a continuous function, passing to a limit, we have:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \varphi_{p_n}\Big(\Delta^n w(k-n)\Big)\,\Delta^n v(k-n)\\
&+\sum_{i=1}^{n-1} a_i \sum_{k\in\mathbb{Z}} \varphi_{p_{n-i}}\Big(\Delta^{n-i} w\big(k-(n-i)\big)\Big)\Delta^{n-i} v\big(k-(n-i)\big) \qquad (7.2.15)\\
&+\sum_{k\in\mathbb{Z}} V(k)\,\varphi_q\big(w(k)\big)v(k) - \lambda\sum_{k\in\mathbb{Z}} f\big(k,w(k)\big)\,v(k) = 0,\ \forall v\in\ell_0^q.
\end{aligned}$$

Considering the density of ℓ_0^q in ℓ^q, we have the previous equality satisfied for all $v \in \ell^q$. Hence, w is a critical point of the functional J. Thus, w is a solution of (7.2.4).

To finish the proof, we have to see that $w \neq 0$.

Let us assume that $w = 0$. In such a case, we have:

$$\lim_{m\to+\infty} |u_m|_\infty = \lim_{m\to+\infty} |w_m|_\infty = \lim_{m\to+\infty} \max\Big\{|w_m| \;\Big|\; k\in\mathbb{Z}\Big\} = 0. \qquad (7.2.16)$$

Using (F_4), we know that for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $|x| < \delta$ then, for every $k \in [0, T-1]$, the following inequalities are fulfilled:

$$\Big|F(k,x)\Big| \leqslant \varepsilon\,|x|^q \quad\text{and}\quad \Big|f(k,x)\,x\Big| \leqslant \varepsilon\,|x|^q\,. \qquad (7.2.17)$$

From (7.2.16), for every $k \in [0, T-1]$, we can ensure the existence of a positive integer M_k, such that if $m > M_k$, then $\big|w_m(k)\big| < \delta$.

By the construction of w_m, the maximum value of $\big|w_m\big|$ is attained at $i_m \in [0, T-1]$, we conclude that for $m > M = \max\Big\{M_k \;\Big|\; k\in[0,T-1]\Big\}$ we have:

$$\big|w_m(k)\big| \leqslant \big|w_m(i_m)\big| < \delta.$$

Hence, from (7.2.17), for all $m > M$ we have:

$$\Big|F\big(k, w_m(k)\big)\Big| \leqslant \varepsilon\big|w_m(k)\big|^q \text{ and } \Big|f\big(k,w_m(k)\big)\,w_m(k)\Big| \leqslant \varepsilon\,\big|w_m(k)\big|^q, \forall k\in\mathbb{Z}.$$

This implies that:

$$\begin{aligned}
0 \le qJ(w_m) =& \frac{q}{p_n}\sum_{k\in\mathbb{Z}}\Big|\Delta^n w_m(k-n)\Big|^{p_n} + \sum_{i=1}^{n-1}\frac{a_i\,q}{p_{n-i}}\sum_{k\in\mathbb{Z}}\Big|\Delta^{n-i} w_m\big(k-(n-i)\big)\Big|^{p_{n-i}}\\
&+\sum_{k\in\mathbb{Z}} V(k)\big|w_m(k)\big|^q - \lambda q\sum_{k\in\mathbb{Z}} F\big(k,w_m(k)\big),
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{k\in\mathbb{Z}} \left(\left|\Delta^n w_m(k-n)\right|^{p_n} + \sum_{i=1}^{n-1} a_i \left|\Delta^{n-i} w_m\big(k-(n-i)\big)\right|^{p_{n-i}} + V(k)\left|w_m(k)\right|^q \right) \\
&\quad - \lambda \sum_{k\in\mathbb{Z}} f\big(k, w_m(k)\big) w_m(k) \\
&\quad - \lambda \sum_{k\in\mathbb{Z}} \Big(qF\big(k, w_m(k)\big) - f\big(k, w_m(k)\big) w_m(k) \Big), \\
&\leqslant \big\langle J'(w_m), w_m \big\rangle + \lambda \sum_{k\in\mathbb{Z}} \Big(q\varepsilon \left|w_m(k)\right|^q + \varepsilon \left|w_m(k)\right|^q \Big), \\
&= \big\langle J'(w_m), w_m \big\rangle + \lambda\, \varepsilon (q+1) \left|w_m\right|_q^q, \\
&\leqslant \left\| J'(w_m) \right\| \left\| w_m \right\|_q + \lambda\, \varepsilon \frac{q\,(q+1)}{V_0} \left\| w_m \right\|_q^q.
\end{aligned}
$$

Since $(w_m) \in \ell^q$ is bounded, $\lim\limits_{m\to+\infty} J'(w_m) = 0$ and we can choose $\varepsilon > 0$ arbitrarily small, we arrive to a contradiction with the fact that:

$$
\lim_{m\to+\infty} J\big(w_m\big) = \lim_{m\to+\infty} J\big(u_m\big) = c > 0. \qquad \square
$$

Example 7.2.7. *Let $r > p_i \geqslant q > 1$ for $i = 1, \dots, n$ and $b\colon \mathbb{Z} \to \mathbb{R}$ a positive T-periodic function.*

Consider $f(k,t) = b(k)\varphi_r(t)$. Let us verify that such f satisfies $(F_1) - (F_4)$.

(F_1) *Obviously f is continuous as a function of t and T-periodic as a function of k,*

(F_2) *$F(k,t) = b(k)\,\Phi_r(t)$. There exist $r \in \mathbb{R}$, such that $r > p_i \geqslant q > 1$, for $i = 1, 2$ such that:*

$$
r\,F(k,t) = b(k)\,|t|^r = t\,b(k)\,t\,|t|^{r-2} = t\,f(k,t), \ \forall k \in \mathbb{Z},\ t \neq 0.
$$

(F_3) *$F(k,t) > 0$, $\forall k \in \mathbb{Z}$, $\forall t > 0$.*

(F_4) *Since $r > q$, we have:*

$$
\lim_{|t|\to 0} \frac{f(k,t)}{|t|^{q-1}} = \lim_{|t|\to 0} b(k)\,t\,|t|^{r-q-1} = 0.
$$

Then, for V, $b\colon \mathbb{Z} \to \mathbb{R}$ two positive T-periodic functions the problem:

$$
\begin{aligned}
\Delta^n \Big[\varphi_q\Big(\Delta^n u(k-n)\Big) \Big] + \sum_{i=1}^{n-1} (-1)^i a_i\, \Delta^{n-i} \Big[\varphi_q\Big(\Delta^{n-i} u\big(k-(n-i)\big)\Big) \Big] & \\
+(-1)^n \Big(V(k)\,\varphi_q\big(u(k)\big) - \lambda\, b(k)\,\varphi_r\big(u(k)\big) \Big) &= 0,
\end{aligned}
$$

$$
\lim_{|k|\to+\infty} |u(k)| = 0,
$$

where $r > p_i \geqslant q > 1$ for $i = 1, \dots, n$, with $a_i \geqslant 0$ for $i = 1, \dots, n-1$, has a non-trivial homoclinic solution for every $\lambda > 0$.

7.2.2 Boundary value problems

In this part we study the existence of multiple solutions for the following boundary value problem:

$$\Delta^n\left[\varphi_q\Big(\Delta^n u(k-2)\Big)\right]+\sum_{i=1}^{n-1}(-1)^i a_i \Delta^{n-i}\left[\varphi_q\Big(\Delta^{n-i}u\big(k-(n-i)\big)\Big)\right] \qquad (7.2.18)$$

$$+(-1)^n\Big(V(k)\,\varphi_q\big(u(k)\big)-\lambda f\big(k,u(k)\big)\Big)=0,$$

$$u(0)=\Delta u(-1)=\Delta^2 u(-2)=\cdots=\Delta^{n-1}u(1-n)=0, \qquad (7.2.19)$$

$$u(T+1)=\Delta u(T+1)=\Delta^2 u(T+1)=\cdots=\Delta^{n-1}u(T+1)=0. \qquad (7.2.20)$$

where $a_i \geqslant 0$ are positive numbers for $i=1\ldots,n-1$, T is a fixed positive integer, $[1,T]=\{1,2,\ldots,T\}$, the difference operators have been introduced in (7.2.2)-(7.2.3) and φ_q has been defined in Definition 7.0.1 for $1<q<+\infty$.

Moreover, we consider $V\colon [1,T]\to\mathbb{R}$, as a positive function. We can consider it as a restriction to the discrete interval $[1,T]$ of the T-periodic function V introduced in the first part of the section.

We also denote $V_0=\min\Big\{V(1),\ldots,V(T)\Big\}$ and $V_1=\max\Big\{V(1),\ldots,V(T)\Big\}$.

Finally, let $f\colon [1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function.

As in the first part, we obtain the existence result by means of variational methods.

In order to construct our variational approach, we consider the following T-dimensional Banach space:

$$\begin{aligned} X_1=&\Big\{u\colon [1-n,T+n]\to\mathbb{R}\;\big|\; u(0)=\cdots=\Delta^{n-1}u(1-n)=0,\\ &\quad u(T+1)=\cdots=\Delta^{n-1}u(T+1)=0\Big\}, \end{aligned} \qquad (7.2.21)$$

coupled with the following norm

$$\|u\|_{X_1}=\left(\sum_{k=1}^{T+n}\Big|\Delta^n(k-n)\Big|^q+\sum_{i=1}^{n-1}a_i\sum_{k=1}^{T+n-i}\Big|\Delta^{n-i}u\big(k-(n-i)\big)\Big|^q+\sum_{k=1}^{T}V(k)\,\big|u(k)\big|^q\right)^{1/q}.$$

We have the following result, in terms of the norm $\|\cdot\|_{X_1}$.

Lemma 7.2.8. *For every $u\in X_1$, the following inequality holds:*

$$\max_{k\in[1,T]}\big|u(k)\big|\leqslant \rho\,\|u\|_{X_1},$$

where:

$$\rho=\frac{\displaystyle\prod_{j=1}^{n}(T+j)}{\left(2^{qn}(T+n)+\displaystyle\sum_{i=1}^{n-1}a_i 2^{q(n-i)}(T+n-i)\prod_{j=n-i+1}^{n}(T+j)^q+V_0\prod_{j=1}^{n}(T+j)^q\right)^{(1/q)}}. \qquad (7.2.22)$$

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Proof. First, we have:

$$\max_{k\in[1,T]} \left|u(k)\right|^q \leqslant \sum_{k=1}^{T} \left|u(k)\right|^q \leqslant \frac{1}{V_0}\sum_{k=1}^{T} V(k)\left|u(k)\right|^q. \tag{7.2.23}$$

Secondly, taking into account the boundary conditions (7.2.19)-(7.2.20), for every $u \in X$ and all $j = 1, \dots, T$, we have:

$$\begin{aligned}\sum_{k=1}^{T+1} \left|\Delta u(k-1)\right| &= \sum_{k=1}^{j} \left|u(k)-u(k-1)\right| + \sum_{k=j+1}^{T+1} \left|u(k-1)-u(k)\right| \\ &\geqslant \sum_{k=1}^{j} \Big(\left|u(k)\right| - \left|u(k-1)\right|\Big) + \sum_{k=j+1}^{T+1} \Big(\left|u(k-1)\right| - \left|u(k)\right|\Big) \\ &= 2\left|u(j)\right| - \left|u(0)\right| - \left|u(T+1)\right| = 2\left|u(j)\right|.\end{aligned} \tag{7.2.24}$$

Analogously, for every $u \in X$ and all $j = 1, \dots, T+i$, with $i = 1, \dots, n$, we obtain:

$$\sum_{k=1}^{T+i} \left|\Delta^i u(k-i)\right| \geqslant 2\left|\Delta^{i-1} u\big(j-(i-1)\big)\right|. \tag{7.2.25}$$

From (7.2.24), we deduce that:

$$\max_{k\in[1,T]} \left|u(k)\right| \leqslant \frac{1}{2}\sum_{k=1}^{T+1} \left|\Delta u(k-1)\right|. \tag{7.2.26}$$

Now, let us see that for $2 \leqslant i \leqslant n$, the following inequality is fulfilled.

$$\max_{k\in[1,T]} \left|u(k)\right| \leqslant \frac{1}{2^i}\prod_{j=1}^{i-1}(T+j)\sum_{k=1}^{T+i} \left|\Delta^i u(k-i)\right|. \tag{7.2.27}$$

For $i = 2$, from (7.2.25) and (7.2.26), we have:

$$\begin{aligned}\max_{k\in[1,T]} \left|u(k)\right| &\leqslant \frac{1}{2}\sum_{k=1}^{T+1} \left|\Delta u(k-1)\right| \leqslant \frac{1}{2}(T+1)\max_{k\in[1,T+1]} \left|\Delta u(k-1)\right| \\ &\leqslant \frac{T+1}{4}\sum_{k=1}^{T+2} \left|\Delta^2 u(k-2)\right|.\end{aligned}$$

Now, assume, as induction hypothesis, that (7.2.27) is fulfilled, let us see what happens for $i+1$. By using (7.2.25), we obtain:

$$\begin{aligned}\max_{k\in[1,T]} \left|u(k)\right| &\leqslant \frac{1}{2^i}\prod_{j=1}^{i-1}(T+j)\sum_{k=1}^{T+i} \left|\Delta^i u(k-i)\right| \leqslant \frac{1}{2^i}\prod_{j=1}^{i-1}(T+j)\,(T+i)\max_{k\in[1,T+i]} \left|\Delta^i u(k-i)\right| \\ &\leqslant \frac{1}{2^{i+1}}\prod_{j=1}^{i}(T+j)\sum_{k=1}^{T+i+1} \left|\Delta^{i+1} u\big(k-(i+1)\big)\right|,\end{aligned}$$

hence, (7.2.27) is proved.

Assuming that $\prod_{j=1}^{0}(T+j)=1$, then (7.2.27) is true for $i\in\{1,\ldots,n\}$.

Now, by using the discrete Hölder inequality, see [57] for instance, we have that for all $i=1,\ldots,n$:

$$\max_{k\in[1,T]}\left|u(k)\right|\leqslant\frac{1}{2^i}\prod_{j=1}^{i-1}(T+j)(T+i)^{\frac{q-1}{q}}\left(\sum_{k=1}^{T+i}\left|\Delta^i u(k-i)\right|^q\right)^{\frac{1}{q}}.$$

Thus, combining this with (7.2.23), we obtain:

$$\left(\frac{2^{q\,n}}{\prod_{j=1}^{n-1}(T+j)^q(T+n)^{q-1}}+\sum_{i=1}^{n-1}\frac{a_i\,2^{q\,(n-i)}}{(T+n-i)^{q-1}\prod_{j=1}^{n-i-1}(T+j)^q}+V_0\right)\max_{k\in[1,T]}\left|u(k)\right|^q\leqslant\|u\|_X^q\,,$$

and the result is proved, taking into account that:

$$\frac{2^{q\,n}}{\prod_{j=1}^{n-1}(T+j)^q(T+n)^{q-1}}+\sum_{i=1}^{n-1}\frac{a_i\,2^{q\,(n-i)}}{(T+n-i)^{q-1}\prod_{j=1}^{n-i-1}(T+j)^q}+V_0=\frac{1}{\rho^q}.$$

□

Remark 7.2.9. *The expression of ρ, given in* (7.2.22) *seems to be complicated to assimilate. However, it can be calculated for each particular problem.*

Moreover, from its expression it can be seen that it is clearly bounded as follows:

$$\rho\leqslant\frac{\prod_{j=1}^{n}(T+j)}{\left(V_0\prod_{j=1}^{n}(T+j)^q\right)^{1/q}}=\left(\frac{1}{V_0}\right)^{1/q}.$$

Realise that Lemma 7.2.8 remains applicable if we replace ρ by the previous one upper bound.

Now, let us consider $J_1\colon X_1\longrightarrow\mathbb{R}$, the functional:

$$J_1(u)=\Phi_1(u)-\lambda\sum_{k=1}^{T}F\big(k,u(k)\big),$$

where $F(k,t)=\int_0^t f(k,s)\,\mathrm{d}\,s$ for every $(k,t)\in[1,T]\times\mathbb{R}$ and:

$$\Phi_1(u)=\sum_{k=1}^{T+n}\phi_q\Big(\Delta^n\,u(k-n)\Big)+\sum_{i=1}^{n-1}a_i\sum_{k=1}^{T+n-i}\phi_q\Big(\Delta\,u\big(k-(n-i)\big)\Big)+\sum_{k=1}^{T}V(k)\,\phi_q\big(u(k)\big).$$

Remark 7.2.10. *Realise that* $\Phi_1(u) = \dfrac{\|u\|_{X_1}^q}{q}$*, hence trivially* Φ_1 *is coercive, that is,*

$$\lim_{\|u\|_{X_1} \to +\infty} \Phi_1(u) = +\infty.$$

We have the analogous result for this case to Lemma 7.2.1

Lemma 7.2.11. *The functional* $J_1 \colon X_1 \to \mathbb{R}$ *is* C^1*-differentiable and its critical points are solutions of* (7.2.18)–(7.2.20)*.*

Proof. For all $v \in X$, we have:

$$\begin{aligned}
\langle J_1'(u), v\rangle &= \lim_{h \to 0} \frac{J_1(u + h\,v) - J_1(u)}{h} \\
&= \sum_{k=1}^{T+n} \varphi_q\Big(\Delta^n u(k-n)\Big)\,\Delta^n v(k-n) \\
&\quad + \sum_{i=1}^{n-1} a_i \sum_{k=1}^{T+i} \varphi_q\Big(\Delta^{n-i} u\big(k-(n-i)\big)\Big)\Delta^{n-i} v\big(k-(n-i)\big) \\
&\quad + \sum_{k=1}^{T} V(k)\,\varphi_q\big(u(k)\big)v(k) - \lambda \sum_{k=1}^{T} f\big(k, u(k)\big)\,v(k).
\end{aligned} \tag{7.2.28}$$

On the other hand, let $g \colon \mathbb{R} \to \mathbb{R}$ be a continuous functions. We will see that the following equality holds for all $v \in X_1$ and every $i = 1, \ldots, n$:

$$\sum_{k=1}^{T+i} g\big(u(k-i)\big)\Delta^i v(k-i) = (-1)^i \sum_{k=1}^{T} \Delta^i g\big(u(k-i)\big)v(k) \tag{7.2.29}$$

First, for every $u,\, v \in X_1$, by direct calculations, taking into account the boundary conditions (7.2.19)-(7.2.20), we have:

$$\sum_{k=1}^{T+i} g\big(u(k-i)\big)\Delta^i v(k-i) = -\sum_{k=1}^{T+i-1} \Delta g\big(u(k-i)\big)\Delta^{i-1} v\big(k-(i-1)\big), \tag{7.2.30}$$

Thus, for $i = 1$, (7.2.29) is true.
Now, assume that (7.2.29) is fulfilled for a fixed $i \in \{1, \ldots, n\}$. From (7.2.30), we have:

$$\begin{aligned}
\sum_{k=1}^{T+i+1} g\Big(u\big(k-(i+1)\big)\Big)\Delta^{i+1} v\big(k-(i+1)\big) &= -\sum_{k=1}^{T+i} \Delta g\Big(u\big(k-(i+1)\big)\Big)\Delta^i v(k-i), \\
&= (-1)^{i+1} \sum_{k=1}^{T} \Delta^i \Delta g\Big(u\big(k-(i+1)\big)\Big)v(k),
\end{aligned}$$

and, clearly (7.2.29) is satisfied.

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Substituting (7.2.29) in (7.2.28), we have for all $u, v \in X_1$:

$$\begin{aligned}\langle J_1'(u), v\rangle = & \sum_{k=1}^{T}(-1)^n \Delta^n \Big[\varphi_q\Big(\Delta^n u(k-n)\Big)\Big] v(k) \\ & + \sum_{i=1}^{n-1}(-1)^{n-i} a_i \sum_{k=1}^{T} \Delta^{n-i}\Big[\varphi_q\Big(\Delta^{n-i} u\big(k-(n-i)\big)\Big)\Big] v(k) \\ & + (-1)^n \sum_{k=1}^{T}\Big(V(k)\,\varphi_q\big(u(k)\big) - \lambda \sum_{k=1}^{T} f\big(k, u(k)\big)\Big) v(k).\end{aligned} \tag{7.2.31}$$

Since $v \in X$ is arbitrary, we conclude that if u is a critical point of J_1, then it is a solution of problem (7.2.18)–(7.2.20) and the result is proved. □

Denote $\Psi_1(u) = -\sum_{k=1}^{T} F\big(k, u(k)\big)$.

Notation 7.2.12. *Let X be a finite dimensional Banach space and consider $E\colon X \to \mathbb{R}$ a functional of the type*

$$E(u) = \Xi(u) + \lambda\,\Psi(u),$$

where Ξ, $\Psi\colon X \to \mathbb{R}$ are of class $C^1(X)$ and Ξ is coercive.

Remark 7.2.13. *Realise that $J_1(u) = \Phi_1(u) + \lambda\Psi_1(u)$ satisfies this condition.*

Now, for every $r > \inf_E \Xi$, let us define:

$$\begin{aligned}\psi_1(r) &:= \inf_{u \in \Xi^{-1}((-\infty, r))} \frac{\Psi(u) - \inf_{\Xi^{-1}((-\infty, r])} \Psi}{r - \Xi(u)}, \\ \psi_2(r) &:= \inf_{u \in \Xi^{-1}((-\infty, r))} \sup_{v \in \Xi^{-1}([r, +\infty))} \frac{\Psi(u) - \Psi(v)}{\Xi(v) - \Xi(u)}.\end{aligned}$$

Now, we can apply [36, Theorem 2.1] to our problem.

Theorem 7.2.14 ([36, Theorem 2.1]). *Let X and E be as in Notation 7.2.12. Assume that:*

1. *there exists $r > \inf_X \Xi$ such that $\psi_1(r) < \psi_2(r)$.*
2. *for each $\lambda \in \left(\frac{1}{\psi_2(r)}, \frac{1}{\psi_1(r)}\right)$, we have $\lim_{\|u\|_X \to +\infty} E(u) = +\infty$.*

Then, for each $\lambda \in \left(\frac{1}{\psi_2(r)}, \frac{1}{\psi_1(r)}\right)$, E has at least three critical points.

Let us define, for c, $d > 0$,

$$\Theta(c) := \frac{\displaystyle\sum_{k=1}^{T} \sup_{|s| \leqslant c} F(k,s)}{c^q} > 0,$$

$$\Lambda(d) := \frac{\displaystyle\sum_{k=1}^{T} \left(F(k,d) - \sup_{|s| \leqslant c} F(k,s) \right)}{d^q} > 0.$$

Now, we can state the main result of this section.

