The Owen and Banzhaf-Owen values revisited

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Abstract

In this work, we consider games with coalitional structure. We afford two new parallel axiomatic characterizations for the well-known Owen and Banzhaf-Owen coalitional values. Two properties are common to both characterizations: a property of balanced contributions and a property of neutrality. The results prove that the main difference between these two coalitional values is that the former is efficient while the latter verifies a property of 2-efficiency.

Keywords: cooperative game, Shapley value, Banzhaf value, coalition structure, coalitional value, balanced contributions.

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1 Introduction

The main objective of the cooperative game theory is the investigation of solutions (or values) for games with transferable utility (TU games). A value decides the payoffs allocated to each player in a cooperative game. It can be applied to share costs in

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economic problems or to measure the power of each agent in a collective decision-making system. The Shapley value and the Banzhaf value are two of the best known concepts in this respect. The relevant difference between these two values is that the Shapley value is efficient while the Banzhaf value satisfies a property of 2-efficiency.

When enlightenment on affinity among players is disposable, coalitional values are the most appropriate tools to settle the payoffs. These agreements among players are modeled by a set of a priori unions, i.e., a partition of the set of players. The appraisal of the impact that derives from the action of forming unions is of great interest to game theorists. *Games with a coalition structure* were first considered by Aumann and Drèze [10]. A different approach was used by Owen [24] when introducing and axiomatically characterizing the *Owen* value. In this case, the unions play a game among themselves (the quotient game), and after that the players of a union play an internal game. Both payoffs, in the quotient game for unions and in the internal games for players of the same union, are given by the Shapley value. If both payoffs are given by the Banzhaf value, the *Banzhaf-Owen* value (cf. Owen [25]) is obtained.

In the characterization of the Owen value (Owen [24]), properties of efficiency, additivity, symmetry and null player are used. Later, in a new characterization, Vázquez-Brage et al. [27] introduced a property of balanced contributions in the unions and characterized the Owen value as the only coalitional value for singletons satisfying this property and the quotient game property. Other characterizations of the Owen value can be found in Hart and Kurz [17], in Winter [29], and in Hamiache [16], which make use of properties of consistency, or in Albizuri [2]. The principle of balanced contributions has also been used to characterize the Owen value in Amer and Carreras [8], Calvo et al. [13], and Gómez-Rúa and Vidal-Puga ([14] and [15]).

The Banzhaf-Owen value was first characterized in Albizuri [1], for the particular case of simple games, employing, among others, a property called delegation, which was already used by Lehrer [19] for a characterization of the Banzhaf value without using the additivity axiom. The first characterization of the Banzhaf-Owen value for the family of TU games was provided by Amer et al. [9]. In the characterization, they introduced the properties of delegation neutrality and delegation transfer. Laruelle and Valenciano [18] showed that the Banzhaf-Owen value of a particular player i of a union P_k can be identified as the Banzhaf value of a modified TU game. This game is played by the unions other than P_k and by the players in P_k. Alonso-Meijide et al. [3] provide an axiomatic characterization of the Banzhaf-Owen value. They use a property stronger than the balanced contributions property and a property weaker than the quotient game property. The axiomatic system used there is also compared with parallel axiomatizations of the Owen value. The parallel axiomatizations are useful to decide which value should be chosen, taking into account the context where they will be applied. It is also appropriate to make a reference to the paper of van den Brink and van der Laan [12]. In this paper, a class of share values for games with coalitional structure is axiomatized by using a multiplication property. This class contains both the Owen coalitional share function and the Banzhaf coalitional share function. The first one corresponds with the Owen coalitional value, but with sum of payoffs (shares) normalized to one, but the second one is different from the Banzhaf-Owen coalitional (share) value, that does not satisfy the multiplication property.

The aim of the current paper is to provide a new comparison of these two mentioned coalitional values from the point of view of their properties. To this aim, we present two parallel axiomatic characterizations for the Owen and Banzhaf-Owen values. The property of balanced contributions, introduced in Vázquez-Brage et al. [27], and the property of neutrality for reduced games (it is a similar property to delegation neutrality, that was defined by Amer et al. [9]), appear in both characterizations.

2 Preliminaries

We recollect here some basic concepts.

2.1 Games and values

A cooperative TU game with a finite number of *players* $N = \{1, 2, \ldots, n\}$ (or simply a *game*) is a pair (N, w), where $w : 2^N \to \mathbb{R}$ is a function that allocates to each *coalition* of players $S \subseteq N$ a real number $w(S)$ which represents the utility that every coalition can obtain and verifies $w(\emptyset) = 0$. For a finite set A, we denote $a =\mid A \mid$. From now on, we will denote by $\mathcal{G}_N$ the family of all games with a given set of player N and by $\mathcal{G}$ the family of all games.

We will denote by $f : \mathcal{G} \to \mathbb{R}^N$, the *value* that allocates to every game (N, w), a vector $f(N, w) = (f_1(N, w), \ldots, f_n(N, w)) \in \mathbb{R}^N$ whose components represent the payoffs assigned to every player.