Theorem 7.2.15. *Assume that there exist four positive constants b, c, d, and p, such that $c < d$ and $p < q$ satisfying:*

(d_1) $\Theta(c) < \dfrac{\Lambda(d)}{\left(2^n + \displaystyle\sum_{i=1}^{n-1} 2^{n-i} a_i + T\,V_1\right) \rho^q}$, *where ρ has been introduced in* (7.2.22).

(d_2) $F(k,t) \leqslant b\,(1 + |t|^p)$ *for all* $(k,t) \in [1,T] \times \mathbb{R}$.

Then, for every $\lambda \in \left(\dfrac{2^n + \displaystyle\sum_{i=1}^{n-1} 2^{n-i} a_i + T\,V_1}{q\,\Lambda(d)}, \dfrac{1}{q\,\rho^q\,\Theta(c)} \right)$, *problem* (7.2.18)–(7.2.20) *admits at least three solutions which are critical points of J_1.*

Proof. We just need to find $r > \inf_{X_1} \Phi_1$ such that the hypotheses of Theorem 7.2.14 are satisfied.

Let:

$$r = \frac{c^q}{q\,\rho^q}.$$

Taking into account the relationship between Φ_1 and the norm $\|\cdot\|_{X_1}$, we have:

$$\psi_1(r) = \inf_{\|u\|_X < (q\,r)^{1/q}} \frac{\Psi_1(u) - \displaystyle\inf_{\|u\|_X \leqslant (q\,r)^{1/q}} \Psi_1(u)}{r - \Phi_1(u)} \leqslant \frac{- \displaystyle\inf_{\|u\|_X \leqslant (q\,r)^{1/q}} \Psi_1(u)}{r}$$
$$= \frac{\displaystyle\sup_{\|u\|_X \leqslant (q\,r)^{1/q}} \sum_{k=1}^{T} F\big(k, u(k)\big)}{r}.$$

Now, from Lemma 7.2.8, if $\|u\|_X \leqslant (q\,r)^{1/q}$, then for all $k \in [1,T]$:

$$|u(k)| \leqslant \rho\,(q\,r)^{1/q} = c,$$

thus,

$$\psi_1(r) \leqslant \frac{\displaystyle\sum_{k=1}^{T} \sup_{|u(k)|\leqslant c} F\big(k, u(k)\big)}{r} = q\,\rho^q\,\Theta(c).$$

Let us see that $c^q < \rho^q \left(2^n + \displaystyle\sum_{i=1}^{n-1} 2^{n-i} a_i + V_0\right) d^q$.

To this end, let us choose $k^* \in [1, T]$, such that $V(k^*) = V_0$, and consider:

$$v_c(k) := \begin{cases} c, & \text{if } k = k^*, \\ \\ 0, & \text{if } k \neq k^*. \end{cases}$$

From Lemma 7.2.8, since $c < d$, we have:

$$c^q \leqslant \rho^q\,\|v_c\|_X^q = \rho^q \left(2^n + \sum_{i=1}^{n-1} 2^{n-i} a_i + V_0\right) c^q < \rho^q \left(2^n + \sum_{i=1}^{n-1} 2^{n-i} a_i + V_0\right) d^q.$$

Now, consider $v_d \in X$, such that:

$$v_d(k) := \begin{cases} d, & \text{if } k \in [1, T], \\ \\ 0, & \text{otherwise.} \end{cases}$$

We have:

$$\|v_d\|_X^q = \left(2^n + \sum_{i=1}^{n-1} 2^{n-i} a_i + \sum_{k=1}^{T} V(k)\right) d^q \geqslant \left(2^n + \sum_{i=1}^{n-1} 2^{n-i} a_i + V_0\right) d^q > \left(\frac{c}{\rho}\right)^q = q\,r.$$

Hence,

$$\psi_2(r) = \inf_{\|u\|_X < (q\,r)^{1/q}} \sup_{\|v\|_X \geqslant (q\,r)^{1/q}} \frac{\Psi_1(u) - \Psi_1(v)}{\Phi_1(v) - \Phi_1(u)} \geqslant \inf_{\|u\|_X < (q\,r)^{1/q}} \frac{\Psi_1(u) - \Psi_1(v_d)}{\Phi_1(v_d) - \Phi_1(u)}$$

$$= q \inf_{\|u\|_X < (q\,r)^{1/q}} \frac{\displaystyle\sum_{k=1}^{T} F(k, d) - \sum_{k=1}^{T} F\big(k, u(k)\big)}{\left(2^n + \displaystyle\sum_{i=1}^{n-1} 2^{n-i} a_i + \sum_{k=1}^{T} V(k)\right) d^p - \|u\|_X^q}$$

$$\geqslant q\, \frac{\displaystyle\sum_{k=1}^{T} F(k, d) - \sum_{k=1}^{T} \sup_{\left|u(k)\right| \leq c} F\big(k, u(k)\big)}{\left(2^n + \displaystyle\sum_{i=1}^{n-1} 2^{n-i} a_i + \sum_{k=1}^{T} V(k)\right) d^p}.$$

Now, taking into account that $\sum_{k=1}^{T} V(k) \leqslant T\,V_1$, we have:

$$\psi_2(r) \geqslant \frac{q\,\Lambda(d)}{2^n + \sum_{i=1}^{n-1} 2^{n-i} a_i + T\,V_1}.$$

Thus, from condition (d_1), we conclude:

$$\psi_2(r) > q\,\rho^q \Theta(c) \geqslant \psi_1(r).$$

By another hand, from condition (d_2), we have:

$$J_1(u) = \frac{\|u\|_X^q}{q} - \lambda \sum_{k=1}^{T} F(k,u(k)) \geqslant \frac{\|u\|_X^q}{q} - \lambda\, b \sum_{k=1}^{T} \Big(1 + \big|u(k)\big|^p\Big).$$

Using again Lemma 7.2.8, we conclude:

$$J_1(u) \geqslant \frac{\|u\|_X^q}{q} - \lambda\,T\,b - \lambda\,T\,\rho^p\,\|u\|_X^p\,,$$

which, since $p < q$, ensures that $\lim_{\|u\|_X \to +\infty} J_1(u) = +\infty$.

Therefore, by applying Theorem 7.2.14 the result is proved. □

Example 7.2.16. *Let $n = 2$, $T = 8$ and $V(k) = 6(k+6)^2$ for each $k \in [1,T]$. Then, in this case $V_0 = 294$ and $V_1 = 1176$.*

Moreover, consider $f(k,t) = k\,g(t)$, where:

$$g(t) = \begin{cases} e^t, & t \leqslant 14, \\ e^{14}, & t > 14, \end{cases}$$

then, $F(k,t) = k^2\,G(t)$, where:

$$G(t) = \begin{cases} e^t, & t \leqslant 14, \\ e^{14}(t-13), & t > 14. \end{cases}$$

So, we can see that $F(k,t) \leqslant 8\,e^{14}\Big(1 + |t|^p\Big)$ for all $p > 1$.

Let us choose $c = 3$, $d = 14$ and $q = 3$. We have:

$$\Theta(3) := \frac{\sum_{k=1}^{8} \sup_{|s| \leqslant 3} F(k,s)}{3^3} = \frac{4}{3}\,e^3, \tag{7.2.32}$$

$$\Lambda(14) := \frac{\displaystyle\sum_{k=1}^{8}\left(F(k,14) - \sup_{|s|\le 3} F(k,s)\right)}{14^3} = \frac{9}{686}\Big(e^{14} - e^3\Big). \tag{7.2.33}$$

Now, consider the problem:

$$\begin{aligned} \Delta^2\Big[\varphi_3\Big(\Delta^2 u(k-2)\Big)\Big] - 10\,\Delta\Big[\varphi_3\Big(\Delta u(k-1)\Big)\Big] \\ +6(k+6)^2\varphi_3\big(u(k)\big) - \lambda k g\big(u(k)\big) = 0, \quad k \in [1,8], \end{aligned} \tag{7.2.34}$$

coupled with the boundary conditions (7.2.19)-(7.2.20) *for* $n = 2$.

In this case, $\rho^3 = \dfrac{18225}{5376166} \approxeq 0.00338996 < \dfrac{1}{294} \approxeq 0.0034.$

Moreover, we have:

$$\frac{\Lambda(14)}{(4+20+8\cdot 1176)\ \rho^3} = \frac{2688083}{655123140}\Big(e^{14} - e^3\Big) \approxeq 493.44 > \frac{4}{3}e^3 \approxeq 26.78.$$

Then, we can apply Theorem 7.2.15 to conclude that for each:

$$\lambda \in \left(\frac{718928}{3\big(e^{14}-e^3\big)}, \frac{2688083}{36450\,e^3}\right) = (0.1993, 3.672),$$

problem (7.2.34) *has at least three solutions.*

Now, let us consider a higher order example, for instance let us choose $n = 4$.

Example 7.2.17. *Let us choose T, $V(k)$ and $f(k,t)$ as in Example 7.2.16. For the choice $c = 3$, $d = 14$ and $q = 3$,* (7.2.32)-(7.2.33) *are still true.*

Now, consider the problem:

$$\begin{aligned} \Delta^4\Big[\varphi_3\Big(\Delta^4 u(k-4)\Big)\Big] - \Delta^3\Big[\varphi_3\Big(\Delta^3 u(k-3)\Big)\Big] + 2\Delta^2\Big[\varphi_3\Big(\Delta^2 u(k-2)\Big)\Big] \\ -3\Delta\Big[\varphi_3\Big(\Delta u(k-1)\Big)\Big] + 6(k+6)^2\varphi_3\big(u(k)\big) - \lambda k g\big(u(k)\big) = 0, \end{aligned} \tag{7.2.35}$$

for $k \in [1,8]$, *coupled with the boundary conditions* (7.2.19)-(7.2.20) *for* $n = 4$.

Observe that $a_i = i$ *for* $i = 1,2,3$.

In this case, the expression of ρ is more complicated than in the case $n = 2$. It is given by:

$$\begin{aligned} \rho^3 &= \frac{\displaystyle\prod_{j=1}^{4}(8+j)^3}{\displaystyle 12\cdot 2^{12} + \sum_{i=1}^{3} i2^{3(4-i)}(12-i)\prod_{j=5-i}^{4}(8+j)^3 + 294\prod_{j=1}^{3}(8+j)^3} \\ &= \frac{1091586375}{321251750258} \approxeq 0.003398 < \frac{1}{294}. \end{aligned}$$

Moreover, we have:

$$\frac{\Lambda(14)}{\left(2^4+\displaystyle\sum_{i=1}^{3} i\,2^{4-i}+8\cdot 1176\right)\rho^3}=\frac{42833567}{1047915804474}\big(e^{14}-e^3\big)\approxeq 491.556>\frac{4}{3}e^3.$$

Then, we can apply Theorem 7.2.15 to conclude that for each:

$$\lambda\in\left(\frac{6479956}{27\big(e^{14}-e^3\big)},\frac{428335667}{5821794\,e^3}\right)=(0.1995,3.663),$$

problem (7.2.35) *has at least three solutions.*

Finally, let us consider an example where the interval for λ, where three solutions are found, is a smaller one.

Example 7.2.18. *Let $n=2$, $T=8$ and $V(k)=k$ for each $k\in[1,T]$. Then, in this case $V_0=1$ and $V_1=8$.*

Moreover, consider $f(k,t)=\dfrac{k}{k+1}\,g(t)$, where $g(t)$ has been introduced in Example 7.2.16.

So, we can see that $F(k,t)\leqslant\dfrac{8}{9}e^{14}\Big(1+|t|^p\Big)$ for all $p>1$.

Let us consider $c=1$, $d=14$ and $q=3$, we have:

$$\Theta(1):=\frac{\displaystyle\sum_{k=1}^{8}\sup_{|s|\le 1}F(k,s)}{1^3}=\frac{15551}{2520}\,e,$$

$$\Lambda(14):=\frac{\displaystyle\sum_{k=1}^{8}\left(F(k,14)-\sup_{|s|\le 1}F(k,s)\right)}{14^3}=\frac{15551}{6914880}\Big(e^{14}-e\Big).$$

Now, let us consider the problem:

$$\begin{aligned}\Delta^2\Big[\varphi_3\Big(\Delta^2u(k-2)\Big)\Big]&-10\Delta\Big[\varphi_3\Big(\Delta u(k-1)\Big)\Big]\\&+V(k)\varphi_3\big(u(k)\big)-\lambda f\big(k,u(k)\big)=0,\;k\in[1,8],\end{aligned}\tag{7.2.36}$$

coupled with the boundary conditions (7.2.19)-(7.2.20) *for $n=2$.*

In this case, $\rho^3=\dfrac{18225}{36241}<1$.

Moreover, we have:

$$\frac{\Lambda(14)}{(4+20+8^2)\,\rho^3}=\frac{563583791}{11090084544000}\Big(e^{14}-e\Big)\approxeq 61.11>\frac{15551}{2520}e\approxeq 16.67,$$

then, we can apply Theorem 7.2.15 to conclude that for each:

$$\lambda \in \left(\frac{202836480}{15551\left(e^{14}-e\right)}, \frac{2029496}{18894465\, e} \right) = (0.0108459, 0.0395147),$$

problem (7.2.36) *has at least three solutions.*

Remark 7.2.19. *Finally, we want to point out that the bound given in Remark* 7.2.9 *is a good one for the two first considered examples. Indeed, we have:*

$$\lim_{T \to +\infty} \rho^q = \frac{1}{V_0}.$$

If we take $\dfrac{1}{V_0} = \dfrac{1}{124}$ *instead the exact value of* ρ*, the intervals obtained would have suffer only a slight modification.*

In the first case, we would have warranted the existence of three solutions for:

$$\lambda \in \left(\frac{718928}{3\left(e^{14}-e^3\right)}, \frac{147}{2\, e^3} \right) = (0.1993, 3.659),$$

and, in the second one, we would have:

$$\lambda \in \left(\frac{6479956}{27\left(e^{14}-e^3\right)}, \frac{147}{2\, e^3} \right) = (0.1995, 3.659).$$

Thus, we do not lose a lot of information in none of the both cases.

However, for the third example, the bound is not so good. And, if we consider the interval for the choice $\rho = \dfrac{1}{V_0} = 1$*, we would have proved the existence of three solutions for:*

$$\lambda \in \left(\frac{202836480}{15551\left(e^{14}-e\right)}, \frac{840}{15551\, e} \right) = (0.0108459, 0.0198713),$$

so, we have divided the length of the interval by three, which is a bad approximation. This can be caused by the original length of the interval which was already not so big.

Appendix

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A.

Computation in *Mathematica* of the eigenvalues of the $(k, n-k)$ problems

As we have said in Section 2.1, the eigenvalues of operator $T_n[\bar{M}]$ in X_k, previously introduced in (1.1.8), are given by $\lambda \in \mathbb{R}$ such that $W_{n-k}^n[\bar{M}-\lambda](b) = 0$, where $W_k^n[M]$ is defined in (1.1.7). In some cases the analytical expression of the Wronskian is too complicated or we cannot even obtain it. In such cases, we develop a *Mathematica* program to obtain the closest to 0 eigenvalues of $T_n[\bar{M}]$, after verifying that the linear differential equation (1.0.2) is disconjugate for $M = \bar{M}$.

In the sequel, we will describe the program code.

The initial values which we have to give to the program are the following.

- The order of the considered problem, n.
- The extremes of the interval of definition $I = [a, b]$.
- The coefficients of the operator $T_n[M]$, which are introduced as follows $a1 = a_1(t)$, $\dots$, $an = a_n(t) + \bar{M}$.

```
Clear[n, a, b, A, W, h, Sol, Aut, aut, Ly, y, CF]
```

% We delete all the used elements from the memory.

```
n = n; a = a; b = b;
A = {a1,a2,a3,a4,...,an};
```

% We introduce the needed values. In any coefficient is nul, we should write 0 in the correspondent place.

```
If[Length[A] != n,
Print["Lenght of the vector of coefficients does not fit with
    the order of the problem"]]
```

% If the number of coefficients does not fit with the order of the problem, the program show that the introduced values are wrong.

```
Ly = D[y[t], {t, n}];
Do[Ly = Ly + A[[i]] D[y[t], {t, n - i}], {i, 1, n}]
```

% We build the correspondent operator for $\bar{M}$.

```
Print["Ly \[Congruent] ", Ly]
```

% We print out the studied operator.

% In the sequel, the program obtains the fundamental system of solutions given by the functions $y_k[\bar{M}]$.

```
Do [{CF = {}; Do[{CF = Prepend[CF, Derivative[n - i][y][a]
    == 0]}, {i, 1, n}];
```

% We construct a vector of lenght n, whose coefficients are $y^{(k)}(a) = 0$ for

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$k = 1, \ldots, n$.

```
CF = Delete[CF, n - k + 1];  CF = Insert[CF, Derivative[n - k][y][a] == 1, k];
```

% For each k, we delete the element in possition $n-k-1$ of vector CF, $y^{(n-k)}(a) = 0$ and we add $y^{(n-k)}(a) = 1$ at the end of the vector. In this step, we obtain the boundary conditions related to y_k.

```
{Sol} = DSolve[Prepend[CF, Ly == 0], y[t], t]; y[k, t_] = y[t] /. Sol}, {k, 1, n}]
```

% For each $k = 1, \ldots, n$, we solve the differential equation $L\,y = 0$ coupled with the boundary conditions $y^{(h)}(a) = 0$ for $h = 1, \ldots, n$ if $h \neq n-k$ and $y^{(n-k)}(a) = 1$. We denote the solutions by $y[k,t] = y_k[\bar{M}](t)$.

```
10 Plot[y[1, t], {t, 0, 1}]
```

% It plots the graphic of $y_1[\bar{M}](t) = W_1[\bar{M}](t)$.

```
11 If[FindInstance[y[1, t] == 0 && a < t <= b, t] != {},
12 Print["The equation is not disconjugate"]]
```

% If we find a root of $W_1[\bar{M}](t)$ in $(a,b]$, then it shows that the operator is not disconjugate for $\bar{M} \in \mathbb{R}$.

```
13 h = {y[1, t]};
14 Do[{h = Prepend[h, y[k, t]]; W[t_] = Wronskian[h, t]; Print[
      Plot[{0, W[t]}, {t, a,b}]]; If[FindInstance[W[t] == 0 && 0
      a < t <= b, t] != {}, Print["The equation is not
      disconjugate"]]}, {k, 2, n - 1}]
```

% For each $k \geqslant 2$, we build the vector $(y_1[\bar{M}](t), \ldots, y_k[\bar{M}](t))$ and calculate $W_k[\bar{M}](t)$. Moreover, we plot the graphic of $W_k[\bar{M}](t)$ and if a root on some of the wronskians is found, then it shows that the operator is not disconjugate for $\bar{M}$.
% Now, the calculus of the eigenvalues begins.

```
15 Ly = Ly - \[Lambda] y[t];
```

% We denote Ly = $L\,y(t) - \lambda\,y(t)$

```
16 Do [{CF = {}; Do[{CF = Prepend[CF, Derivative[n - i][y][a] ==
      0]}, {i, 1, n}]; CF = Delete[CF, n - k + 1]; CF = Insert[
      CF, Derivative[n - k][y][a] == 1, k]; {Sol} = DSolve[
      Prepend[CF, Ly == 0], y[t], t]; y[k, t_, \[Lambda]_] = y[t
      ] /. Sol; Print["y", k, "[t]=", y[k, t, \[Lambda]]]}, {k,
      1, n}]
```

% We repeat the calculus of the line 9 to obtain $y_k[\bar{M} - \lambda](t)$ for each $k = 1, \ldots, n$.
% We print out the fundamental system of solutions, $y_k[\bar{M} - \lambda](t)$.

```
17 Print["k=", n - 1]
18 {Aut} = FindRoot[ y[1, b, \[Lambda]], {\[Lambda], 0.1}];
19 aut = \[Lambda] /. Aut;
```

% We use the funtion "FindRoot" to find the closest to 0 eigenvalue, by finding zeros on $y_1[\bar{M} - \lambda](b)$.

```
20 If[ Abs[Im[aut]] < 10^(-4), Print["\[Lambda]=", Re[aut]],
21 Print["It has been a problem on the calculus of the first
      eigenvalue"]];
```

% We look for real eigenvalues, thus we imposse that $|Im(aut)| < 10^{-4}$ to indicate that the real eigenvalue coincide with the real part of the found eigenvalue. Otherwise, probably the obtained eigenvalue is a complex number and the program indicates that it has been a mistake.

```
22 If[Mod[n, 2] == 0, Print[Plot[{0, Piecewise[{{y[1, b, \[
```

```
     Lambda]^n], \[Lambda] > 0}, {y[1, b, -\[Lambda]^n], \[
     Lambda] < 0}, {y[1, b, 0], \[Lambda] == 0}}]}, {\[Lambda
     ], -2 n, 2 n}]], Print[Plot[{0, y[1, b, \[Lambda]^n] },
     {\[Lambda], -2 n, 2 n}]]]
     % We plot the graphic of the function y1[M̄ + λ^n](b) whenever λ < 0 and
     y1[M̄ − λ^n](b) whenever λ ⩾ 0 if n is even. If n is odd, then we plot the function
     y1[M̄ − λ^n](b). In order to obtain, by using the graphic, the desired eigenvalues we
     have to elevate to the n^th power and conserve the sign.
23 h = {y[1, t, \[Lambda]]};
24 Do[{Print["k=", n - k]; h = Prepend[h, y[k, t, \[Lambda]]];
     W[t_, \[Lambda]_] = Wronskian[h, t]; {Aut} = FindRoot[W[
     b, \[Lambda]], {\[Lambda], 0.1}]; aut = \[Lambda] /. Aut;
       If[Abs[Im[aut]] < 10^(-4), Print["\[Lambda]=", Re[aut]],
       Print["It has been a problem on the calculus of the
      first eigenvalue"]]; If[Mod[n, 2] == 0, Print[Plot[{0,
     Piecewise[{{W[b, \[Lambda]^n], \[Lambda] > 0}, {W[b, -\[
     Lambda]^n], \[Lambda] < 0}, {W[b, 0], \[Lambda] == 0}}]},
      {\[Lambda], -2 n, 2 n}]], Print[Plot[{0, W[b, \[Lambda]^
     n]}, {\[Lambda], -2 n, 2 n}]]]}, {k, 2, n - 1}]
     % For each k, we obtain the correspondent wronskian. Then, we repeat the steps from
     17 to 22 to obtain the different eigenvalues.
```

In the sequel, we show an example where we study a problem with non-constant coefficients. We consider:

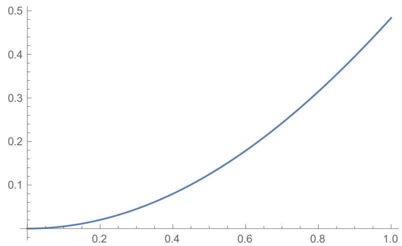
$$T_3^2[M]\,u(t) \equiv u^{(3)}(t) + t\,u'(t) + M\,u(t), \quad t \in [0,1], \tag{A.1}$$

previously introduced in (2.1.17).

In this case, $n = 3$, $a = 0$, $b = 1$, $a1 = 0$, $a2 = t$ and $a3 = 0$.

Once that we have executed the program, we obtain:

Out[1] $Ly = t\,y'[t] + y^{(3)}[t]$

Out[2] 

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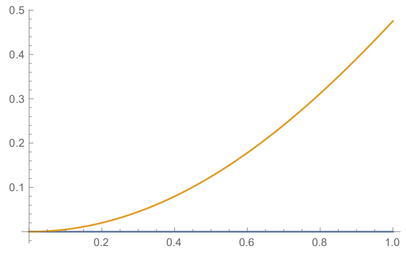
Out[3] 

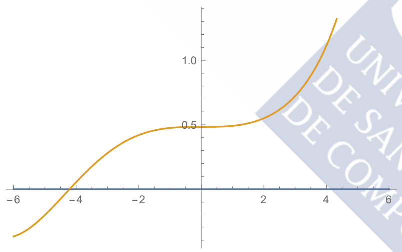
Out[4] $y1[t] = \frac{1}{2}t^2 HypergeometricPFQ\left[\left\{\frac{2}{3} - \frac{\lambda}{3}\right\}, \left\{\frac{4}{3}, \frac{5}{3}\right\}, -\frac{t^3}{9}\right]$

Out[5] $y2[t] = tHypergeometricPFQ\left[\left\{\frac{1}{3} - \frac{\lambda}{3}\right\}, \left\{\frac{2}{3}, \frac{4}{3}\right\}, -\frac{t^3}{9}\right]$

Out[6] $y3[t] = HypergeometricPFQ\left[\left\{-\frac{\lambda}{3}\right\}, \left\{\frac{1}{3}, \frac{2}{3}\right\}, -\frac{t^3}{9}\right]$

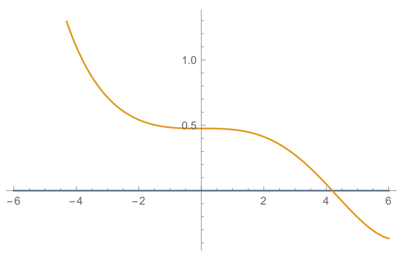
Out[7] $k = 2$

Out[8] $\lambda = -74.7543$

Out[9] 

Out[10] $k = 1$

Out[11] $\lambda = 73.7543$

Out[12] 

Hence, we have that $\lambda_1 = (4.19369)^3 = 73.7543$ is the least positive eigenvalue of $T_3[0]$ in X_1 and $\lambda_2 = -(4.21255)^3 = -74.7543$ is the biggest negative eigenvalue in X_2.

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B. Non-disconjugacy criteria for operator $T_8^1[M] = \dfrac{\mathrm{d}^8}{\mathrm{d}\,t^8} + 11^4 \dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4} + M$

The aim of this Appendix is to develop the analytic calculus which we have mentioned in Remark 2.2.12. We prove the different steps in the following order:

1. $W_4^8[M]$ does not have any zero on $(0,1]$ for every $M \geqslant 0$.
2. $W_1^8[M]$ does not have any zero on $(0,1]$ for every $M \leqslant M_1 = -\dfrac{235165923}{4}$.
   - 2.1. $W_1^8[M_1](t) \neq 0$ for all $t \in (0,1]$.
   - 2.2. If $W_1^8[\bar{M}_1](t) \neq 0$ for all $t \in (0,1]$ for a fixed $\bar{M}_1 \in \mathbb{R}$, then $W_1[M]$ does not have any zero on $(0,1]$ for every $M \leqslant \bar{M}_1$.
3. $W_2^8[M]$ does not have any zero on $(0,1]$ for $M = M_1$.
4. $W_3^8[M]$ does not have any zero on $(0,1]$ for $M = 0$.

In the sequel, we describe the different calculus following the previous steps related to the equation.

$$T_8^1[M]\,u(t) = u^{(8)}(t) + 11^4\,u^{(4)}(t) + M\,u(t) = 0,\ t \in I = [0,1], \tag{B.1}$$

which we have introduced in (2.2.2).

1. $W_4^8[M]$ does not have any zero on $(0,1]$ for every $M \geqslant 0$.

If $W_4^8[M](c) = 0$ for any $c \in (0,1]$, from Proposition 1.1.10, there exists $u \in C^8\big([0,c]\big)$ a non-trivial function satisfying (B.1) coupled with the following boundary conditions:

$$\begin{aligned} u(0) = u'(0) = u''(0) = u^{(3)}(0) = 0,\\ u(c) = u'(c) = u''(c) = u^{(3)}(c) = 0. \end{aligned} \tag{B.2}$$

Then, multiplying (B.1) by u and integrating on $[0,c]$, we have:

$$\int_0^c \left(u^{(8)}(t) + 11^4 u^{(4)}(t)\right) u(t)\,\mathrm{d}\,t = -M \int_0^c u^2(t)\,\mathrm{d}\,t.$$

Now, integrating by parts and taking into account the boundary conditions (B.2), we obtain

$$\int_0^c \left(u^{(4)}(t)\right)^2 \mathrm{d}\,t + 11^4 \int_0^c \left(u''(t)\right)^2 \mathrm{d}\,t = -M \int_0^c u^2(t)\,\mathrm{d}\,t. \tag{B.3}$$

Since u is a non-trivial function on $[0,c]$, from the boundary conditions, we conclude that u'' is also a non-trivial function on $[0,c]$. Thus, (B.3) cannot be fulfilled for any $M \geqslant 0$ and the first item is proved.

2. $W_1^8[M]$ does not have any zero on $(0,1]$ for every $M \leqslant M_1 = -\dfrac{235165923}{4}$.

2.1. $W_1^8[M_1](t) \neq 0$ for all $t \in (0,1]$.

Consider $M_1 = -\dfrac{235165923}{4}$, we have $W_1^8[M_1](t) = f_1(t)$, where:

$$\begin{aligned} f_1(t) = &-\frac{\sin\left(\frac{9\,t}{\sqrt[4]{2}}\right)}{15456258\sqrt[4]{2}} + \frac{\sinh\left(\frac{9\,t}{\sqrt[4]{2}}\right)}{15456258\sqrt[4]{2}} + \frac{\cos\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\sinh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{10601\cdot 71686^{3/4}} \\ &- \frac{\sin\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\cosh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{10601\cdot 71686^{3/4}}. \end{aligned} \tag{B.4}$$

We will prove that this function is strictly positive for $t \in (0,1]$.

We know that $f_1(0) = f_1'(0) = f_1''(0) = f_1'''(0) = f_1^{(4)}(0) = 0$, and we have:

$$f_1^{(5)}(t) = \frac{-81\cos\left(\frac{9\,t}{\sqrt[4]{2}}\right) + 81\cosh\left(\frac{9\,t}{\sqrt[4]{2}}\right) + 2\sqrt{35843}\sin\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\sinh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{42404\sqrt{2}},$$

this function is positive on $(0,1]$ while $\sin\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right) > 0$, i. e., this property is fulfilled for every $t < \dfrac{2^{3/4}\pi}{\sqrt[4]{35843}} \approxeq 0.383991$. Then, all previous derivatives also are, since they all vanish at $t = 0$.

Now, we study the following function:

$$f_1^{(3)}(t) = \frac{\cos\left(\frac{9\,t}{\sqrt[4]{2}}\right)}{42404} + \frac{\cosh\left(\frac{9\,t}{\sqrt[4]{2}}\right)}{42404} - \frac{\cos\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\cosh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{21202},$$

which is positive whenever $\cos\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right) < 0$, i.e., for:

$$t \in \left(\frac{\pi}{\sqrt[4]{71686}}, \frac{3\pi}{\sqrt[4]{71686}}\right) \approxeq (0.191996, 0.575987).$$

Since, it also is positive for $t < \dfrac{2^{3/4}\pi}{\sqrt[4]{35843}}$, it is positive for $0 < t < \dfrac{3\pi}{\sqrt[4]{71686}}$, and the derivatives of less order also do.

Let us consider the following expression:

$$f_1'(t) = -\frac{\cos\left(\dfrac{9\,t}{\sqrt[4]{2}}\right)}{1717362\sqrt{2}} + \frac{\cosh\left(\dfrac{9\,t}{\sqrt[4]{2}}\right)}{1717362\sqrt{2}} - \frac{\sin\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\sinh\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{10601\sqrt{71686}},$$

that is positive if $\sin\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right) < 0$, i.e., for:

$$t \in \left(\frac{2^{3/4}\pi}{\sqrt[4]{35843}}, \frac{2\,2^{3/4}\pi}{\sqrt[4]{35843}}\right) \approxeq (0.38399, 0.76798).$$

For $t \in [0.75, 0.82]$ we can make the following bound:

$$f_1'(t) > -\frac{\sinh\left(\dfrac{41\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)\sin\left(\dfrac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{10601\sqrt{71686}} - \frac{\cos\left(\dfrac{27}{4\sqrt[4]{2}}\right)}{1717362\sqrt{2}} + \frac{\cosh\left(\dfrac{27}{4\sqrt[4]{2}}\right)}{1717362\sqrt{2}},$$

and this function is positive in this interval.

Again, we can conclude that $f_1'(t) > 0$ on $(0, 0.82)$ and also $f_1(t) > 0$ in this interval.

Now, we have to study $f_1(t)$ on $[0.85, 1]$.

We have that $f(1)$ is given by the next expression:

$$\frac{\sinh\left(\frac{9}{\sqrt[4]{2}}\right) - \sin\left(\frac{9}{\sqrt[4]{2}}\right)}{15456258\sqrt[4]{2}} + \frac{\cos\left(\frac{\sqrt[4]{35843}}{2^{3/4}}\right)\sinh\left(\frac{\sqrt[4]{35843}}{2^{3/4}}\right) - \sin\left(\frac{\sqrt[4]{35843}}{2^{3/4}}\right)\cosh\left(\frac{\sqrt[4]{35843}}{2^{3/4}}\right)}{10601\cdot 71686^{3/4}} > 0,$$

then, if we have that $f_1'(t) < 0$ on a neighbourhood of $t = 1$, we can ensure that $f_1(t) > 0$ in such a neighbourhood.

For $t \in [0.96, 1]$, we can bound superiorly $f_1'(t)$, as follows:

$$f_1'(t) < -\frac{\sinh\left(\dfrac{12\sqrt[4]{71686}}{25}\right)\sin\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{10601\sqrt{71686}} - \frac{\cos\left(\dfrac{9}{\sqrt[4]{2}}\right) + \cosh\left(\dfrac{9}{\sqrt[4]{2}}\right)}{1717362\sqrt{2}} < 0.$$

Then, for $t \in [0.92, 0.96]$ we have:

$$f_1'(t) < -\frac{\sinh\left(\dfrac{23\sqrt[4]{35843}}{25\cdot 2^{3/4}}\right)\sin\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{10601\sqrt{71686}} - \frac{\cos\left(\dfrac{108\cdot 2^{3/4}}{25}\right) + \cosh\left(\dfrac{108\cdot 2^{3/4}}{25}\right)}{1717362\sqrt{2}} < 0.$$

So, we can conclude, that $f_1(t) > 0$ for $t \in [0.92, 1]$.

Now, we will see that:

$$f_1''(t) = \frac{\sin\left(\frac{9\,t}{\sqrt[4]{2}}\right) + \sinh\left(\frac{9\,t}{\sqrt[4]{2}}\right)}{190818 \cdot 2^{3/4}} - \frac{\cos\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\sinh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}}$$
$$- \frac{\sin\left(\frac{\sqrt[4]{35843}t}{2^{3/4}}\right)\cosh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}},$$

is a negative function for all $t \in [0.84, 0.92]$ and, then we will be able to affirm that $f_1'(t)$ has only a zero on that interval, which implies that $f_1(t)$ cannot have any, since $f_1'(0.84) > 0$ and $f_1'(0.92) < 0$.

For $t \in [0.87, 0.92]$, we have that $f_1''(t)$ is bounded superiorly by:

$$-\frac{\sinh\left(\frac{87\sqrt[4]{35843}}{100 \cdot 2^{3/4}}\right)\cos\left(\frac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}} + \frac{\sin\left(\frac{207}{25\sqrt[4]{2}}\right) + \sinh\left(\frac{207}{25\sqrt[4]{2}}\right)}{190818 \cdot 2^{3/4}}$$
$$-\frac{\sin\left(\frac{87\sqrt[4]{35843}}{100\ 2^{3/4}}\right)\cosh\left(\frac{87\sqrt[4]{35843}}{100 \cdot 2^{3/4}}\right)}{21202\sqrt[4]{71686}} < 0.$$

Eventually, for $t \in [0.82, 0.87]$, $f_1''(t)$ is bounded superiorly by:

$$-\frac{\sinh\left(\frac{41\sqrt[4]{35843}}{50 \cdot 2^{3/4}}\right)\cos\left(\frac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}} + \frac{\sin\left(\frac{783}{100\sqrt[4]{2}}\right) + \sinh\left(\frac{783}{100\sqrt[4]{2}}\right)}{190818 \cdot 2^{3/4}}$$
$$-\frac{\sin\left(\frac{41\sqrt[4]{35843}}{50 \cdot 2^{3/4}}\right)\cosh\left(\frac{41\sqrt[4]{35843}}{50 \cdot 2^{3/4}}\right)}{21202\sqrt[4]{71686}} < 0,$$

so, $f_1''(t) < 0$ on $[0.82, 0.87]$.

Hence, we can conclude that $f_1(t) > 0$ on $(0, 1]$.

Thus, $W_1[M_1](t) \neq 0$ for all $t \in (0, 1]$.

<u>2.</u> $W_1^8[M]$ does not have any zero on $(0, 1]$ for every $M \leqslant M_1 = -\dfrac{235165923}{4}$.

<u>2.2.</u> If $W_1^8[\bar{M}_1](t) \neq 0$ for all $t \in (0, 1]$ for a fixed $\bar{M}_1 \in \mathbb{R}$, then $W_1^8[M]$ does not have any zero on $(0, 1]$ for every $M \leqslant \bar{M}_1$.

Let us assume that there exists $\bar{\bar{M}}_1 \leqslant \bar{M}_1$ such that $W_1^8[\bar{\bar{M}}_1]$ has at least a zero on $(0, 1]$. Let us choose $c \in (0, 1]$ such that $W_1^8[\bar{\bar{M}}_1](c) = 0$ and $W_1^8[\bar{\bar{M}}_1](t) \neq 0$ for all $t \in (0, c)$.

Then, $\bar{u}(t) = W_1^8[\bar{\bar{M}}_1](t)$ satisfies the equation (B.1) on $(0, c)$ for $M = \bar{\bar{M}}_1$ coupled with the boundary conditions:

$$\begin{aligned} \bar{u}(0) = \bar{u}'(0) = \bar{u}''(0) = \bar{u}^{(3)}(0) = \bar{u}^{(4)}(0) = \bar{u}^{(5)}(0) = \bar{u}^{(6)}(0) = 0, \\ \bar{u}(c) = 0. \end{aligned} \tag{B.5}$$

Now, consider the related Green's function of operator $T_8^1[\bar{M}_1]$ related to the boundary conditions (B.5), $g_{\bar{M}_1}(t, s)$.

The operator $T_8^1[M]$ is self-adjoint, thus $T_8^{1*}[M] = T_8^1[M]$ for all $M \in \mathbb{R}$. This fact, coupled with Lemma 3.5.3, implies that $\widehat{w}_{\bar{M}_1}(s) = \dfrac{\partial^7}{\partial t^7} g_{\bar{M}_1}(t, s)_{|t=0}$ satisfies equation (B.1) for $M = \bar{M}_1$ and $t \in (0, c)$ coupled with the boundary conditions:

$$\begin{aligned} \widehat{w}_{\bar{M}_1}(0) = -1, \\ \widehat{w}_{\bar{M}_1}(c) = \widehat{w}'_{\bar{M}_1}(c) = \widehat{w}''_{\bar{M}_1}(c) = \widehat{w}^{(3)}_{\bar{M}_1}(c) = \widehat{w}^{(4)}_{\bar{M}_1}(c) = \widehat{w}^{(5)}_{\bar{M}_1}(c) = \widehat{w}^{(6)}_{\bar{M}_1}(c) = 0. \end{aligned}$$

Furthermore, since the operator is self-adjoint and taking into account that $X_1^* = X_7$, we have that $W_1^8[M](t) = 0$ if, and only if, $W_7^8[M](t) = 0$ for any $t \in (0, 1]$. Thus, $W_7^8[\bar{M}_1](t) \neq 0$ for any $t \in (0, 1]$. Using this fact, we have that $\widehat{w}_{\bar{M}_1}(t) < 0$ for $t \in [0, c)$. Otherwise, if there exists $c^* \in (0, c)$ such that $\widehat{w}_{\bar{M}_1}(c^*) = 0$, then $\bar{w}(t) = \widehat{w}_{\bar{M}_1}(t - c^*)$ is a solution of (B.1) coupled with the boundary conditions:

$$\begin{aligned} \bar{w}(0) = 0, \\ \bar{w}(c - c^*) = \bar{w}'(c - c^*) = \cdots = \bar{w}^{(6)}(c - c^*) = 0. \end{aligned} \tag{B.6}$$

Hence, from Proposition 1.1.10, $W_7^8[\bar{M}_1](c - c^*) = 0$ which is a contradiction.

We can rewrite (B.1), to obtain that $\bar{u}$ is a solution of the following equation:

$$u^{(8)}(t) + 11^4 u^{(4)}(t) + \bar{M}_1\, u(t) = \big(\bar{M}_1 - \bar{\bar{M}}_1\big) u(t),$$

thus, from the properties of the Green's function, we have:

$$\bar{u}(t) = \int_0^c g_{\bar{M}_1}(t, s) \big(\bar{M}_1 - \bar{\bar{M}}_1\big) \bar{u}(s) \, \mathrm{d}\, s.$$

In particular,

$$\bar{u}^{(7)}(0) = \int_0^c \widehat{w}_{\bar{M}_1}(s) \big(\bar{M}_1 - \bar{\bar{M}}_1\big) \bar{u}(s) \, \mathrm{d}\, s.$$

Since $\bar{u}$ is of constant sign and it satisfies the boundary conditions (B.5), we have that

$$\widehat{w}_{\bar{M}_1}(s) \left(\bar{M}_1 - \bar{\bar{M}}_1\right) \geqslant 0,$$

thus, clearly, $\bar{M}_1 - \bar{\bar{M}}_1 \leqslant 0$, which is a contradiction with the assumption $\bar{\bar{M}}_1 \leqslant \bar{M}_1$.

Hence, from steps 2.1 and 2.2, we have proved that $W_1^8[M]$ does not have any zero on $(0, 1]$ for every $M \leqslant M_1$.

3. $W_2^8[M]$ does not have any zero on $(0,1]$ for $M = M_1$.

For $M_1 = -\dfrac{235165923}{4}$, we have:

$$W_2^8[\tilde{M}](t) = f_1(t)\, f_1''(t) - \left(f_1'(t)\right)^2,$$

where f_1 has been introduced in (B.4).

We will see that $W_2^8[M_1](t)$ is a strictly negative function for $t \in (0,1]$.

While $f_1''(t) < 0$ the previous assertion is true, since $f_1(t) > 0$ for every $t \in (0,1]$.

We have seen that $f_1''(t) < 0$ for every $t \in [0.84, 0.92]$, now, for $t \in [0.92, 0.96]$, we have:

$$f_1''(t) < -\frac{\sin\left(\dfrac{23\sqrt[4]{35843}}{25\cdot 2^{3/4}}\right)\cosh\left(\dfrac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}} + \frac{\sin\left(\dfrac{108\cdot 2^{3/4}}{25}\right)+\sinh\left(\dfrac{108\cdot 2^{3/4}}{25}\right)}{190818\cdot 2^{3/4}} < 0.$$

Also,

$$\begin{aligned} 0 > f_1''(1) = & \frac{\sin\left(\dfrac{9}{\sqrt[4]{2}}\right)+\sinh\left(\dfrac{9}{\sqrt[4]{2}}\right)}{190818\cdot 2^{3/4}} - \frac{\cos\left(\dfrac{\sqrt[4]{35843}}{2^{3/4}}\right)\sinh\left(\dfrac{\sqrt[4]{35843}}{2^{3/4}}\right)}{21202\sqrt[4]{71686}} \\ & - \frac{\sin\left(\dfrac{\sqrt[4]{35843}}{2^{3/4}}\right)\cosh\left(\dfrac{\sqrt[4]{35843}}{2^{3/4}}\right)}{21202\sqrt[4]{71686}}, \end{aligned}$$

then, if $f_1'''(t) > 0$ for $t \in [0.96, 1]$, we have that $f_1''(t) < 0$ in this interval.

$$f_1'''(t) = \frac{\cos\left(\dfrac{9\,t}{\sqrt[4]{2}}\right)}{42404} + \frac{\cosh\left(\dfrac{9\,t}{\sqrt[4]{2}}\right)}{42404} - \frac{\cos\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\cosh\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{21202},$$

which is positive while $\cos\left(\dfrac{\sqrt[4]{35843}\,t}{2^{3/4}}\right) < 0$, this is true, in particular, if $t \in \left(\dfrac{5\,\pi}{\sqrt[4]{71686}}, 1\right]$, where $\dfrac{5\,\pi}{\sqrt[4]{71686}} \approxeq 0.959978 < 0.96$.

Now, for $t \in [0.79, 0.84]$, we have $f_1''(t)$ bounded from above by:

$$\begin{aligned} & \frac{\sin\left(\dfrac{189}{25\sqrt[4]{2}}\right)+\sinh\left(\dfrac{189}{25\sqrt[4]{2}}\right)}{190818\cdot 2^{3/4}} - \frac{\sin\left(\dfrac{79\sqrt[4]{35843}}{100\cdot 2^{3/4}}\right)\cosh\left(\dfrac{79\sqrt[4]{35843}}{100\cdot 2^{3/4}}\right)}{21202\sqrt[4]{71686}} \\ & - \frac{\cos\left(\dfrac{21\sqrt[4]{35843}}{25\cdot 2^{3/4}}\right)\sinh\left(\dfrac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}} < 0. \end{aligned}$$

While $\sin\left(\frac{\sqrt[4]{35843}t}{2^{3/4}}\right) \geqslant 0$, i.e. for $t \in \left[\frac{2\cdot 2^{3/4}\pi}{\sqrt[4]{35843}}, 0.79\right]$, where $\frac{2\cdot 2^{3/4}\pi}{\sqrt[4]{35843}} \approxeq 0.767982$, we have:

$$f_1''(t) < \frac{\sin\left(\frac{711}{100\sqrt[4]{2}}\right) + \sinh\left(\frac{711}{100\sqrt[4]{2}}\right)}{190818\cdot 2^{3/4}} - \frac{\cos\left(\frac{79\sqrt[4]{35843}}{100\cdot 2^{3/4}}\right)\sinh\left(\frac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}} < 0.$$

Finally, for $t \in \left[0.75, \frac{2\cdot 2^{3/4}\pi}{\sqrt[4]{35843}}\right]$, we have that $f_1''(t)$ follows the expression:

$$-\frac{\cos\left(\frac{3\sqrt[4]{35843}}{4\cdot 2^{3/4}}\right)\sinh\left(\frac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{21202\sqrt[4]{71686}} + \frac{\sin\left(\frac{693}{100\sqrt[4]{2}}\right) + \sinh\left(\frac{693}{100\sqrt[4]{2}}\right)}{190818\cdot 2^{3/4}}$$
$$-\frac{\sin\left(\frac{3\sqrt[4]{35843}}{4\cdot 2^{3/4}}\right)\cosh\left(\frac{77\sqrt[4]{35843}}{100\cdot 2^{3/4}}\right)}{21202\sqrt[4]{71686}} < 0.$$

So, we have that $f_1''(t) < 0$ for $t \in [0.75, 1]$, and, then also does $W_2^8[\tilde{M}](t) < 0$.

Now, we have to see what happens on $[0, 0.75]$. We compare this two functions:

$$f_{11}(t) = f_1(t)\, f_1''(t) \;\text{ and }\; f_{12}(t) = \left(f_1'(t)\right)^2.$$

We know that $f_{11}(0) = f_{12}(0)$, we will see that both functions are convex on a neighbourhood of $t = 0$.

$f_{11}''(t) = \left(f_1''(t)\right)^2 + 2\, f_1'(t)\, f_1'''(t) + f_1(t)\, f_1^{(4)}(t)$ and $f_{12}''(t) = 2\left(f_1''(t)\right)^2 + f_1'''(t)\, f_1'(t)$,

then both derivatives will be positives while:

$$f_1^{(4)}(t) = -\frac{9\sin\left(\frac{9t}{\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}} + \frac{9\sinh\left(\frac{9t}{\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}} - \frac{\sqrt[4]{35843}\cos\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\sinh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{21202\cdot 2^{3/4}}$$
$$+\frac{\sqrt[4]{35843}\sin\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)\cosh\left(\frac{\sqrt[4]{35843}\,t}{2^{3/4}}\right)}{21202\cdot 2^{3/4}} > 0.$$

We know that $f_1^{(5)}(t) > 0$ for $t < \frac{2^{3/4}\pi}{\sqrt[4]{35843}} \approxeq 0.383991$, so $f_1^{(4)}(t)$ is also positive.

For $t \in [0.38, 0.46]$ we have the following boundedness:

$$f_1^{(4)}(t) > \frac{\sqrt[4]{35843}\sin\left(\frac{23\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)\cosh\left(\frac{\sqrt[4]{35843}t}{2^{3/4}}\right)}{21202\cdot 2^{3/4}} - \frac{9\sin\left(\frac{171}{50\sqrt[4]{2}}\right) - 9\sinh\left(\frac{171}{50\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}}$$
$$-\frac{\sqrt[4]{35843}\cos\left(\frac{23\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)\sinh\left(\frac{19\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}} > 0.$$

For $t \in [0.46, 0.49]$:

$$f_1^{(4)}(t) > \frac{\sqrt[4]{35843} \sin\left(\frac{49\sqrt[4]{35843}}{100 \cdot 2^{3/4}}\right) \cosh\left(\frac{\sqrt[4]{35843} t}{2^{3/4}}\right)}{21202 \cdot 2^{3/4}} - \frac{9 \sin\left(\frac{207}{50\sqrt[4]{2}}\right) - 9 \sinh\left(\frac{207}{50\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}}$$
$$- \frac{\sqrt[4]{35843} \cos\left(\frac{49\sqrt[4]{35843}}{100 \cdot 2^{3/4}}\right) \sinh\left(\frac{23\sqrt[4]{35843}}{50 \cdot 2^{3/4}}\right)}{21202 \cdot 2^{3/4}} > 0.$$

Finally, for $t \in [0.49, 0.5]$ we have:

$$f_1^{(4)}(t) > \frac{\sqrt[4]{35843} \sin\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right) \cosh\left(\frac{\sqrt[4]{35843}\, t}{2^{3/4}}\right)}{21202 \cdot 2^{3/4}} - \frac{9 \sin\left(\frac{441}{100\sqrt[4]{2}}\right) - 9 \sinh\left(\frac{441}{100\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}}$$
$$- \frac{\sqrt[4]{35843} \cos\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right) \sinh\left(\frac{49\sqrt[4]{35843}}{100 \cdot 2^{3/4}}\right)}{21202 \cdot 2^{3/4}} > 0,$$

hence, $f_1^{(4)}(t) > 0$, for $t \in [0, 0.5]$ and then ${f_1}_1''(t) > 0$ and ${f_1}_2''(t) > 0$.

We can see that the 11 first derivatives of $W_2^8[\bar{M}(t)]$ are null at $t = 0$, and:

$$\frac{\mathrm{d}^{12}}{\mathrm{d}\,t^{12}} W_2^8[\bar{M}(t)]_{|t=0} = -132 \left(f_1^{(7)}(0)\right)^2 = -132,$$

so, in a neighbourhood of $t = 0$, ${f_1}_2(t) < {f_1}_1(t)$.

Also:

$$W_2^8[\bar{M}](0.5) = \frac{1}{211425830888108184} \left(-35843 \cosh^2\left(\frac{9}{2\sqrt[4]{2}}\right) + \frac{35843}{2} \cosh\left(\frac{9}{\sqrt[4]{2}}\right) - \frac{3}{2}\left(27095 + 4374 \cos\left(\frac{\sqrt[4]{35843}}{2^{3/4}}\right)\right)\right.$$
$$+ \cosh\left(\frac{9}{2\sqrt[4]{2}}\right)\left(71686 \cos\left(\frac{9}{2\sqrt[4]{2}}\right) + 324\sqrt{35843} \sin\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right) \sinh\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right)\right) + 7101 \cosh\left(\frac{\sqrt[4]{35843}}{2^{3/4}}\right)$$
$$+ \frac{3}{2}\left(6\sqrt[4]{35843}\left(\sinh\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right)\left(36\sqrt[4]{35843} \sin\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right) \cos\left(\frac{9}{2\sqrt[4]{2}}\right)\right.\right.\right.$$
$$\left. - \sqrt{2} \cos\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right)\left(\left(81 + \sqrt{35843}\right) \sin\left(\frac{9}{2\sqrt[4]{2}}\right) - \left(\sqrt{35843} - 81\right) \sinh\left(\frac{9}{2\sqrt[4]{2}}\right)\right)\right)$$
$$\left.\left.\left.\left. + \sqrt{2} \sin\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right) \cosh\left(\frac{\sqrt[4]{35843}}{2 \cdot 2^{3/4}}\right)\left(\left(81 - \sqrt{35843}\right) \sin\left(\frac{9}{2\sqrt[4]{2}}\right) + \left(81 + \sqrt{35843}\right) \sinh\left(\frac{9}{2\sqrt[4]{2}}\right)\right)\right)\right)\right)\right)$$
$$< 0.$$

Hence ${f_1}_2(0.5) < {f_1}_1(0.5)$ and, then we can conclude that ${f_1}_2(t) < {f_1}_1(t)$ for all $t \in [0, 0.5]$, so, we can affirm $W_2^8[M_1(t)] < 0$ for $t \in [0, 0.5]$.

We only have to see, what happens for $t \in [0.5, 0.75]$.

Now, we study $\dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} W_2^8[\bar{M}(t)] = -\left(f_1''(t)\right)^2 + f_1(t)\, f_1^{(4)}(t)$, while $f_1^{(4)}(t) < 0$, that function is also negative.

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For $t \in [0.65, 0.75]$ we have:

$$f_1^{(4)}(t) < \frac{9\sinh\left(\frac{27}{4\sqrt[4]{2}}\right) - 9\sin\left(\frac{27}{4\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}} - \frac{\sqrt[4]{35843}\cos\left(\frac{13\sqrt[4]{35843}}{20\cdot 2^{3/4}}\right)\sinh\left(\frac{13\sqrt[4]{35843}}{20\, 2^{3/4}}\right)}{21202\cdot 2^{3/4}}$$
$$+ \frac{\sqrt[4]{35843}\sin\left(\frac{3\sqrt[4]{35843}}{4\cdot 2^{3/4}}\right)\cosh\left(\frac{13\sqrt[4]{35843}}{20\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}} < 0.$$

Now, for $t \in [0.57, 0.65]$ we have the following boundedness:

$$f_1^{(4)}(t) < \frac{9\sinh\left(\frac{117}{20\sqrt[4]{2}}\right) - 9\sin\left(\frac{117}{20\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}} - \frac{\sqrt[4]{35843}\cos\left(\frac{57\sqrt[4]{35843}}{100\cdot 2^{3/4}}\right)\sinh\left(\frac{13\sqrt[4]{35843}}{20\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}}$$
$$+ \frac{\sqrt[4]{35843}\sin\left(\frac{13\sqrt[4]{35843}}{20\cdot 2^{3/4}}\right)\cosh\left(\frac{57\sqrt[4]{35843}}{100\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}} + 6 < 0.$$

For, $t \in [0.54, 0.57]$, we obtain:

$$f_1^{(4)}(t) < \frac{9\sinh\left(\frac{513}{100\sqrt[4]{2}}\right) - 9\sin\left(\frac{513}{100\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}} - \frac{\sqrt[4]{35843}\cos\left(\frac{27\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)\sinh\left(\frac{57\sqrt[4]{35843}}{100\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}}$$
$$+ \frac{\sqrt[4]{35843}\sin\left(\frac{27\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)\cosh\left(\frac{27\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}} < 0.$$

If we take $t \in [0.525, 0.54]$, we have:

$$f_1^{(4)}(t) < \frac{9\sinh\left(\frac{243}{50\sqrt[4]{2}}\right) - 9\sin\left(\frac{243}{50\sqrt[4]{2}}\right)}{42404\sqrt[4]{2}} - \frac{\sqrt[4]{35843}\cos\left(\frac{21\sqrt[4]{35843}}{40\cdot 2^{3/4}}\right)\sinh\left(\frac{27\sqrt[4]{35843}}{50\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}}$$
$$+ \frac{\sqrt[4]{35843}\sin\left(\frac{21\sqrt[4]{35843}}{40\cdot 2^{3/4}}\right)\cosh\left(\frac{21\sqrt[4]{35843}}{40\cdot 2^{3/4}}\right)}{21202\cdot 2^{3/4}} < 0.$$

So, we only need to see if the Wronskian is negative for $t \in [0.5, 0.525]$.

We will see that $\frac{\mathrm{d}}{\mathrm{d}\,t} W_2^8[M_1](t) = -f_1'(t)\, f_1''(t) + f_1(t)\, f_1^{(3)}(t) < 0$ for $t \in [0.5, 0.525]$, then, $\frac{\mathrm{d}}{\mathrm{d}\,t} W_2^8[M_1](t) < 0$ for $t \in [0.5, 0.75]$ and also does $W_2^8[M_1](t) < 0$.

Since $f_1^{(3)}(t) > 0$, we know that $f_1(t)$, $f_1'(t)$ and $f_1''(t)$ are increasing functions for $t \in [0, 0.525]$, then we can make the following bound:

$$-f_1'(t)\, f_1''(t) + f_1(t)\, f_1^{(3)}(t) < -f_1'(0.5)\, f_1''(0.5) + f_1(0.525)\, f_1^{(3)}(t), \quad t \in [0.5, 0.525],$$

also, we have:

$$f_1^{(3)}(t) < \frac{\cos\left(\frac{189}{40\sqrt[4]{2}}\right) + \cosh\left(\frac{189}{40\sqrt[4]{2}}\right)}{42404} - \frac{\cos\left(\frac{\sqrt[4]{35843}}{2\cdot 2^{3/4}}\right)\cosh\left(\frac{21\sqrt[4]{35843}}{40\cdot 2^{3/4}}\right)}{21202},$$

and, finally:

$$-f_1'(0.5)\,f_1''(0.5) + f_1(0.525)\left(\frac{\cos\left(\frac{189}{40\sqrt[4]{2}}\right) + \cosh\left(\frac{189}{40\sqrt[4]{2}}\right)}{42404} - \frac{\cos\left(\frac{\sqrt[4]{35843}}{2\cdot 2^{3/4}}\right)\cosh\left(\frac{21\sqrt[4]{35843}}{40\cdot 2^{3/4}}\right)}{21202}\right),$$

which is negative.

So, we have proved that $W_2^8[M_1](t) < 0$ for $t \in [0,1]$.

4. $W_3^8[M]$ does not have any zero on $(0,1]$ for $M = 0$.

If we take $M = 0$, we have:

$$w_3(t) = W_3^8[0](t) = \frac{r_1(t) + r_2(t) + r_3(t)}{100253956065975624},$$

where:

$$\begin{aligned} r_1(t) =& 6\sqrt{2}\left(121t^2 - 2\right)\sin\left(11\sqrt{2}t\right) + 11\left(24 - 121t^2\right)t\cos\left(11\sqrt{2}t\right) + 6\sqrt{2}\left(121t^2 + 2\right)\sinh\left(11\sqrt{2}t\right) \\ &- 11\left(121t^2 + 24\right)t\cosh\left(11\sqrt{2}t\right), \\ r_2(t) =& -7986t^3 + \sin\left(\frac{11t}{\sqrt{2}}\right)\left(\sqrt{2}\left(-14641t^4 + 2904t^2 + 24\right)\cosh\left(\frac{11t}{\sqrt{2}}\right) - 528t\sinh\left(\frac{11t}{\sqrt{2}}\right)\right), \\ r_3(t) =& \cos\left(\frac{11t}{\sqrt{2}}\right)\left(\sqrt{2}\left(14641t^4 + 2904t^2 - 24\right)\sinh\left(\frac{11t}{\sqrt{2}}\right) - 21296t^3\cosh\left(\frac{11t}{\sqrt{2}}\right)\right). \end{aligned}$$

We have $w_3(0) = w_3'(0) = \cdots = w_3^{(14)}(0) = 0$. We study $w_3^{(14)}(t)$, while it is negative, since they all vanish at $t = 0$, the derivatives of less order also are.

$$w_3^{(14)}(t) = \frac{g_1(t) + g_2(t) + g_3(t) + g_4(t)}{264},$$

where:

$$\begin{aligned} g_1(t) =& 1408t\left(121t^2 - 129\right)\cos\left(11\sqrt{2}t\right) - 1408t\left(121t^2 + 129\right)\cosh\left(11\sqrt{2}t\right), \\ g_2(t) =& 384\sqrt{2}\left(605t^2 - 52\right)\sin\left(11\sqrt{2}t\right) - 384\sqrt{2}\left(605t^2 + 52\right)\sinh\left(11\sqrt{2}t\right), \\ g_3(t) =& \sin\left(\frac{11t}{\sqrt{2}}\right)\left(127776t^3\sinh\left(\frac{11t}{\sqrt{2}}\right) + \sqrt{2}\left(14641t^4 + 94380t^2 - 10608\right)\cosh\left(\frac{11t}{\sqrt{2}}\right)\right), \\ g_4(t) =& \cos\left(\frac{11t}{\sqrt{2}}\right)\left(\sqrt{2}\left(14641t^4 - 94380t^2 - 10608\right)\sinh\left(\frac{11t}{\sqrt{2}}\right) - 110352t\cosh\left(\frac{11t}{\sqrt{2}}\right)\right). \end{aligned}$$

Trivially, $g_1(t) < 0$ and $g_2(t) < 0$ for every $t > 0$.

Now, for $t < \dfrac{1}{11}\sqrt{2\left(195+\sqrt{40677}\right)} \approxeq 2.56$, $\left(14641t^4 - 94380t^2 - 10608\right) < 0$, then $g_4(t) < 0$ while $\cos\left(\dfrac{11t}{\sqrt{2}}\right) \geqslant 0$, i. e., for $t \in \left[0, \dfrac{\pi}{11\sqrt{2}}\right]$, where $\dfrac{\pi}{11\sqrt{2}} \approxeq 0.202$.

If $t \in \left[0, \dfrac{\pi}{11\sqrt{2}}\right]$, we have that $\sin\left(\dfrac{11t}{\sqrt{2}}\right) \geqslant 0$. Let us see that:

$$g_{31}(t) = \left(127776t^3 \sinh\left(\frac{11t}{\sqrt{2}}\right) + \sqrt{2}\left(14641t^4 + 94380t^2 - 10608\right)\cosh\left(\frac{11t}{\sqrt{2}}\right)\right) < 0.$$

Since $\sinh\left(\dfrac{11t}{\sqrt{2}}\right) < \cosh\left(\dfrac{11t}{\sqrt{2}}\right)$, we can bound:

$$g_{31}(t) \leqslant \left(127776t^3 + \sqrt{2}\left(14641t^4 + 94380t^2 - 10608\right)\right)\cosh\left(\frac{11t}{\sqrt{2}}\right),$$

which for $t \in \left[0, \dfrac{\pi}{11\sqrt{2}}\right]$ is non-positive.

So, $w_3^{(14)}(t) < 0$ for every $t \in \left[0, \dfrac{\pi}{11\sqrt{2}}\right]$ and also do the derivatives of less order.

Now, we study $w_3^{(11)}(t)$, for $t \geqslant \dfrac{\pi}{11\sqrt{2}}$, we have:

$$w_3^{(11)}(t) = \frac{h_1(t)+h_2(t)+h_3(t)+h_4(t)}{175692},$$

where:

$$\begin{aligned}
h_1(t) =& 48\left(847t^2-25\right)\cos\left(11\sqrt{2}t\right) - 48\left(847t^2+25\right)\cosh\left(11\sqrt{2}t\right),\\
h_2(t) =& 16\sqrt{2}\Big(\left(57-121t^2\right)\sin\left(11\sqrt{2}t\right) - \left(121t^2+57\right)\sinh\left(11\sqrt{2}t\right)\Big),\\
h_3(t) =& \cos\left(\frac{11t}{\sqrt{2}}\right)\left(\left(2400-14641t^4\right)\cosh\left(\frac{11t}{\sqrt{2}}\right) + 66\sqrt{2}t\left(152-363t^2\right)\sinh\left(\frac{11t}{\sqrt{2}}\right)\right),\\
h_4(t) =& 66t\sin\left(\frac{11t}{\sqrt{2}}\right)\left(-\sqrt{2}\left(363t^2+152\right)\cosh\left(\frac{11t}{\sqrt{2}}\right) - 770t\sinh\left(\frac{11t}{\sqrt{2}}\right)\right).
\end{aligned}$$

Obviously, $h_1(t) < 0$ and $h_2(t) < 0$ for every $t > 0$.

$h_4(t) < 0$ while $\sin\left(\dfrac{11t}{\sqrt{2}}\right) > 0$, i.e. for $t \in \left[0, \dfrac{\pi\sqrt{2}}{11}\right]$, where $\dfrac{\pi\sqrt{2}}{11} \approxeq 0.4039$.

If $\cos\left(\dfrac{11t}{\sqrt{2}}\right) < 0$ for $t \in \left[\dfrac{\pi}{11\sqrt{2}}, \dfrac{3\pi}{11\sqrt{2}}\right]$, if we have both $t \leqslant \dfrac{2}{11}\sqrt{5}\sqrt[4]{6} \approxeq 0.64$ and $t \leqslant \dfrac{2\sqrt{\frac{38}{3}}}{11} \approxeq 0.65$, then $h_3(t) < 0$.

Hence for $t \in \left[\frac{\pi}{11\sqrt{2}}, \frac{\pi\sqrt{2}}{11}\right]$, $w_3^{(11)}(t) < 0$ and, since we have seen that $w_3^{(11)}(t) < 0$ for $t \in \left[0, \frac{\pi}{11\sqrt{2}}\right]$, we have that $w_3^{(11)}(t) < 0$ for every $t \in \left[0, \frac{\pi\sqrt{2}}{11}\right]$ and the derivatives of less order also do.

Now, we study $w_3^{(8)}(t)$:

$$w_3^{(8)}(t) = \frac{\ell_1(t) + \ell_2(t) + \ell_3(t) + \ell_4(t)}{467692104},$$

where:

$$\begin{aligned}
\ell_1(t) =& 176t\left(12 - 121t^2\right)\cos\left(11\sqrt{2}t\right) - 176t\left(121t^2 + 12\right)\cosh\left(11\sqrt{2}t\right),\\
\ell_2(t) =& -11616\sqrt{2}t^2\left(\sin\left(11\sqrt{2}t\right) + \sinh\left(11\sqrt{2}t\right)\right),\\
\ell_3(t) =& \cos\left(\frac{11t}{\sqrt{2}}\right)\left(63888t^3\cosh\left(\frac{11t}{\sqrt{2}}\right) + \sqrt{2}\left(14641t^4 + 20328t^2 - 168\right)\sinh\left(\frac{11t}{\sqrt{2}}\right)\right),\\
\ell_4(t) =& \sin\left(\frac{11t}{\sqrt{2}}\right)\left(\sqrt{2}\left(-14641t^4 + 20328t^2 + 168\right)\cosh\left(\frac{11t}{\sqrt{2}}\right) + 7920t\sinh\left(\frac{11t}{\sqrt{2}}\right)\right).
\end{aligned}$$

As before, $\ell_1(t) < 0$ and $\ell_2(t) < 0$ for every $t > 0$.

Now, for:

$$\frac{1}{11}\sqrt{2\left(\sqrt{1806} - 42\right)} < t < \frac{1}{11}\sqrt{2\left(42 + \sqrt{1806}\right)},$$

where $\frac{1}{11}\sqrt{2\left(\sqrt{1806} - 42\right)} \approxeq 0.09$ and $\frac{1}{11}\sqrt{2\left(42 + \sqrt{1806}\right)} \approxeq 1.18$, if $\cos\left(\frac{11t}{\sqrt{2}}\right) \leqslant 0$, then $\ell_3(t) \leqslant 0$ and, whenever $\sin\left(\frac{11t}{\sqrt{2}}\right) \leqslant 0$, then $\ell_4(t) \leqslant 0$. For $t \in \left[\frac{\pi\sqrt{2}}{11}, \frac{3\pi}{11\sqrt{2}}\right]$ both are non-positive, so $w_3^{(8)}(t) < 0$.

Hence, we can conclude that $w_3^{(8)}(t) < 0$ for $t \in \left[0, \frac{3\pi}{11\sqrt{2}}\right]$, where $\frac{3\pi}{11\sqrt{2}} \approxeq 0.6058$ and the derivatives of less order also do.

Now, let us see what happens for $w_3^{(5)}(t)$.

$$w_3^{(5)}(t) = \frac{m_1(t) + m_2(t) + m_3(t) + m_4(t)}{311249095212},$$

where:

$$\begin{aligned}
m_1(t) =& 6\left(2 - 121t^2\right)\cos\left(11\sqrt{2}t\right) - 6\left(121t^2 + 2\right)\cosh\left(11\sqrt{2}t\right),\\
m_2(t) =& 22\sqrt{2}t\left(121t^2 + 6\right)\sin\left(11\sqrt{2}t\right) + 22\sqrt{2}t\left(6 - 121t^2\right)\sinh\left(11\sqrt{2}t\right),\\
m_3(t) =& -66t\cos\left(\frac{11t}{\sqrt{2}}\right)\left(\sqrt{2}\left(121t^2 - 2\right)\sinh\left(\frac{11t}{\sqrt{2}}\right) + 44t\cosh\left(\frac{11t}{\sqrt{2}}\right)\right),
\end{aligned}$$

$$m_4(t) = \sin\left(\frac{11t}{\sqrt{2}}\right)\left(\left(14641t^4 - 24\right)\sinh\left(\frac{11t}{\sqrt{2}}\right) + 66\sqrt{2}t\left(121t^2+2\right)\cosh\left(\frac{11t}{\sqrt{2}}\right)\right).$$

Trivially, $m_1(t) < 0$ for $t > 0$. If $t > \dfrac{\sqrt{6}}{11} \approxeq 0.22$, then $m_2(t) < 0$.

If $t > \dfrac{\sqrt{2}}{11} \approxeq 013$, then $m_3(t) < 0$ whenever $\cos\left(\dfrac{11t}{\sqrt{2}}\right) > 0$, i. e., this property is fulfilled for $t \in \left[\dfrac{3\,\pi}{11\,\sqrt{2}}, \dfrac{5\,\pi}{11\,\sqrt{2}}\right]$.

If $t > \dfrac{1}{11} \cdot 2^{3/4}\sqrt[4]{3} \approxeq 0.20$ and $\sin\left(\dfrac{11t}{\sqrt{2}}\right) > 0$, then $m_3(t) < 0$ if, i. e., this property is satisfied for $t \in \left[\dfrac{\pi\sqrt{2}}{11}, \dfrac{2\,\pi\sqrt{2}}{11}\right]$.

Then, $w_3^{(5)}(t) < 0$ if $t \in \left[0, \dfrac{2\,\pi\sqrt{2}}{11}\right]$.

Now, we study $w_3'''(t)$, given by the following expression:

$$w_3'''(t) = \frac{p_1(t) + p_2(t) + p_3(t) + p_4(t)}{37661140520652},$$

where:

$$p_1(t) = 11\sqrt{2}t\left(-121t^2 - 3\right)\sin\left(11\sqrt{2}t\right) - 3\left(121t^2+5\right)\cos\left(11\sqrt{2}t\right) - 18,$$

$$p_2(t) = 11\sqrt{2}t\left(3 - 121t^2\right)\sinh\left(11\sqrt{2}t\right) + 3\left(121t^2 - 5\right)\cosh\left(11\sqrt{2}t\right),$$

$$p_3(t) = \cos\left(\frac{11t}{\sqrt{2}}\right)\left(\left(48 - 14641t^4\right)\cosh\left(\frac{11t}{\sqrt{2}}\right) - 2662\sqrt{2}t^3\sinh\left(\frac{11t}{\sqrt{2}}\right)\right),$$

$$p_4(t) = 242t^2\sin\left(\frac{11t}{\sqrt{2}}\right)\left(6\sinh\left(\frac{11t}{\sqrt{2}}\right) - 11\sqrt{2}t\cosh\left(\frac{11t}{\sqrt{2}}\right)\right).$$

$p_1(t) < 0$ while $\sin\left(11\sqrt{2}t\right) > 0$ and $\cos\left(11\sqrt{2}t\right) > 0$. This is true, in particular, for every $t \in \left[\dfrac{2\sqrt{2}\,\pi}{11}, \dfrac{9\pi}{22\,\sqrt{2}}\right]$.

We have $p_2'(t) = -22\Big(11t\big(121t^2 - 6\big)\cosh\left(11\sqrt{2}t\right) + 6\sqrt{2}\sinh\left(11\sqrt{2}t\right)\Big)$, which is negative if $t > \dfrac{\sqrt{6}}{11} \approxeq 0.22$. Moreover:

$$p_2\left(\frac{2\sqrt{2}\,\pi}{11}\right) = 4\pi\left(3 - 8\pi^2\right)\sinh(4\pi) + 3\left(8\pi^2 - 5\right)\cosh(4\pi) < 0,$$

then $p_2(t) < 0$ for every $t > \dfrac{2\sqrt{2}\,\pi}{11}$.

For $t > \dfrac{2\sqrt[4]{3}}{11} \approxeq 0.23$, $p_3(t) < 0$ if $\cos\left(\dfrac{11t}{\sqrt{2}}\right) > 0$, in particular if $t \in \left[\dfrac{2\sqrt{2}\,\pi}{11}, \dfrac{9\pi}{22\,\sqrt{2}}\right]$.

Now, for $t \in \left[\dfrac{2\sqrt{2}\,\pi}{11}, \dfrac{9\pi}{22\,\sqrt{2}}\right]$, we have that $\sin\left(\dfrac{11t}{\sqrt{2}}\right) > 0$, so we only have to see that:

$$p_{41}(t) = 6\sinh\left(\frac{11t}{\sqrt{2}}\right) - 11\sqrt{2}t\cosh\left(\frac{11t}{\sqrt{2}}\right) < 0,$$

we can bound it as follows:

$$p_{41}(t) < \left(6 - 11\sqrt{2}t\right)\cosh\left(\frac{11t}{\sqrt{2}}\right),$$

and this is negative for $t > \dfrac{6}{11\sqrt{2}} \approxeq 0.39$, then $p_{41}(t) < 0$ and also does $p_4(t)$.

So, we can affirm that $w_3'''(t) < 0$ for $t \in \left[0, \dfrac{9\,\pi}{22\,\sqrt{2}}\right]$ where $\dfrac{9\,\pi}{22\,\sqrt{2}} \approxeq 0.9088$.

Finally, let us study the expression of $w_3''(t)$, this is:

$$w_3''(t) = \frac{q_1(t) + q_2(t) + q_3(t) + q_4(t)}{828545091454344},$$

where:

$$\begin{aligned}
q_1(t) =& -6\sqrt{2}\left(121t^2 + 2\right)\sin\left(11\sqrt{2}t\right) + 22\left(121t^2 - 3\right)t\cos\left(11\sqrt{2}t\right) - 396t,\\
q_2(t) =& 6\sqrt{2}\left(121t^2 - 2\right)\sinh\left(11\sqrt{2}t\right) - 22t\left(121t^2 + 3\right)\cosh\left(11\sqrt{2}t\right),\\
q_3(t) =& \cos\left(\frac{11t}{\sqrt{2}}\right)\left(\sqrt{2}\Big(24 - 121t^2\left(121t^2 + 12\right)\Big)\sinh\left(\frac{11t}{\sqrt{2}}\right) + 528t\cosh\left(\frac{11t}{\sqrt{2}}\right)\right),\\
q_4(t) =& \sqrt{2}\left(-14641t^4 + 1452t^2 + 24\right)\sin\left(\frac{11t}{\sqrt{2}}\right)\cosh\left(\frac{11t}{\sqrt{2}}\right).
\end{aligned}$$

If $t > \dfrac{\sqrt{3}}{11} \approxeq 0.15$, $\sin\left(11\sqrt{2}t\right) > 0$ and $\cos\left(11\sqrt{2}t\right) < 0$, then $q_1(t) < 0$. This happens if $t \in \left[\dfrac{9\,\pi}{22\,\sqrt{2}}, \dfrac{5\,\pi}{11\,\sqrt{2}}\right]$.

If $t > \dfrac{\sqrt{2}}{11} \approxeq 0.13$, we can bound $q_2(t)$ as follows:

$$q_2(t) < 2\left(3\sqrt{2}\left(121t^2 - 2\right) - 11t\left(121t^2 + 3\right)\right)\cosh\left(11\sqrt{2}t\right),$$

which is negative for every $t > 0$. Then, $q_2(t) < 0$.

If $t \in \left[\dfrac{9\,\pi}{22\,\sqrt{2}}, \dfrac{5\,\pi}{11\,\sqrt{2}}\right]$, $\cos\left(\dfrac{11t}{\sqrt{2}}\right) > 0$, then we only have to study:

$$q_{31}(t) = \sqrt{2}\Big(24 - 121t^2\left(121t^2 + 12\right)\Big)\sinh\left(\frac{11t}{\sqrt{2}}\right) + 528t\cosh\left(\frac{11t}{\sqrt{2}}\right),$$

and we have:

$$q_{3_1}'(t) = -11\left(5324\sqrt{2}t^3\sinh\left(\frac{11t}{\sqrt{2}}\right) + \left(14641t^4 + 1452t^2 - 72\right)\cosh\left(\frac{11t}{\sqrt{2}}\right)\right),$$

which is negative for $t > \frac{1}{11}\sqrt{6\left(\sqrt{3}-1\right)} \approxeq 0.19$ and:

$$q_{3_1}\left(\frac{9\,\pi}{22\,\sqrt{2}}\right) = \sqrt{2}\left(24 - \frac{81}{8}\pi^2\left(12 + \frac{81\pi^2}{8}\right)\right)\sinh\left(\frac{9\pi}{4}\right) + 108\sqrt{2}\pi\cosh\left(\frac{9\pi}{4}\right) < 0.$$

Then $q_{3_1}(t) < 0$ and $q_3(t)$ also does.

Finally, if $t > \frac{1}{11}\sqrt{2\left(3+\sqrt{15}\right)} \approxeq 0.34$ and $\sin\left(\frac{11t}{\sqrt{2}}\right) < 0$, which is satisfied for $t \in \left[\frac{9\,\pi}{22\,\sqrt{2}}, \frac{5\,\pi}{11\,\sqrt{2}}\right]$, we have that $q_4(t) < 0$.

So, we can conclude that $w_3''(t) < 0$ for $t \in \left[0, \frac{5\,\pi}{11\,\sqrt{2}}\right]$, where $\frac{5\,\pi}{11\,\sqrt{2}} \approxeq 1.01$, then $w_3''(t) < 0$ on $[0, 1]$, and also do $w_3'(t)$ and $w_3(t) = W_3^8[0](t)$.

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C. Non-disconjugacy criteria for operator $T_4^6[M]=\dfrac{\mathrm{d}^4}{\mathrm{d}\,t^4}+1000\dfrac{\mathrm{d}}{\mathrm{d}\,t}+M$

In the sequel, we show the explicit calculus for applying Corollary 2.2.7 to operator $T_4^6[M]$ previously introduced in (2.2.3). As we will see, proving the non-disconjugacy property is really complicated for this case. Moreover, we will prove that the parameter $\bar{M}$ for which the non-disconjugacy criterion given in Theorem 2.2.8 follows is unique.

We present the followed steps below.

1. Corollary 2.2.7 is fulfilled.
   1.1. $W_1^4[0]$ has at least a zero on $(0,1]$.
   1.2. $\exists\, M^*<0$ such that $W_1^4[M^*](t)\neq 0$ for all $t\in(0,1]$.
   1.3. $W_1^4[M](1)\neq 0$ for all $M\in[M^*,0]$.
   1.4. $\exists\,\widehat{M}\in[M^*,0]$ such that $W_1^4[\widehat{M}]$ has a double zero at $c\in(0,1]$.
2. If $M\neq\widehat{M}$, then the hypotheses of Theorem 2.2.8 are not fulfilled.
   2.1. $W_1^4[M]$ is of constant sign on $(0,1]$ for $M\leqslant\widehat{M}$.
   2.2. $W_3^4[M]$ is of constant sign on $(0,1]$ for $M\leqslant 0$.
   2.3. $W_2^4[M]$ is of constant sign on $(0,1]$ for $M\geqslant\widehat{M}$.

<u>1.1.</u> $W_1^4[0]$ has at least a zero on $(0,1]$.

For $\bar{M}=0$, the solution of (2.2.3) with the initial conditions $u(0)=u'(0)=u''(0)=0$ and $u'''(0)=1$, which represents the first Wronskian, is given by the following expression:

$$W_1^4[0](t)=-\frac{e^{-10\,t}\left(1-3\,e^{10\,t}+2\,e^{15\,t}\,\cos\left(5\sqrt{3}\,t\right)\right)}{3000}. \tag{C.1}$$

From the fact that:

$$W_1^4[0]\left(\frac{\pi}{5\sqrt{3}}\right)=\frac{3-e^{-\frac{2\pi}{\sqrt{3}}}+2e^{\frac{\pi}{\sqrt{3}}}}{3000}>0>\frac{3-e^{-\frac{4\pi}{\sqrt{3}}}-2e^{\frac{2\pi}{\sqrt{3}}}}{3000}=W_1^4[0]\left(\frac{2\,\pi}{5\sqrt{3}}\right),$$

we deduce that $W_1^4[0]$ vanish at least at one point in $(0,1)$.

Numerically, we see that it has two zeros on $(0,1)$, $t_0\approxeq 0.554944$ and $t_1\approxeq 0.905023$, see Figure C.1.

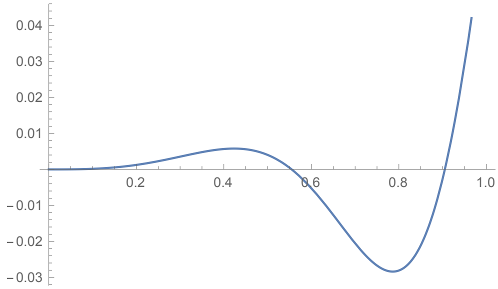

Figure C.1: $W_1[0]u(t)$ with $t \in [0,1]$.

<u>1.2.</u> $\exists\, M^* < 0$ such that $W_1^4[M^*](t) \neq 0$ for all $t \in (0,1]$.

Now, if we denote $\bar{M} = -s^4$, then we have that the solution of (2.2.3) with the initial conditions $u(0) = u'(0) = u''(0) = 0$ and $u'''(0) = 1$, $u_m(t) = W_1^4[-s^4]\,(t)$ follows the next expression:

$$\begin{aligned} u_m(t) =& \frac{m e^{\left(-\sqrt{\frac{250-m^3}{m}}-m\right)t}}{8\sqrt{\dfrac{250-m^3}{m}}\,(2m^6+15625)} \left(\left(\left(\sqrt{\frac{250-m^3}{m}} + m \right) m^2 + 125 \right) e^{2\sqrt{\frac{250-m^3}{m}}t} \right. \\ &+ 2\sqrt{m}\sqrt{\frac{250-m^3}{m}}\, e^{\left(\sqrt{\frac{250-m^3}{m}}+2m\right)t} \left(\frac{(m^3-125)\sin\left(\frac{\sqrt{m^3+250}t}{\sqrt{m}}\right)}{\sqrt{m^3+250}} \right. \\ &\left. - m^{3/2}\cos\left(\frac{\sqrt{m^3+250}t}{\sqrt{m}}\right) \right) + \left. \left(\sqrt{\frac{250-m^3}{m}} - m \right) m^2 - 125 \right), \end{aligned} \tag{C.2}$$

where m is given as a function of s in the following sense:

$$m(s) = \frac{\sqrt{\dfrac{\sqrt[3]{\sqrt{3}\sqrt{s^{12}+105468750000}+562500}}{3^{2/3}} - \dfrac{s^4}{\sqrt[3]{3}\sqrt[3]{\sqrt{3}\sqrt{s^{12}+105468750000}+562500}}}}{\sqrt{2}}. \tag{C.3}$$

It is clear that m is a continuous positive function on $[0,+\infty)$ which satisfies:

$$m(0) = 5, \qquad \lim_{s\to\infty} m(s) = 0.$$

Let us see that m is a strictly decreasing function for $s > 0$.

$$m'(s) = \frac{2s^3 f_m(s)}{3^{5/6}\sqrt{s^{12}+105468750000}\left(\sqrt{3}\sqrt{s^{12}+105468750000}+562500\right)^{4/3} m(s)},$$

where:

$$f_m(s) := f_{m1}(s) + f_{m2}(s),$$
$$f_{m1}(s) := 3^{2/3}s^8\left(\sqrt{3}\sqrt{s^{12}+105468750000}+562500\right)^{2/3} - 3s^{12},$$
$$f_{m2}(s) := -3s^{12} - 1125000\left(\sqrt{3}\sqrt{s^{12}+105468750000}+562500\right)$$
$$\leqslant -1125000\left(\sqrt{3}\sqrt{s^{12}+105468750000}+562500\right).$$

In order to see that $m'(s) < 0$, we only have to consider the function f_m.
First, we study $f_{m1}(s)$, for which we have the following property:

$$f_{m1}'(s) = 12s^3\left(\frac{\sqrt[6]{3}s^8}{\sqrt{s^{12}+105468750000}\sqrt[3]{\sqrt{3}\sqrt{s^{12}+105468750000}+562500}} - 1\right) < 0.$$

Also, we have $f_{m1}(0) = 22500$, so, since f_{m1} is a decreasing function and we have that:

$$f_m(s) \leq f_{m1}(0) - 1125000\left(\sqrt{3}\sqrt{s^{12}+105468750000}+562500\right)$$
$$= -22500\left(50\sqrt{3}\sqrt{s^{12}+105468750000}+28124999\right) < 0,$$

we can affirm that $f_m(s) < 0$ and also does $m'(s)$ for every $s > 0$.

Hence, $m(s) \leqslant 5 < 5\sqrt[3]{2}$, a condition which is needed to ensure that the expression (C.2) is well-defined. In particular, there is a unique $s_0 > 0$ such that $m(s_0) = 2$.

For $m(s_0) = 2$, we have that $W_1^4[-s_0^4]\,(t)$ follows expression (C.2) with $m = 2$ that is:

$$W_1^4[-s_0^4]\,(t) = -\frac{e^{-13t}}{7788} + \frac{e^{9t}}{3916} - \frac{e^{2t}\left(39\sqrt{129}\sin\left(\sqrt{129}t\right) + 172\cos\left(\sqrt{129}t\right)\right)}{1354758}.$$

Let us see that it is strictly positive for $t \in (0,1]$, see Figure C.2.

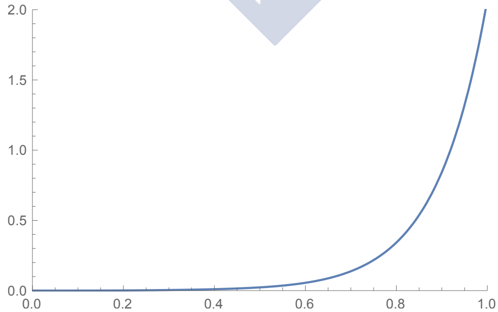

Figure C.2: $W_1[-s_0^4]u(t)$ with $t \in [0,1]$.

Denote $v(t) = W_1^4[-s_0^4](t)$, if we prove that $v^{(3)}(t) > 0$ on $[0,1]$ then, taking into account the fact that $v(0) = v'(0) = v''(0) = 0$, we deduce that $v''(t)$, $v'(t)$ and $v(t)$ are strictly increasing functions on $[0,1]$. In particular $v(t) > 0$ on $(0,1]$.

We have the following expression for $v^{(3)}(t)$:

$$v^{(3)}(t) = \frac{2197e^{-13t}}{7788} + \frac{729e^{9t}}{3916} + \frac{e^{2t}\left(9750\sqrt{129}\sin\left(\sqrt{129}t\right) + 720379\cos\left(\sqrt{129}t\right)\right)}{1354758}.$$

It is obvious that $v^{(3)}(t) > f(t)$, where:

$$f(t) := \frac{729e^{9t}}{3916} + \frac{e^{2t}\left(9750\sqrt{129}\sin\left(\sqrt{129}t\right) + 720379\cos\left(\sqrt{129}t\right)\right)}{1354758}.$$

It is immediate to verify that $f(t) > 0$ for $t \in \left(0, \frac{\pi}{2\sqrt{129}}\right]$, where $\frac{\pi}{2\sqrt{129}} \approxeq 0.138301$.

If $t \in \left(\frac{\pi}{2\sqrt{129}}, \frac{2\pi}{3\sqrt{129}}\right]$ $\left(\text{with } \frac{2\pi}{3\sqrt{129}} \approxeq 0.184401\right)$, f can be bounded from above by:

$$f(t) \geqslant \frac{729e^{9t}}{3916} - \frac{1}{2}e^{2t}\frac{720379}{1354758} =: g(t).$$

And $g(t) > 0$ for $t > \frac{1}{7}\log\left(\frac{184283}{129033}\right) \approxeq 0.0509149$, in particular this is fulfilled for $t \in \left(\frac{\pi}{2\sqrt{129}}, \frac{2\pi}{3\sqrt{129}}\right]$.

On the other hand, for every $t \in [0,1]$ we can bound f from below in the following way

$$f(t) \geqslant \frac{729e^{9t}}{3916} - \frac{e^{2t}\left(9750\sqrt{129} + 720379\right)}{1354758} =: \tilde{f}(t).$$

It is satisfied that $\tilde{f}(t) > 0$ for every $t > \frac{1}{7}\log\left(\frac{22\left(720379 + 9750\sqrt{129}\right)}{5548419}\right) \approxeq 0.170364.$

Since $\frac{1}{7}\log\left(\frac{22\left(720379 + 9750\sqrt{129}\right)}{5548419}\right) < \frac{2\pi}{3\sqrt{129}}$, we have proved that $f(t) > 0$ for every $t \in [0,1]$, and then $v^{(3)}(t) > 0$ on $[0,1]$

So $v(t) > 0$ for every $t \in (0,1]$.

1.3. $W_1^4[M](1) \neq 0$ for all $M \in [M^*, 0]$.

Now, we are interested into know if $W_1[M](1)$ is, or not, of constant sign for all M in the interval $[-s_0^4, 0]$.

Denote $W_1^4[M](1) = f_1(m)\big(g_1(m) + h_1(m)\big)$, where:

$$f_1(m) = \frac{e^{-\sqrt{\frac{250-m^3}{m}}-m}\,m}{8\sqrt{\frac{250-m^3}{m}}\left(2m^6 + 15625\right)},$$

$$g_1(m) = \left(\sqrt{\frac{250-m^3}{m}} - m\right)m^2 + e^{2\sqrt{\frac{250-m^3}{m}}}\left(\left(\sqrt{\frac{250-m^3}{m}} + m\right)m^2 + 125\right) - 125,$$

$$h_1(m) = 2e^{\left(2m+\sqrt{\frac{250-m^3}{m}}\right)}\sqrt{m}\sqrt{\frac{250-m^3}{m}}\left(\frac{(m^3-125)\sin\left(\frac{\sqrt{m^3+250}}{\sqrt{m}}\right)}{\sqrt{m^3+250}} - m^{3/2}\cos\left(\frac{\sqrt{m^3+250}}{\sqrt{m}}\right)\right),$$

with $2 \leqslant m \leqslant 5$, defined as $m(s)$ in (C.3).

Trivially $f_1(m) > 0$ for $m \leqslant 5$.

Now, we study the sign of $g_1(m)$ and $h_1(m)$.

We have that $\left(\sqrt{\frac{250-m^3}{m}} - m\right) m^2 \geqslant 0$ and $\left(\sqrt{\frac{250-m^3}{m}} + m\right) m^2 > 0$ for $m \in [2,5]$, then:

$$g_1(m) > 125\left(e^{2\sqrt{\frac{250-m^3}{m}}} - 1\right) > 0.$$

Now, we only need to verify that $h_1(m) \geqslant 0$. Obviously next inequality is satisfied:

$$2e^{\left(\sqrt{\frac{250-m^3}{m}}+2m\right)}\sqrt{m}\sqrt{\frac{250-m^3}{m}} > 0, \quad m \in [2,5],$$

then we only have to analyse:

$$h_2(m) = \frac{(m^3-125)\sin\left(\frac{\sqrt{m^3+250}}{\sqrt{m}}\right)}{\sqrt{m^3+250}} - m^{3/2}\cos\left(\frac{\sqrt{m^3+250}}{\sqrt{m}}\right).$$

Denote $q(m) = \dfrac{\sqrt{m^3+250}}{\sqrt{m}}$, we can see that it is a decreasing function in $[2,5]$.

While $m \in [2,5]$, $q(m) \in \left[5\sqrt{3}, \sqrt{129}\right] \subset \left(\dfrac{11\,\pi}{4}, \dfrac{11\,\pi}{3}\right)$. First, note that if $q(m)$ is in the interval $\left[3\,\pi, \dfrac{7\,\pi}{2}\right]$, since both $\sin\left(q(m)\right)$ and $\cos\left(q(m)\right)$ are negative, it is clear that $h_2(m) > 0$.

While $q(m) \in \left(\dfrac{7\pi}{2}, \dfrac{11\pi}{3}\right)$, the following inequalities are satisfied:

$$\sin\left(q(m)\right) \leq -\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos\left(q(m)\right) \leqslant \frac{1}{2},$$

hence we can bound $h_2(m)$ on such an interval as follows:

$$h_2(m) \geqslant -\frac{\sqrt{3}\left(m^3-125\right)}{2\sqrt{m^3+250}} - \frac{\sqrt{m^3}}{2},$$

which is positive for $m > 5\sqrt[3]{\dfrac{1}{2}}\left(4-\sqrt{10}\right)$. Then $h_2(m)$ is positive whenever:

$$q(m) < q\left(5\sqrt[3]{\frac{1}{2}\left(4-\sqrt{10}\right)}\right) = 5\sqrt[6]{\frac{247}{6} - \frac{7\sqrt{10}}{3}} \approxeq 8.99.$$

Now, since $\frac{7\pi}{2} > 5\sqrt[6]{\frac{247}{6} - \frac{7\sqrt{10}}{3}}$, we have that $h_2(m) > 0$ when $q(m) \in \left(\frac{7\pi}{2}, \frac{11\pi}{3}\right)$.

So, we only need to verify what happens if $q(m) \in \left(\frac{11\pi}{4}, 3\pi\right)$. We have the following inequalities:

$$\sin\big(q(m)\big) \leqslant \frac{\sqrt{2}}{2} \qquad \text{and} \qquad \cos\big(q(m)\big) \leq -\frac{\sqrt{2}}{2},$$

we can bound $h_2(m)$ as follows:

$$h_2(m) \geqslant \frac{m^3 - 125}{\sqrt{2}\sqrt{m^3+250}} + \frac{\sqrt{m^3}}{\sqrt{2}},$$

which is positive for $m > \frac{5}{2^{2/3}}$. So, it is positive if $q(m) < q\left(\frac{5}{2^{2/3}}\right) = \frac{15}{2^{2/3}} \approxeq 9.44941$.

Since $3\,\pi < \frac{15}{2^{2/3}}$, we conclude that $W_1[M](1) > 0$ for all $M \in [-s_0^4, 0]$.

<u>1.4.</u> $\exists\, \widehat{M} \in [M^*, 0]$ such that $W_1^4[\widehat{M}]$ has a double zero at $c \in (0,1]$.

Since $W_1^4[M]\,(t)$ is a continuous function on M, we can affirm that there exist $\widehat{M} \in (-s_0^4, 0)$ and $\widehat{t} \in (0,1)$ such that $W_1^4[\widehat{M}](t)$ has a double zero at $\widehat{t}$ and that $W_1^4[M](t)$ oscillates on $[0,1]$ for all $0 \geqslant M > \widehat{M}$.

We can obtain numerically that $\widehat{M} \approxeq -7.78318^4$ and $\widehat{t} \approxeq 0.75996$.

Hence, we can conclude, using Corollary 2.2.7, that the linear differential equation (2.2.3) is not disconjugate for every $M \in \mathbb{R}$.

<u>2.</u> If $M \neq \widehat{M}$, then the hypotheses of Theorem 2.2.8 are not fulfilled.

In addition to the previous proof, we will verify that there is not any $M \in \mathbb{R}$, $M \neq \widehat{M}$ such that we can apply the non-disconjugacy criteria given in Theorem 2.2.8.

<u>2.1.</u> $W_1^4[M]$ is of constant sign on $(0,1]$ for $M \leqslant \widehat{M}$.

Suppose that there exists $\widehat{\widehat{M}} < \widehat{M}$ and $\widehat{\widehat{t}} \in (0,1]$ such that $W_1^4\left[\widehat{\widehat{M}}\right]\left(\widehat{\widehat{t}}\right) = 0$. If there are more than one points where the Wronskian vanish, let us choose the least of them. Since there always exist an interval $\left[0, \alpha_{\widehat{\widehat{M}}}(0)\right]$ such that the linear differential equation (2.2.3) is disconjugate, by using Proposition 1.1.3, we can affirm $\widehat{\widehat{t}} \geqslant \alpha_{\widehat{\widehat{M}}}(0) > 0$.

First, let us assume that $\widehat{\widehat{t}} \neq \widehat{t}$. Hence, we can affirm that $g_{\widehat{M}}(t,s)$, the related Green's function to the operator $T[\widehat{M}]$ in the space $X_{3\left[0,\widehat{\widehat{t}}\right]}$, is well-defined.

Let us see that, in fact $\frac{\partial^3}{\partial t^3} g_{\widehat{M}}(t,s)_{|t=0} \leqslant 0$ for all $s \in \left[0, \widehat{\widehat{t}}\right]$.

The adjoint operator of $T_4^6[M]$ is given by:

$$T_4^{6^*}[M]v(t) = v^{(4)}(t) - 1000\,v'(t) + M\,v(t),\ t \in \left[0, \widehat{\widehat{t}}\right],$$

where v is defined on the space $X_3^*{}_{\left[0,\widehat{\widehat{t}}\,\right]} = X_1{}_{\left[0,\widehat{\widehat{t}}\,\right]}$.

Denote $w(t) = \frac{\partial^3}{\partial s^3} g^*_{\widehat{M}}(t,s)_{|s=0}$. From (1.3.3), we have that $w(s) = \frac{\partial^3}{\partial t^3} g_{\widehat{M}}(t,s)_{|t=0}$.

As in Lemma 3.5.3, we obtain that $T_4^{6\,*}\left[\widehat{\widehat{M}}\right] w(t) = 0$ for all $t \in \left(0,\widehat{\widehat{t}}\,\right]$ and:

$$w\left(\widehat{\widehat{t}}\right) = w'\left(\widehat{\widehat{t}}\right) = w''\left(\widehat{\widehat{t}}\right) = 0,$$

moreover $w(0) = -1$.

However if we consider $w_1(t) = W_1^4\left[\widehat{M}\right]\left(\widehat{\widehat{t}} - t\right)$, we observe that $T_4^{6\,*}\left[\widehat{\widehat{M}}\right] w_1(t) = 0$ for all $t \in \left[0,\widehat{\widehat{t}}\,\right]$ and $w_1\left(\widehat{\widehat{t}}\right) = w_1'\left(\widehat{\widehat{t}}\right) = w_1''\left(\widehat{\widehat{t}}\right) = 0$. Moreover, we know that $w_1 \geqslant 0$ for all $t \in \left[0,\widehat{\widehat{t}}\,\right]$ and that $w_1(0) = W_1^4[\widehat{M}]\left(\widehat{\widehat{t}}\right) \neq 0$, since $\widehat{\widehat{t}} \neq \widehat{t}$. Thus, we conclude that $w(t) = \alpha\, w_1(t)$ for $\alpha \in \mathbb{R}$. Thus, w is a constant sign function on $\left[0,\widehat{\widehat{t}}\,\right]$. Since $w(0) = -1$, $w(s) = \dfrac{\partial^3}{\partial t^3} g_{\widehat{M}}(t,s)_{|t=0} \leqslant 0$ for all $s \in \left[0,\widehat{\widehat{t}}\,\right]$.

On another hand, under our assumptions, $\widehat{M} - \widehat{\widehat{M}}$ is an eigenvalue of $T_4^6[\widehat{M}]$ on $X_3{}_{\left[0,\widehat{\widehat{t}}\,\right]}$, with related eigenfunction $W_1^4\left[\widehat{\widehat{M}}\right]$. By the choice of $\widehat{\widehat{t}}$, we can affirm that $W_1\left[\widehat{\widehat{M}}\right](t) > 0$ for all $t \in \left(0,\widehat{\widehat{t}}\right)$.

We can express $W_1^4\left[\widehat{\widehat{M}}\right](t)$ by means of $g_{\widehat{M}}$ as follows:

$$W_1^4\left[\widehat{\widehat{M}}\right](t) = \int_0^{\widehat{\widehat{t}}} g_{\widehat{M}}(t,s)\left(\widehat{M} - \widehat{\widehat{M}}\right) W_1^4\left[\widehat{\widehat{M}}\right](s)\,\mathrm{d}\,s,\ \forall\, t \in \left[0,\widehat{\widehat{t}}\,\right],$$

and, in particular, we have:

$$1 = W_1^4\left[\widehat{\widehat{M}}\right]^{(3)}(0) = \int_0^{\widehat{\widehat{t}}} \frac{\partial^3}{\partial t^3} g_{\widehat{M}}(t,s)_{|t=0}\left(\widehat{M} - \widehat{\widehat{M}}\right) W_1^4\left[\widehat{\widehat{M}}\right](s)\,\mathrm{d}\,s,\ \forall\, t \in \left[0,\widehat{\widehat{t}}\,\right],$$

however, this is not possible for $\widehat{\widehat{M}} < \widehat{M}$, because $W_1^4\left[\widehat{\widehat{M}}\right](t) > 0$ for all $t \in [0,1]$ and $\dfrac{\partial^3}{\partial t^3} g_{\widehat{M}}(t,s)_{|t=0} \leqslant 0$ for all $s \in \left[0,\widehat{\widehat{t}}\,\right]$.

Now let us see what happens if $\widehat{\widehat{t}} = \widehat{t}$. By the previous assumptions, we can deduce that for all $M \leqslant \widehat{M}$ and for all $t \in (0,\widehat{t})$, $W_1^4[M](t) > 0$.

If there exists $M \in \left(\widehat{\widehat{M}}, \widehat{M}\right)$ such that $W_1^4[M]\left(\widehat{t}\right) \neq 0$, we can repeat the previous arguments, since $W_1^4[M](t) \neq 0$ for all $t \in [0,\widehat{t}]$ and then, the related Green's function $g_M(t,s)$ is well-defined in $X_3{}_{[0,\widehat{t}]}$. So, repeating the arguments we achieve that $\dfrac{\partial^3}{\partial t^3} g_M(t,s)_{|t=0} \leqslant 0$

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for all $s \in \left[0, \widehat{t}\right]$ and this contradicts the existence of a positive eigenvalue $M - \widehat{\widehat{M}}$ of $T_4^6[\widehat{M}]$ in $X_{3}{}_{\left[0,\widehat{t}\right]}$. As a consequence, for all $M \in \left(\widehat{\widehat{M}}, \widehat{M}\right)$, it is satisfied that $W_1^4[M]\left(\widehat{t}\right) = 0$. But, this contradicts the discrete character of the spectrum of operator $T_4^6[\widehat{M}]$ in $X_{3}{}_{\left[0,\widehat{t}\right]}$. Hence, we can affirm that $W_1^4[M](t) > 0$ for all $t \in (0,1]$ and $M < \widehat{M}$.

<u>2.2.</u> $W_3^4[M]$ is of constant sign on $(0,1]$ for $M \leqslant 0$.

With a similar argument we can conclude that $W_3^4[M](t)$ is of constant sign for every $M \leqslant 0$.

First, let us see that for $M = 0$, $W_3^4[M](t) > 0$, for all $t \in (0,1]$. We have:

$$W_3^4[0](t) = -\frac{e^{-5t}\left(-3e^{5t} + e^{15t} + 2\cos\left(5\sqrt{3}t\right)\right)}{3000},$$

Moreover, $W_3^4[0](0) = W_3[0]'(0) = W_3''[0](0) = 0$ and

$$W_3^4[0]'''(t) = -\frac{1}{3}e^{-5t}\left(e^{15t} + 2\cos\left(5\sqrt{3}t\right)\right),$$

which is negative for $t \leqslant \dfrac{\pi}{10\sqrt{3}} \approxeq 0.18$.

On another hand,

$$W_3^4[0]'''(t) \leqslant -\frac{1}{3}e^{-5t}\left(e^{15t} - 2\right),$$

which is negative for $t > \dfrac{\log(2)}{15} \approxeq 0.046$. Then $W_3^4[0](t) < 0$ for all $t \in (0,1]$.

In this case, we can verify that $W_3^4[M](t)$ is a solution of $T_4^{6*}[M]\,v = 0$ coupled with the boundary conditions $W_3^4[M](0) = W_3^4[M]'(0) = W_3^4[M]''(0) = 0$ and $W_3^4[M]'''(0) = 1$.

Then, if there exists $\tilde{M} < 0$ such that there exist $\tilde{t} \in (0,1]$ satisfying $W_3^4\left[\tilde{M}\right]\left(\tilde{t}\right) = 0$, repeating the arguments of step 2.1 we arrive to the conclusion that $\dfrac{\partial^3}{\partial\, t^3} g_0^*(t,s)_{|t=0} \leqslant 0$ for all $t \in \left[0, \tilde{t}\right]$, where $g_M^*(t,s)$ is the related Green's function of $T_4^{6*}[M]$ in $X_{3}{}_{\left[0,\tilde{t}\right]}$.

Now, repeating the arguments done for $W_1^4[M]$ we arrive to a contradiction of supposing that there exist $\tilde{M} < 0$ and $\tilde{t} \in (0,1]$ satisfying $W_3^4\left[\tilde{M}\right]\left(\tilde{t}\right) = 0$. Hence, $W_3^4[M](t) \neq 0$ for all $t \in (0,1]$ and $M \leqslant 0$.

<u>2.3.</u> $W_2^4[M]$ is of constant sign on $(0,1]$ for $M \geqslant \widehat{M}$.

For all $t \in (0,1]$, we have seen that $W_1^4[M](t) \neq 0$ for $M \leqslant \widehat{M} < 0$ and that $W_3^4[M](t) \neq 0$ for all $M \leqslant 0$. Let us see that $W_2^4[M]$ is of constant sign for $M \geqslant \widehat{M}$.

<u>2.3.1.</u> $W_2^4[M]$ constant sign on $(0,1]$ for $M \geqslant 0$.

First, we will see that $W_2^4[M](t) \neq 0$ for all $t \in (0,1]$ and every $M \geqslant 0$. Suppose that the previous assertion is false, i.e., there exist $\tilde{\tilde{M}} \geqslant 0$ and $\tilde{\tilde{t}} \in (0,1]$ such that $W_2^4\left[\tilde{\tilde{M}}\right]\left(\tilde{\tilde{t}}\right) = 0$.

From Proposition 1.1.10, this fact implies that there exists a solution of (2.2.3) coupled with the boundary conditions $u(0)=u'(0)=u\left(\tilde{\tilde{t}}\right)=u'\left(\tilde{\tilde{t}}\right)$.

Multiplying (2.2.3) by u and integrating, we obtain:

$$\int_0^{\tilde{\tilde{t}}} u^{(4)}(t)\,u(t)\,\mathrm{d}\,t+1000\int_0^{\tilde{\tilde{t}}} u'(t)\,u(t)\,\mathrm{d}\,t+\tilde{\tilde{M}}\int_0^{\tilde{\tilde{t}}} u^2(t)=0,\ t\in\left[0,\tilde{\tilde{t}}\right],$$

integrating by parts:

$$u(t)\,u'''(t)|_0^{\tilde{\tilde{t}}}-\,u'(t)\,u''(t)|_0^{\tilde{\tilde{t}}}+\int_0^{\tilde{\tilde{t}}}\left(u^{(2)}(t)\right)^2\mathrm{d}\,t+500\ u^2(t)|_0^{\tilde{\tilde{t}}}$$
$$+1000\int_0^{\tilde{\tilde{t}}}\left(u'(t)\right)^2\mathrm{d}\,t+\tilde{\tilde{M}}\int_0^{\tilde{\tilde{t}}} u^2(t)=0,\ t\in\left[0,\tilde{\tilde{t}}\right],$$

and, taking into account the boundary conditions, we attain

$$\int_0^{\tilde{\tilde{t}}}\left(u^{(2)}(t)\right)^2\mathrm{d}\,t+1000\int_0^{\tilde{\tilde{t}}}\left(u'(t)\right)^2\mathrm{d}\,t+\tilde{\tilde{M}}\int_0^{\tilde{\tilde{t}}} u^2(t)=0,\ t\in\left[0,\tilde{\tilde{t}}\right].$$

It is obvious that the previous expression cannot be fulfilled by any non-trivial function when $\tilde{\tilde{M}}\geqslant 0$. Thus, we conclude that $W_2^4[M](t)\neq 0$ for all $t\in(0,1]$ and every $M\geqslant 0$.

2.3.2 $W_2^4[M]$ constant sign on $(0,1]$ for $M\in\left[\widehat{M},0\right]$.

If we study $W_2^4[-s_1^4](t)$, where s_1 is the unique positive value such that $m(s_1)=3$, then the corresponding Wronskian is expressed by:

$$\frac{9e^{-\frac{2}{3}\left(9+\sqrt{669}\right)t}}{16883743888}\left(-81\sqrt{61771}\left(e^{\left(6+\sqrt{\frac{223}{3}}\right)t}-e^{\left(6+\sqrt{669}\right)t}\right)\sin\left(\sqrt{\frac{277}{3}}t\right)\right.$$
$$\left.-61771e^{2\sqrt{\frac{223}{3}}t}\left(e^{12t}+1\right)+61771\left(e^{\left(6+\sqrt{\frac{223}{3}}\right)t}+e^{\left(6+\sqrt{669}\right)t}\right)\cos\left(\sqrt{\frac{277}{3}}t\right)\right),$$

and it satisfies that $W_2^4[-s_1^4](0.7)\approxeq 0.012157>0$ and $W_2^4[-s_1^4](1)\approxeq -0.203874<0$. So, it necessarily changes its sign.

We will see that $W_2^4[M](1)<0$ for all $M\in[-s_1^4,0]$.

We have that the Wronskian at $t=1$ can be expressed as follows:

$$W_2^4[-s^4](1)=\frac{m(s)^3}{64e^{2\left(\sqrt{\frac{250-m(s)^3}{m(s)}}+m(s)\right)}\left(250-m(s)^3\right)\left(2m(s)^6+15625\right)^2}f\big(m(s)\big),$$

we can obtain the analytic expression of f, but it has a complicated expression to manage. However, it is strictly less than $-2\cdot 10^{19}$, so clearly negative, see Figure C.3. Thus, $W_2^4[-s^4](1)<0$ for every $s\in[0,s_1]$.

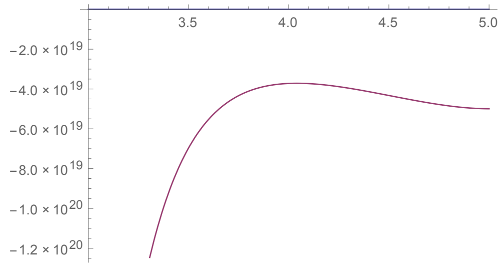

Figure C.3: $f(m(s))$ with $m(s) \in [3,5]$.

Hence, it must exist $M^* \in \big(-s_1^4, 0\big)$ such that $W_2^4[M^*]$ has a double zero on $c \in (0,1)$ and it is of constant sign. Also, we can affirm that $W_2^4[M]$ is negative for $0 \geqslant M \geqslant M^*$.

From Lemma 2.2.11, since $W_3^4[M](c) \neq 0$ for all $M \leqslant 0$, in particular for $M = M^*$, we can affirm that $W_1^4[M^*](c) = 0$. Hence, $M^* \geqslant \widehat{M}$.

Suppose that $M^* > \widehat{M}$.

We have the following expression of the Wronskian:

$$W_2^4[M](t) = \begin{vmatrix} y_1[M](t) & y_2[M](t) \\ \\ y_1[M]'(t) & y_2[M]'(t) \end{vmatrix}, \tag{C.4}$$

where $y_k[M](t)$ are defined in (1.1.6).

Firstly, we must verify that $y_2[M](t) = y_1[M]'(t)$. Trivially, $y_1[M]'(t)$ is a solution of (2.2.3), since it is a problem with constant coefficients.

Also,the following equalities are fulfilled:

$$\begin{aligned} y_2[M](0) &= y_1[M]'(0) = 0,\\ y_2[M]'(0) &= y_1[M]''(0) = 0,\\ y_2[M]''(0) &= y_1[M]^{(3)}(0) = 1,\\ y_2[M]^{(3)}(0) &= y_1[M]^{(4)}(0) = -1000\, y_1[M]'(0) - M\, y_1[M](0) = 0. \end{aligned}$$

Hence, by the expression of the Wronskian, since $W_1^4[M^*](c) = W_2^4[M^*](c) = 0$, we conclude that $W_1^4[M^*]'(c) = y_1[M^*]'(c) = 0$.

Moreover $y_1[M^*]''(c) \neq 0$, because $W_3^4[M^*](c) \neq 0$.

Then, in a neighbourhood of $t = c$ the function $y_1[M^*](t)$ does not change its sign, but, since $M^* > \widehat{M}$, $y_1[M^*](t)$ has to change sign necessarily. And we have seen that $y_1[M^*](1) > 0$. Also, by definition, $y_1[M^*](t)$ is positive on a neighbourhood of $t = 0$. So, since it has at least a change of sign, it must have at least two more. Hence it has to have at least seven zeros, having three at $t = 0$, two at $t = c$ and two simple zeros at $t = t_1$ y $t = t_2$.

Now, we will study the maximum number of zeros which has $y_1[0](t)$ on $[0,1]$. It is given by the expression (C.1), and its third order derivative follows the expression:

$$y_1[0]'''(t) = \frac{1}{3}e^{-10t}\left(2e^{15t}\cos\left(5\sqrt{3}t\right) + 1\right),$$

which only can have zeros if $g(t) := 2e^{15t}\cos\left(5\sqrt{3}t\right) + 1$ do it. However, the function:

$$g'(t) = -10e^{15t}\left(\sqrt{3}\sin\left(5\sqrt{3}t\right) - 3\cos\left(5\sqrt{3}\right)\right),$$

has three zeros on the interval $[0,1]$, and satisfy $g'(0) = 30 > 0$, coupled with:

$$g'(1) = -10\,e^{15}\left(\sqrt{3}\sin\left(5\sqrt{3}\right) - 3\cos\left(5\sqrt{3}\right)\right) < 0.$$

So, $g(t)$ and also $y_1[0]'''(t)$ can have at most 4 zeros if $g(0) < 0$ and $g(1) < 0$, but we have that $g(0) = 3 > 0$. Hence $y_1[0]'''(t)$ has at most 3 zeros on $[0,1]$.

As direct consequence $y_1[0](t)$ can have at most 6 zeros. But $y_1[M^*](t)$ has at least 7. Then, since $y_1[M](1) > 0$ for $M \in [M^*, 0]$, it must exist $M^{**} \in (M^*, 0)$, such that $y_1[M^{**}](t)$ has a double zero at $t^{**} \in (0,1)$.

But, this fact, from the expression (C.4) implies that $W_2^4[M^{**}](t^{**}) = 0$, and this is a contradiction with $M^{**} < M^*$. So, necessarily, $M^* \leqslant \widehat{M}$. Thus, $M^* = \widehat{M}$.

Then, we conclude that $W_2^4[M](t) \neq 0$ on $(0,1]$ for $M > \widehat{M}$.

So, we have seen that the hypotheses of Corollaries 2.2.5, 2.2.6 and 2.2.7 and Theorem 2.2.8 are only satisfied for a unique $\widehat{M} \in \mathbb{R}$.

In this case, in terms of Theorem 2.2.9, we have that 0 is an eigenvalue of $T_4^6[\widehat{M}]$ in $X_{3[0,\tilde{t}]}$ and in $X_{2[0,\tilde{t}]}$.

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Resumo

Esta Tese, enmarcada baixo o título “Existencia de solucións de problemas de contorno non lineais”, contén unha recompilación pormenorizada dos diferentes resultados probados pola autora ó longo da súa etapa predoutoral.

O interese das ecuacións diferenciais non lineais é ben coñecido debido a que a práctica totalidade dos fenómenos físicos, así como outros moitos en economía, medicina, bioloxía ou química veñen modelados por ecuacións deste tipo.

É de vital importancia facer un estudo, o máis preciso posible, das propiedades de existencia de solución, así como da súa unicidade ou multiplicidade. Neste traballo centrarémonos no estudo cualitativo de diversos problemas diferenciais de contorno, tanto lineais como non lineais. En particular, na maioría de ocasións o noso interese fundamental será poder garantir a existencia de solucións con signo constante no seu intervalo de definición. Isto é debido a que moitas das magnitudes modeladas mediante ecuacións diferenciais só poden tomar valores non negativos.

A pesar de que no título soamente se fai referencia ó estudo de problemas non lineais; isto, en moitos dos casos, non é posible sen facer previamente unha exhaustiva análise dun problema lineal asociado. Por este motivo a primeira parte da tese está adicada ó estudo de problemas de contorno lineais. Unha vez coñecidos os resultados para os problemas lineais, procédese ó estudo de problemas de contorno non lineais.

Aínda que o estudo dos problemas de contorno lineais foi presentado como unha ferramenta para poder abordar posteriormente a análise de certos problemas non lineais, este tipo de problemas e o seu estudo teñen un interese propio en sí mesmos. De feito, ó longo dos primeiros cinco capítulos, aparecen numerosos exemplos onde se pode ver claramente a utilidade dos diferentes resultados.

Capítulo 1: Resultados previos

Coa intención de construír un traballo autosuficiente, o Capítulo 1 contén unha escolma de conceptos e resultados que se usarán ó longo do resto de capítulos.

Nun primeiro momento, encadraremos os nosos intereses arredor do concepto de disconxugación para unha ecuación diferencial de orde n:

$$T_n[M]\,u(t) = u^{(n)}(t) + p_1(t)\,u^{(n-1)}(t) + \cdots + p_{n-1}(t)\,u'(t) + \big(p_n(t) + M\big)\,u(t) = 0, \quad (1)$$

onde $t \in I \equiv [a,b]$ e $p_j \in C^{n-j}(I)$.

Seguindo a monografía de Coppel, véxase [42, Capítulo 3], establécense certas propiedades dunha ecuación diferencial en relación con este concepto de disconxugación, que ten que ver co máximo número de ceros que pode ter calquera solución non trivial da ecuación diferencial dada.

O obxectivo da Sección 1.2 é introducir o concepto de función de Green asociada a un problema diferencial lineal de orde n con condicións de dous puntos:

$$\begin{aligned} T_n[M]\, u(t) &= \sigma(t), \quad t \in I, \\ U_i(u) &= 0, \quad i = 1, \dots, n, \end{aligned} \tag{2}$$

onde:

$$U_i(u) = \sum_{j=0}^{n-1} \left(\omega_j^i \, u^{(j)}(a) + \nu_j^i \, u^{(j)}(b) \right), \quad i = 1, \dots, n,$$

sendo ω_j^i e ν_j^i constantes reais para $i = 1, \dots, n$, $j = 0, \dots, n-1$ e $\sigma \in C(I)$.

É ben coñecido, véxase Teorema 1.2.4, que o problema (2) ten unha única solución se, e só se, existe unha única función de Green asociada, $g_M(t,s)$. Neste caso, a única solución vén dada por:

$$u(t) = \int_a^b g_M(t,s)\, \sigma(s) \, \mathrm{d}\, s.$$

Desta expresión dedúcese claramente que se a función de Green é de signo constante, entón a solución do problema (2) asociado vai a ter o mesmo signo constante para todo $\sigma \geqslant 0$. Isto é o que se coñece como carácter inverso positivo ou negativo, dependendo do signo.

Nesta sección presentase unha escolla de resultados relacionados coa función de Green que aparecen recollidos en [16]. Así mesmo, introdúcese unha relación entre a disconxugación e o signo constante da función de Green asociada ó problema:

$$\begin{aligned} T_n[M]\, u(t) &= \sigma(t), \quad t \in I, \\ u(a) = u'(a) = \cdots &= u^{(k-1)}(a) = 0, \\ u(b) = u'(b) = \cdots &= u^{(n-k-1)}(b) = 0, \end{aligned} \tag{3}$$

véxase Lema 1.2.14.

As condicións de contorno consideradas en (3) son as coñecidas como $(k, n-k)$ e denótase por X_k ó espazo de funcións de clase $C^n(I)$ que verifican ditas condicións.

Para finalizar o capítulo preliminar introdúcese a expresión do operador adxunto a un operador dado, así como unha relación entre as funcións de Green de ambos problemas:

$$g_M^*(t,s) = g_M(s,t),$$

sendo $g_M^*(t,s)$ a función de Green relativa ó problema adxunto asociado ó problema (2).

Capítulo 2: Disconxugación

Unha vez establecida a relación entre o signo constante da función de Green para os problemas $(k, n-k)$ e a disconxugación da ecuación diferencial asociada dada polo Lema 1.2.14, este capítulo enfócase en caracterizar cando unha ecuación é ou non disconxugada. As caracterizacións son obtidas mediante teoría espectral.

Primeiro, caracterízase o conxunto de parámetros para os cales a ecuación diferencial (1) é disconxugada.

Teorema 1 (Theorem 2.1.1)**.** *Sexan $\bar{M} \in \mathbb{R}$ e $n \geqslant 2$ tales que* (1) *é unha ecuación disconxugada en I para $M = \bar{M}$. Entón,* (1) *é disconxugada en I se, e só se, $M \in (\bar{M} - \lambda_1, \bar{M} - \lambda_2)$ onde:*

- *$\lambda_1 = +\infty$ se $n = 2$ e, para $n > 2$, $\lambda_1 > 0$ é o mínimo dos menores autovalores positivos de $T_n[\bar{M}]$ en X_k con $n - k$ par.*
- *$\lambda_2 < 0$ é o máximo dos maiores autovalores negativos de $T_n[\bar{M}]$ en X_k con $n - k$ impar.*

Dado que a principal hipótese utilizada no teorema anterior é que exista un $\bar{M} \in \mathbb{R}$ tal que a ecuación (1) sexa disconxugada para $M = \bar{M}$, é importante poder garantir cando o conxunto de disconxugación é, ou non, baleiro. A segunda parte do Capítulo 2 contén varios resultados nesta liña, rematando con dous resultados equivalentes nos que se establece unha caracterización espectral para dita propiedade.

Teorema 2 (Theorem 2.2.9)**.** *A ecuación diferencial* (1) *é non disconxugada en I para todo $M \in \mathbb{R}$ se, e só se, existen $c_1, c_2 \in (a, b]$ e $\bar{M} \in \mathbb{R}$ tales que:*

- *Existe $k^* \in \{1, \dots, n-1\}$ tal que $n - k^*$ é par e $\lambda_1 \leqslant 0$ é un autovalor de $T_n[\bar{M}]$ en $X_{k^*[a,c_1]}$.*
- *Existe $k^{**} \in \{1, \dots, n-1\}$ tal que $n - k^{**}$ é impar e $\lambda_2 \geqslant 0$ é un autovalor de $T_n[\bar{M}]$ en $X_{k^{**}[a,c_2]}$.*

Observación 3. *Nótese que no anterior resultado $X_{k[c,d]}$ está denotando ó espazo X_k no intervalo $[c, d] \subset [a, b]$.*

O Teorema 2.2.10 é unha reformulación equivalente do Teorema 2 no que se establece unha caracterización espectral para garantir cando existe un mínimo intervalo de disconxugación.

Os resultados deste capítulo aparecen recollidos en [28, 31].

Capítulo 3: Funcións de Green

Este é o capítulo máis longo da tese, no que aparecen recollidos algúns dos resultados máis importantes deste proxecto. Ó mesmo tempo, moitos destes resultados son técnicos e cunha notación dificultosa, polo que ó longo do capítulo se ilustran cun exemplo particular recorrente. O obxectivo final é caracterizar o signo constante da función de Green asociada ó problema:

$$\begin{aligned} T_n[M] = \sigma(t), \quad t \in I \equiv [a, b], \\ u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0, \end{aligned} \tag{4}$$

onde $\sigma_i, \varepsilon_j \in \mathbb{Z}$ para $i = 1, \dots, k$, $j = 1, \dots, n-k$ e:

$$0 \leqslant \sigma_1 < \sigma_2 < \cdots < \sigma_k \leqslant n-1, \quad 0 \leqslant \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leqslant n-1.$$

Observación 4. *Denotamos por $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ ó espazo das funcións de clase $C^n(I)$ verificando as condicións de fronteira consideradas. Denotaremos, a maiores α, $\beta \in \{0,1,\dots,n-1\}$ tales que:*

$$\alpha \notin \{\sigma_1,\dots,\sigma_k\},\ \textit{e se } \alpha \neq 0,\ \textit{entón } \{0,\dots,\alpha-1\} \subset \{\sigma_1,\dots,\sigma_k\},$$
$$\beta \notin \{\varepsilon_1,\dots,\varepsilon_{n-k}\},\ \textit{e se } \beta \neq 0,\ \textit{entón } \{0,\dots,\beta-1\} \subset \{\varepsilon_1,\dots,\varepsilon_{n-k}\}.$$

En realidade caracterizaremos cando a función de Green cumpre certas propiedades máis fortes có signo constante:

(P_{g_1}) Existen tres funcións continuas ϕ, k_1 e k_2 tales que $\phi(s) > 0$ para todo $s \in (a,b)$ e $0 < k_1(t) < k_2(t)$ para todo $t \in (a,b)$, verificando:

$$\phi(s)\, k_1(t) \leqslant g_M(t,s) \leqslant \phi(s)\, k_2(t), \quad \text{para todo } (t,s) \in I \times I.$$

(N_{g_1}) Existen tres funcións continuas ϕ, k_1 e k_2 tales que $\phi(s) > 0$ para todo $s \in (a,b)$ e $k_2(t) < k_1(t) < 0$ para todo $t \in (a,b)$, verificando:

$$\phi(s)\, k_2(t) \leqslant g_M(t,s) \leqslant \phi(s)\, k_1(t), \quad \text{para todo } (t,s) \in I \times I.$$

Na Sección 3.3 dedúcense as hipóteses impostas ó operador e ás condicións de contorno que se indican a continuación.

(T_d) Diremos que o operador $T_n[\bar{M}]$ verifica a propiedade (T_d) en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ se, e só se, existe unha descomposición da seguinte forma:

$$T_0\, u(t) = u(t), \quad T_k\, u(t) = \frac{\mathrm{d}}{\mathrm{d}\, t}\left(\frac{T_{k-1}\, u(t)}{v_k(t)}\right),\ k = 1,\dots,n,$$

onde $v_k > 0$, $v_k \in C^n(I)$ son tales que:

$$T_n[\bar{M}]\, u(t) = v_1(t)\, \dots\, v_n(t)\, T_n\, u(t),\ t \in I,$$

e, ademais, para calquera $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ cúmprese:

$$T_{\sigma_1}\, u(a) = \cdots = T_{\sigma_k}\, u(a) = 0,$$
$$T_{\varepsilon_1}\, u(b) = \cdots = T_{\varepsilon_{n-k}}\, u(b) = 0.$$

(N_a) Diremos que $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ verifican a propiedade (N_a) se, e só se,

$$\operatorname{card}\Big\{\ell \in \{\sigma_1,\dots,\sigma_k\} \mid \ell < h\Big\} + \operatorname{card}\Big\{\ell \in \{\varepsilon_1,\dots,\varepsilon_{n-k}\} \mid \ell < h\Big\} \geqslant h,$$

para todo $h \in \{1,\dots,n-1\}$.

- A descomposición dada pola propiedade (T_d) non ten porqué ser única.

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

- Se $T_n[\bar{M}]$ satisfai (T_d) en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$, entón $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ verifican (N_a) se, e só se, $\lambda = 0$ non é autovalor de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
- Na Sección 3.8, próbase que a hipótese (T_d) non pode ser eliminada en xeral.

Tras unha serie de resultados previos, na Sección 3.7 establécese unha caracterización do conxunto de parámetros para os que a función de Green satisfai ou ben $\left(P_{g_1}\right)$ ou ben $\left(N_{g_1}\right)$, sendo esta a propiedade verificada nun entorno do $\bar{M} \in \mathbb{R}$ para o que se cumpre a hipótese (T_d).

Teorema 5 (Theorem 3.7.1)**.** *Sexa $\bar{M} \in \mathbb{R}$ tal que $T_n[\bar{M}]$ verifica a propiedade (T_d) en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ e $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ cumpren (N_a).*

Denotemos $g_M(t,s)$ como a función de Green asociada ó problema (4)*. Cúmprense as seguintes propiedades.*

- *Se $n-k$ é par e $2 \leqslant k \leqslant n-1$, entón $g_M(t,s)$ cumpre a propiedade $\left(P_{g_1}\right)$ se, e só se, $M \in (\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, onde:*
  - * $\lambda_1 > 0$ *é o menor autovalor positivo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * $\lambda_2 < 0$ *é o máximo entre:*
    - · $\lambda_2' < 0$*, o maior autovalor negativo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
    - · $\lambda_2'' < 0$*, o maior autovalor negativo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *Se $k=1$ e n é impar, entón $g_M(t,s)$ cumpre a propiedade $\left(P_{g_1}\right)$ se, e só se, M pertence a $(\bar{M}-\lambda_1, \bar{M}-\lambda_2]$, onde:*
  - * $\lambda_1 > 0$ *é o menor autovalor positivo de $T_n[\bar{M}]$ en $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
  - * $\lambda_2 < 0$ *é o maior autovalor negativo de $T_n[\bar{M}]$ en $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.*
- *Se $n-k$ é impar e $2 \leqslant k \leqslant n-2$, entón $g_M(t,s)$ cumpre a propiedade $\left(N_{g_1}\right)$ se, e só se, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, onde:*
  - * $\lambda_1 < 0$ *é o maior autovalor negativo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - * $\lambda_2 > 0$ *é o mínimo entre:*
    - · $\lambda_2' > 0$*, o menor autovalor positivo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}}$.*
    - · $\lambda_2'' > 0$*, o menor autovalor positivo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_k|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}$.*
- *Se $k=1$ e $n>2$ é par, entón $g_M(t,s)$ cumpre a propiedade $\left(N_{g_1}\right)$ se, e só se, $M \in [\bar{M}-\lambda_2, \bar{M}-\lambda_1)$, onde:*
  - * $\lambda_1 < 0$ *é o maior autovalor negativo de $T_n[\bar{M}]$ en $X_{\{\sigma_1\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}$.*
  - * $\lambda_2 > 0$ *é o menor autovalor positivo de $T_n[\bar{M}]$ en $X_{\{\sigma_1|\alpha\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-2}\}}$.*

- *Se $k = n - 1$ e $n > 2$, entón $g_M(t,s)$ cumpre a propiedade $\left(N_{g_1}\right)$ se, e só se, $M \in [\bar{M} - \lambda_2, \bar{M} - \lambda_1)$, onde:*
    - $*$ *$\lambda_1 < 0$ é o maior autovalor negativo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.*
    - $*$ *$\lambda_2 > 0$ é o menor autovalor positivo de $T_n[\bar{M}]$ en $X_{\{\sigma_1,\dots,\sigma_{n-2}\}}^{\{\varepsilon_1|\beta\}}$.*
- *Se $n = 2$, entón $g_M(t,s)$ cumpre a propiedade $\left(N_{g_1}\right)$ se, e só se, $M \in (-\infty, \bar{M} - \lambda_1)$, onde:*
    - $*$ *$\lambda_1 < 0$ é o maior autovalor negativo de $T_n[\bar{M}]$ en $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.*

Na segunda parte da Sección 3.7, procédese ó estudo da función de Green para un conxunto de parámetros que non contén a $\bar{M}$. Neste caso non se obtén unha caracterización completa do conxunto de parámetros que satisfán unha das propiedades previamente mencionadas, pero si que se dá unha cota superior para este intervalo, xunto con certas propiedades que a función de Green verifica para ese conxunto de parámetros.

Os resultados deste capítulo aparecen recollidos en [29, 30].

Capítulo 4: Operador fortemente inverso positivo (negativo)

Un operador $T_n[M]$ dise fortemente inverso positivo en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ se calquera función $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ tal que $T_n[M]\, u \gneqq 0$ en I, debe cumprir $u > 0$ on (a,b) e, ademais,

$$u^{(\alpha)}(a) > 0 \text{ and } \begin{cases} u^{(\beta)}(b) > 0, & \text{se } \beta \text{ é par,} \\ u^{(\beta)}(b) < 0, & \text{se } \beta \text{ é impar.} \end{cases}$$

Analogamente pódese definir o concepto de operador fortemente inverso positivo.

Establécese, así mesmo, unha relación entre o signo constante da función de Green e o carácter fortemente inverso positivo ou negativo do problema asociado. Utilizando esta relación, no Capítulo 4 obtense unha caracterización para estas propiedades. De feito, poderiamos reescribir o Teorema 5, substituíndo as propiedades $\left(P_{g_1}\right)$ e $\left(N_{g_1}\right)$ por carácter fortemente inverso positivo e negativo, respectivamente, véxase Teorema 4.1.1.

Como consecuencia deste resultado, utilizando o método de sub e sobre solucións, obtense un corolario directo para o seguinte operador:

$$T_n[M,c]\, u(t) = T_n[M]\, u(t) + c(t)\, u(t), \quad t \in I, \tag{5}$$

con $c \in C(I)$.

Corolario 6 (Corollary 4.1.3). *Sexa o operador $T_n[M,c]$ definido en* (5), *$\bar{M} \in \mathbb{R}$ tal que $T_n[\bar{M}]$ cumpre a propiedade (T_d) en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ e $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ verificando (N_a).*

Consideremos λ_1 e λ_2 como no Teorema 5. Entón, cúmprense as seguintes propiedades:

- *Se $n-k$ é par e $-\lambda_1 < c(t) \leqslant -\lambda_2$ para todo $t \in I$, entón $T_n[\bar{M},c]$ é fortemente inverso positivo en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

- *Se $n-k$ é par, $n > 2$ e $-\lambda_2 \leqslant c(t) < -\lambda_1$ para todo $t \in I$, entón $T_n[\bar{M}, c]$ é fortemente inverso positivo en $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

- *Se $n = 2$ e $c(t) < -\lambda_1$ para todo $t \in I$, entón $T_n[\bar{M}, c]$ é fortemente inverso positivo en $X_{\{\sigma_1\}}^{\{\varepsilon_1\}}$.*

Na Sección 4.3 preséntase unha gran escolma de casos particulares onde se ve a aplicabilidade dos resultados obtidos, xa que para obter o conxunto de parámetros para os que o operador é fortemente inverso positivo ou negativo, evitamos o cálculo da expresión da función de Green e o seu posterior estudo. Polo tanto os resultados obtidos móstranse como unha ferramenta moi potente á hora de garantir que unha gran familia de problemas de lineais de contorno de dous puntos ten solución de signo constante.

Na seguinte sección, introducímonos no estudo de problemas con condicións de contorno non homoxéneas. O resultado principal desta sección contén unha caracterización espectral do carácter fortemente positivo ou negativo para un elevado número de problemas de contorno con condicións non homoxéneas.

Finalízase o capítulo presentando unha serie de exemplos nos que se ve de novo a utilidade dos resultados probados previamente. Aparecen reflectidos problemas con coeficientes nonconstantes, nos que se ve aínda máis claramente o interese dos resultados obtidos.

Ó igual ca no Capítulo 3, a maioría dos resultados e exemplos aquí mostrados aparecen publicados en [29, 30].

Capítulo 5: Viga simplemente suxeita

Este capítulo está adicado a un problema de orde catro con condicións de contorno asociadas a unha viga simplemente suxeita. Este é un modelo moi interesante na física, xa que modela o comportamento, entre outros, dunha ponte colgante. Neste caso o signo constante dunha solución é equivalente a que o desprazamento vertical da superficie da ponte se produza nunha soa dirección (no sentido das forzas aplicadas, a forza da gravidade fai que estas se dirixan cara abaixo xeralmente, no caso do signo positivo). Isto é un feito fundamental para poder asegurar a estabilidade da estrutura.

En [53, Sección 2.6.6], considérase unha viga simplemente suxeita (que modela a calzada da ponte colgante) suxeita a forzas non lineais ó longo de cordas (os cables da ponte en suspensión). O modelo unidimensional para o desprazamento vertical da calzada sería:

$$\begin{cases} E\,I\,u^{(4)}(t) - T\,u''(t) + g(u(t)) = q(t), \quad t \in I, \\ u(a) = u(b) = u''(a) = u''(b) = 0, \end{cases} \tag{6}$$

onde a e b son os extremos da ponte, E e I son dúas constantes positivas dadas polo material da viga (o módulo de Young e o momento de inercia), $T \geqslant 0$ é a constante da forza de tensión, $q(t)$ é unha forza cara abaixo distribuída na viga e g é a forza de recuperación. Se consideramos g como unha función non autónoma da forma $g\big(t, u(t)\big) = f(t)\,u(t)$, sendo $f \in C(I)$, dividindo en (6) por $E\,I$ obtemos:

$$T_4[p,c]\,u(t) = u^{(4)}(t) - p\,u''(t) + c(t)\,u(t) = h(t), \quad t \in I,$$

onde $p = \dfrac{T}{E\,I}$, $c(t) = \dfrac{f(t)}{E\,I}$ e $h(t) = \dfrac{q(t)}{E\,I}$.

A segunda parte do capítulo estará adicada a probar a existencia de solucións constantes para o problema:

$$\begin{aligned} T_4[p,c]\,u(t) &= h(t)(\geqslant 0), \quad t \in I \\ u(a) = u''(a) &= u(b) = u''(b) = 0, \end{aligned} \tag{7}$$

para casos máis xerais de c fronte ós vistos anteriormente.

Na primeira parte do capítulo realízase un estudo similar ó xa feito nos capítulos anteriores para o problema particular:

$$\begin{aligned} T_4[M]\,u(t) = u^{(4)}(t) + p_1(t)\,u'''(t) + p_2(t)\,u''(t) + M\,u(t) = h(t), \quad t \in I \\ u(a) = u''(a) = u(b) = u''(b) = 0, \end{aligned}$$

onde $p_1 \in C^3(I)$ e $p_2 \in C^2(I)$.

Os resultados obtidos neste caso para o carácter fortemente inverso positivo son un caso particular do xa visto no Capítulo 4. En troques, neste caso, para o carácter fortemente inverso negativo pódese probar que o intervalo máximo obtido no que esta propiedade se pode satisfacer é, en realidade, o intervalo óptimo para o que o problema estudado é fortemente inverso negativo.

Nesta sección tamén se mostran varios casos particulares nos que se segue a ver a utilidade dos resultados obtidos. Os resultados desta sección aparecen recollidos en [32].

Como xa dixemos, a segunda parte do Capítulo 5 está adicada ó estudo do problema (7). Neste capítulo introdúcese unha técnica diferente ás usadas anteriormente que será máis profundamente desenrolada no Capítulo 7: a formulación variacional.

Tódolos resultados probados anteriormente na liña do Corolario 6, para garantir a existencia de solución de signo constante, esixían que a función $c(t)$ estivese contida entre dous autovalores de $T_4[p,0]$ con diferentes condicións de contorno prefixadas. Neste caso, facendo uso da formulación variacional, podemos superar estes valores en certo sentido. De feito, permítese que a función $c(t)$ supere estes valores tanto como sexa necesario nalgún punto, impoñendo algunha hipótese extra, por exemplo que a integral da parte negativa da función estea limitada por unha cota dada (véxase Teorema 5.2.8). Isto permite construír modelos con funcións moito máis diversas que as permitidas anteriormente.

Para finalizar o capítulo, móstrase un caso particular dunha ponte colgante no que os resultados anteriores son aplicados. Os resultados desta parte pódense ver en [33].

Con este capítulo péchase o estudo cualitativo dun elevado número de problemas de contorno lineais, no que se obtiveron resultados de aplicabilidade probada a través dos numerosos exemplos que aparecen ó longo dos capítulos.

Capítulo 6: Existencia de solución para problemas non lineais vía funcións de Green de signo constante

O obxectivo deste capítulo será usar os resultados anteriormente probados para poder saber cando o seguinte problema non lineal ten unha ou múltiples solucións de signo constante para as distintas eleccións das condicións de contorno.

$$\begin{cases} T_n[M]\,u(t) = f\big(t, u(t)\big),\ t \in I \equiv [a,b], \\ u^{(\sigma_1)}(a) = u^{(\sigma_2)}(a) = \cdots = u^{(\sigma_k)}(a) = 0, \\ u^{(\varepsilon_1)}(b) = u^{(\varepsilon_2)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0. \end{cases} \tag{8}$$

É ben coñecido que as solucións do problema anterior veñen dadas polos puntos fixos do operador integral:

$$\mathcal{L}[M]\,u(t) = \int_a^b g_M(t,s)\, f\big(s, u(s)\big)\,\mathrm{d}\,s, \tag{9}$$

onde $g_M(t,s)$ é a función de Green asociada á parte lineal do problema (8).

Introducindo a hipótese de que $g_M(t,s)$ verifique $\big(P_{g_1}\big)$ ou $\big(N_{g_1}\big)$, usando resultados ben coñecidos como o Teorema de punto fixo de Krasnosel'skiĭ, próbanse diversos resultados que garanten a existencia dun ou múltiples puntos fixos para o operador definido en (9) en certos espazos de Banach.

Unha vez os resultados son establecidos e demostrados para o problema xeral (8), na Sección 6.4, procédese ó estudo de problemas particulares. Aqueles para os que o problema lineal asociado xa foi estudado caracterizando o conxunto de parámetros para os cales a función de Green asociada verifica ou ben, a propiedade $\big(P_{g_1}\big)$, ou ben, $\big(N_{g_1}\big)$.

Un dos problemas estudados é:

$$\begin{cases} u^{(4)}(t) - \pi^4 u(t) = f\big(t, u(t)\big),\ t \in [0,1], \\ u(0) = u''(0) = 0, \\ u'(1) = u''(1) = 0, \end{cases} \tag{10}$$

onde f toma valores positivos en $[0,1] \times (-\infty, 0]$.

Mediante o seguinte resultado garántese a existencia de tres solucións negativas.

Teorema 7 (Theorem 6.4.4)**.** *Sexan p, q e r números positivos tales que:*

$$0 < p < \frac{500}{147}\,p < q < \frac{500}{147}q \leqslant r.$$

Asumamos, a maiores, que a función f verifica as seguintes condicións:

(a) $f(t,u) \leqslant \dfrac{3\,\pi^3}{2}\,r$ *para todo* $t \in [0,1]$ *e* $u \in [-r, 0]$,

(b) $f(t,u) < \dfrac{3\,\pi^3}{2}\,p$ *para todo* $t \in [0,1]$ *e* $u \in \left[-\dfrac{500}{147}p, 0\right]$,

(c) $f(t,u) \geqslant -214\,u$ *para todo* $t \in \left[\dfrac{1}{3}, 1\right]$ *e* $u \in \left[-\dfrac{500}{147}q, -q\right]$.

Entón o problema (10) *ten ó menos tres solucións, u_1, u_2, u_3 tales que $\|u_i\|_{C(I)} \leqslant r$ para $i = 1,2,3$ e:*

$$\min_{t \in \left[\frac{1}{3},1\right]} u_1(t) > -p > \min_{t \in \left[\frac{1}{3},1\right]} u_2(t), \qquad \max_{t \in \left[\frac{1}{3},1\right]} u_2(t) > -q > \max_{t \in \left[\frac{1}{3},1\right]} u_3(t).$$

Analogamente, para o problema:

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) = u''(1) = 0, \end{cases}$$

onde $f(t,u) \geqslant 0$ para todo $(t,u) \in [0,1] \times [0,+\infty)$, próbase a existencia de dúas solucións estritamente positivas mediante o seguinte resultado.

Teorema 8 (Theorem 6.4.16)**.** *Supoñamos que existen números positivos p, q e r tales que:*

$$0 < p < q < r.$$

Asumamos, a maiores, que a función f verifica as seguintes condicións:

(i) $f(t,u) \geqslant \dfrac{15625\,u}{66}$ *para todo* $t \in \left[\frac{1}{5}, \frac{4}{5}\right]$ *e* $u \in \left[r, \frac{125}{48}\, r\right]$*, sendo a desigualdade estrita para* $u = r$*,*

(ii) $f(t,u) \leqslant 72\,q$ *para todo* $t \in [0,1]$ *e* $u \in \left[0, \frac{125}{48}\, q\right]$*, sendo a desigualdade estrita para* $u = q$*,*

(iii) $f(t,u) > \dfrac{1600000\,u}{8073}$ *para todo* $t \in \left[\frac{1}{5}, \frac{4}{5}\right]$ *e* $u \in \left[\frac{48}{125}\, p, p\right]$.

Entón, o problema (6.4.37) *ten ó menos dúas solucións positivas,* u_1 *e* u_2*, tales que:*

$$p < \|u_1\|_{C(I)}, \quad \max_{t \in \left[\frac{1}{5}, \frac{4}{5}\right]} u_1(t) < q < \max_{t \in \left[\frac{1}{5}, \frac{4}{5}\right]} u_2(t), \quad \min_{t \in \left[\frac{1}{5}, \frac{4}{5}\right]} u_2(t) < r.$$

Estes resultados poden ser obtidos para unha gran familia de problemas de contorno, aínda que os cálculos poden ser complicados debido á expresión da función de Green asociada. Aínda así, a existencia dunha solución pode ser garantida se f cumpre unha das seguintes condicións:

$$f_0^+(t) := \limsup_{u \to 0^+} \frac{f(t,u)}{u} = 0, \quad \text{e} \quad f_\infty^-(t) := \liminf_{u \to \infty} \frac{f(t,u)}{u} = +\infty,$$
$$f_0^-(t) := \liminf_{u \to 0^+} \frac{f(t,u)}{u} = +\infty, \quad \text{e} \quad f_\infty^+(t) := \limsup_{u \to \infty} \frac{f(t,u)}{u} = 0,$$

sen necesidade de coñecer a expresión da función de Green, véxanse Corolarios 6.2.4 e 6.2.5.

As técnicas aquí desenroladas son válidas para problemas con condicións de contorno diferentes ás aquí estudadas, sempre e cando se poida garantir que ou ben a propiedade $\big(P_{g_1}\big)$, ou ben $\big(N_{g_1}\big)$ se cumpren.

Capítulo 7: Existencia de solución para problemas non lineais vía formulación variacional

Este capítulo está adicado a desenvolver unha técnica diferente ás usadas anteriormente: a formulación variacional.

Os problemas non lineais estudados neste capítulo son diferentes dos estudados antes, no senso de que implican elementos non lineais en tódolos coeficientes non nulos. Entón non se pode obter un problema lineal asociado para o cal calcular a función de Green asociada coma antes. Neste caso, as solucións dos problemas considerados veñen dados por puntos críticos de operadores asociados en espazos de Banach adecuados.

Estúdanse dous tipos de problemas diferentes e ben diferenciados. Nun caso considérase un problema continuo, mentres que na segunda parte se estudan dous problema discreto. Os resultados da primeira sección aparecen en [83] e os relativos ós dous problemas discretos recóllense en [84].

Problemas de contorno periódicos con p-Laplaciano

Neste capítulo preténdese estudar o problema:

$$\begin{cases} \left[\varphi_p\left(u^{(n)}(t)\right)\right]^{(n)} + \displaystyle\sum_{i=1}^{n-1}(-1)^i a_i \left[\varphi_p\left(u^{(n-i)}(t)\right)\right]^{(n-i)} \\ \qquad\qquad + (-1)^n\Big(f\big(t,u(t)\big) - h\big(t,u(t)\big)\Big) = 0,\ t\in[0,T], \\ u(T)-u(0)=\cdots=u^{(2\,n-1)}(T)-u^{(2\,n-1)}(0)=0, \end{cases} \tag{11}$$

onde $T \geqslant 0$, $a_i \geqslant 0$ para $i = 1,\ldots,n-1$ e φ_p denota o funcional p-Laplaciano, definido como:

$$\varphi_p(t) = \begin{cases} t|t|^{p-2}, & t\neq 0, \\ 0, & t=0. \end{cases}$$

Neste caso todos os coeficientes non nulos presentan termos non lineais. Este é o motivo que fai que para estudar este problema non sexan válida as técnicas usadas no Capítulo 6. Por iso imos a desenvolver unha técnica distinta: a formulación variacional. Para iso necesitamos distinguir entre os conceptos de solución clásica ou forte e solución débil para o problema (11).

Definición 9. *Unha función $u \in C^n([0,T])$ dise solución clásica do problema* (11) *se $\varphi_p\left(u^{(n)}(\cdot)\right) \in C^n([0,T])$ e verifica* (7.1.1).

Previamente a definir o que é unha solución débil, debemos presentar o espazo no que está definida.

$$W_p := \Big\{u \in W^{n,p}(0,T)\ \mid u(T)-u(0)=\cdots=u^{(n-1)}(T)-u^{(n-1)}(0)=0\Big\},$$

onde $W^{n,p}(0,T)$ é o espazo de Sobolev:

$$W^{n,p}(0,T) = \Big\{u\in L^p(0,T)\ \mid u^{(i)}\in L^p(0,T),\ i=1,\ldots,n\Big\}.$$

Definición 10. *Unha función $u \in W_p$ dise solución débil do problema* (11) *se para cada $v \in W_p$ se verifica a seguinte igualdade:*

$$\begin{aligned} \int_0^T \varphi_p \left(u^{(n)}(t)\right) v^{(n)}(t)\,\mathrm{d}\,t + \sum_{i=1}^{n-1} a_i \int_0^T \varphi \left(u^{(n-i)}(t)\right)\, v^{(n-i)}(t)\,\mathrm{d}\,t \\ + \int_0^T \left(f\big(t,u(t)\big) - h\big(t,u(t)\big)\right) v(t)\,\mathrm{d}\,t = 0. \end{aligned} \tag{12}$$

As ferramentas dadas pola formulación variacional permiten garantir a existencia dunha ou múltiples solucións débiles, sempre e cando as funcións f e h satisfagan unha propiedade axeitada. En troques, non sempre podemos asegurar que as solucións débiles obtidas son, de feito, solucións clásicas do problema (11). Nesta sección vense varios resultados que garanten a existencia de solucións, tanto débiles coma clásicas.

Na parte final do capítulo vese como se poderían modificar estes resultados, sen demasiada dificultade, para un problema con impulsos.

Problemas non lineais discretos con p-Laplaciano

Esta sección está adicada a un problema diferente a tódolos estudados anteriormente, no sentido de que estamos a falar dun problema discreto. Vese nese capítulo como as técnicas de formulación variacional desenvoltas na sección anterior son válidas tamén, con modificacións evidentes, para un problema discreto.

Estudaranse dous problemas diferentes nos que aparece a seguinte ecuación en diferencias:

$$\begin{aligned} \Delta^n\left[\varphi_{p_n}\Big(\Delta^n u(k-n)\Big)\right] + \sum_{i=1}^{n-1}(-1)^i a_i\,\Delta^{n-i}\left[\varphi_{p_{n-i}}\Big(\Delta^{n-i}u(k-i)\Big)\right] \\ +(-1)^n\Big(V(k)\,\varphi_q\big(u(k)\big) - \lambda f\big(k,u(k)\big)\Big) = 0, \end{aligned} \tag{13}$$

onde $k \in \mathbb{Z}$ e:

- φ_p foi previamente definido para $p > 1$,
- $V : \mathbb{Z} \to \mathbb{R}$ é unha función positiva T-periódica para T, un enteiro prefixado,
- $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ é unha función dada con condicións adicionais en cada caso,
- $a_i \geqslant 0$ son números reais fixados para cada $i = 1, \ldots, n-1$,
- $p_i \geqslant q \geqslant 1$ para $i = 1, \ldots, n$,
- os operadores diferenciais dados por:

$$\begin{aligned} \Delta u(k) &= u(k+1) - u(k), \\ \Delta^i u(k) &= \Delta^{i-1}u(k+1) - \Delta^{i-1}u(k),\ \text{if } i \geqslant 2. \end{aligned}$$

Nunha primeira parte estúdase a existencia de solucións homoclínicas non nulas para a ecuación (13), é dicir, solución non triviais de (13) que cumpren:

$$\lim_{|k|\to+\infty} \big|u(k)\big| = 0.$$

Na segunda parte do capítulo estúdase o problema de contorno discreto:

$$\begin{aligned}\Delta^n\Big[\varphi_q\Big(\Delta^n u(k-2)\Big)\Big] + \sum_{i=1}^{n-1}(-1)^i a_i \Delta^{n-i}\Big[\varphi_q\Big(\Delta^{n-i}u\big(k-(n-i)\big)\Big)\Big]&\\ +(-1)^n\Big(V(k)\,\varphi_q\big(u(k)\big) - \lambda f\big(k,u(k)\big)\Big) &= 0, \qquad (14)\\ u(0) = \Delta u(-1) = \Delta^2 u(-2) = \cdots = \Delta^{n-1}u(1-n) &= 0,\\ u(T+1) = \Delta u(T+1) = \Delta^2 u(T+1) = \cdots = \Delta^{n-1}u(T+1) &= 0,\end{aligned}$$

onde $k \in \{0,1,\dots,T\}$.

Aquí establecendo hipóteses adecuadas, próbase a existencia de, ó menos, tres solucións para o problema anterior.

Conclusións e problemas futuros

Ó longo dos sete capítulos desta Tese foron probados un gran número de resultados cuxa utilidade foi plasmada nos diferentes exemplos.

Un dos grandes avances foi o de poder caracterizar o conxunto de parámetros para os que a función de Green asociada a un problema de contorno dado ten signo constante sen necesidade de coñecer a expresión de dita función. A función de Green adoita ser complicada de abordar, de feito, para o caso de problemas con coeficientes non-constantes nin sequera temos asegurado poder obter a súa expresión. En troques, o cálculo dos diversos autovalores é relativamente sinxelo en comparación. No caso de operadores con coeficientes constantes redúcese á resolución dun sistema ecuacións lineais e no caso de coeficientes non-constantes pode resolverse mediante ferramentas informáticas, como Mathematica, usando métodos numéricos. Polo tanto as distintas caracterizacións obtidas ó longo dos primeiros cinco capítulos son eficaces e prácticas, en particular a dada no Teorema 5 para as propiedades $\big(P_{g_1}\big)$ e $\big(N_{g_1}\big)$.

Un dos posibles problemas a estudar no futuro sería tratar de xeneralizar estes resultados para condicións de contorno máis diversas, por exemplo as periódicas ou condicións de dous puntos que non cumpren (N_a). Durante o noso estudo, observamos que usar as mesmas técnicas de oscilación que as aquí empregadas non eran válidas para o caso das condicións de dous puntos sen satisfacer a hipótese (N_a). Unha das causas é que para o $\bar{M} \in \mathbb{R}$ para o cal se satisfai a propiedade (T_d), $\lambda = 0$ é sempre un autovalor asociado ó operador coas condicións de contorno. Por outra banda, no caso das condicións periódicas, existen exemplos coñecidos nos que a función de Green non comeza a cambiar de signo en ningún dos vértices de $I \times I$. Polo tanto, a idea sería tratar de atopar unha nova técnica ou modificar a xa coñecida para poder chegar a conclusións similares nestes casos.

Aínda que, como se acaba de mencionar, quedan casos por abordar; a cantidade de problemas que se poden acoller nos resultados probados é moi elevada, inclúense problemas de tódalas ordes cunha gran variedade de condicións de contorno a escoller, por exemplo en orde catro considéranse 40 condicións de contorno diferentes entre as que están as coñecidas como condicións de contorno para unha viga fortemente suxeita ou dunha viga simplemente suxeita, entre outras.

No Capítulo 6, usando os resultados dos capítulos anteriores, garántese a existencia dunha ou múltiples solucións de signo constante para un gran número de condicións de contorno, tódalas estudadas anteriormente. De feito, se os resultados obtidos na primeira parte se puidesen xeneralizar para máis condicións de contorno, os resultados do Capítulo 6 quedarían directamente xeneralizados.

Debido á gran cantidade de problemas de punto fixo existentes, é posible que a aplicación doutros resultados diferentes, nos garanta a existencia de solución baixo condicións diferentes para a función f.

No Capítulo 7, introdúcese unha técnica diferente para levar a cabo a obtención de resultados. Isto é debido á diferente natureza dos problemas a estudar. Ó longo do capítulo, usando a formulación variacional xunto con teoremas de puntos críticos coñecidos anteriormente, chégase a resultados de existencia de solución. Vese, a maiores, neste capítulo, que xeneralizar un resultado, coñecido para un problema dunha orde dada, a ordes superiores non é doado xeralmente.

Un problema a estudar no futuro sería, na liña de [27], combinar as técnicas da formulación variacional co signo constante da función de Green para poder obter resultados que garantan a existencia de solución de diferentes problemas de contorno non lineais.

A maiores das futuras liñas de traballo mencionadas anteriormente, ó longo dos capítulos vanse detallando diferentes problemas abertos xunto coas dificultades que non permitiron solucionalos polo momento.

Chegamos coa fin deste proxecto a unha parada na nosa viaxe simbólica a través do estudo de problemas de contorno lineais e non lineais, con moito camiño percorrido; pero con moito máis por percorrer. E, aínda que os resultados obtidos son interesantes, útiles e potentes, como escribiu Robert Louis Stevenson: "*Eu non viaxo por ir a ningunha parte, senón que por ir. Viaxo polo feito de viaxar. A cuestión é moverse*". Polo tanto, o estudo de problemas de contorno lineais e non lineais continuara para alcanzar novos resultados e novos problemas abertos nun gratificante camiño sen un final determinado.

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