Two of the more studied values that appear in this context are the *Shapley value* (Shapley [26]), which is defined as $\varphi_i(N, w) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!}[w(S \cup \{i\}) - w(S)]$, and the *Banzhaf value* (Banzhaf [11]), that is
$\beta_i(N, w) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}}[w(S \cup \{i\}) - w(S)]$, for all $(N, w) \in \mathcal{G}_N$ and all $i \in N$.

2.2 Games with a coalition structure

Given a finite set of players $N = \{1, 2, \ldots, n\}$, we will express by $P(N)$ the set of all partitions of N. Every $P = \{P_1, P_2, \ldots, P_m\} \in P(N)$, is called a *system of a priori unions* or a *coalition structure* on N. Thus, immediately arise two *trivial coalition structures*, the one formed only by the grand coalition, $P^N = \{N\}$, and the system where every union is composed by a unique *singleton* player, $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$.

Let $i \in N$ be a player, we will denote by $P(i)$ the family of a priori unions over N where $\{i\}$ is a singleton union, that is, there exists a union in the partition which is formed only by the player i. Formally, $P \in P(i)$ if and only if $\{i\} \in P$.

Suppose that we have a player $i \in P_k \in P$. If the player i decides to leave the union he/she belongs to and forms a union by himself, we will denote this new partition by P_{-i}. Formally, $P_{-i} = \{P_h \in P : h \neq k\} \cup \{P_k \setminus \{i\}, \{i\}\}$. Notice that $P_{-i} \in P(i)$.

Given a game $(N, w) \in \mathcal{G}_N$ and a system of unions $P \in P(N)$, we define a *cooperative game with a coalition structure* as the 3-tuple (N, w, P). We will denote by $\mathcal{G}^{cs}$ the set of all cooperative games with a system of unions, and by $\mathcal{G}^{cs}_N$ the subset when the player set is N.

If $(N, w, P) \in \mathcal{G}^{cs}$ and $P = \{P_1, P_2, \ldots, P_m\}$, the *quotient game* is the cooperative game (M, w^P) where the set $M = \{1, 2, \ldots, m\}$ is composed by the representatives of every union and $w^P(R) = w(\cup_{r\in R} P_r)$ for all $R \subseteq M$. In other words, the quotient game is the game played by the unions (considering that every union acts as a representative player of the whole union). Note that whenever $P = P^n$, the quotient game (M, w^P) coincides with (N, w).

Let $(N, w, P) \in \mathcal{G}^{cs}$ be a cooperative game with $P = \{P_1, P_2, \ldots, P_m\}$ the coalition structure. Suppose that two players of the same union, $i, j \in P_k \in P$ with $i \neq j$, decide to merge together and form a new player $p \notin N$. Let us denote the new set of players $N^{\{i,j\}} = (N \setminus \{i, j\}) \cup \{p\}$ and the new system of a priori unions $P^{\{i,j\}} = \left\{P_1^{\{i,j\}}, P_2^{\{i,j\}}, \ldots, P_m^{\{i,j\}}\right\}$ where $P_k^{\{i,j\}} = (P_k \setminus \{i, j\}) \cup \{p\}$ and $P_h^{\{i,j\}} = P_h$ for $h \neq k$. Thus, the $\{i, j\}$-*reduced game* $\left(N^{\{i,j\}}, w^{\{i,j\}}, P^{\{i,j\}}\right)$ of (N, w, P) is defined by $w^{\{i,j\}}(S) = w((S \setminus \{p\}) \cup \{i, j\})$ if $p \in S$ and $w^{\{i,j\}}(S) = w(S)$ otherwise, for every $S \subseteq N^{\{i,j\}}$.

We will denote by $g : \mathcal{G}^{cs} \to \mathbb{R}^N$ the *coalitional value* that allocates to every game (N, w, P) a vector $g(N, w, P) = (g_1(N, w, P), \ldots, g_n(N, w, P)) \in \mathbb{R}^N$, whose components represent the payoffs assigned to every player. We consider here two possibilities of combining the Banzhaf and the Shapley value in two steps, in order to obtain a coalitional value.

Definition 2.1 (Owen [24]) The *Owen value* Φ is the coalitional value defined by:

$$\Phi_i(N, w, P) = \sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i\}} \frac{r!\,(m-r-1)!}{m!} \frac{t!\,(p_k-t-1)!}{p_k!} \Big[w(Q\cup T\cup\{i\}) - w(Q\cup T)\Big],$$

for all $(N, w, P) \in \mathcal{G}^{cs}_N$, $i \in P_k$, and $P_k \in P$, where $Q = \bigcup_{r\in R} P_r$.

Definition 2.2 (Owen [25]) The *Banzhaf-Owen value* Ψ is the coalitional value defined by:

$$\Psi_i(N, w, P) = \sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{p_k-1}} \Big[w(Q \cup T \cup \{i\}) - w(Q \cup T)\Big],$$

for all $(N, w, P) \in \mathcal{G}^{cs}_N$, $i \in P_k$, and $P_k \in P$, where $Q = \bigcup_{r\in R} P_r$.

2.3 Comparison between the two coalitional values in games with a trivial systems of unions

The coalitional values introduced in the previous section can be considered extensions of the Shapley and Banzhaf values in the sense that they coincide with them when the system of unions is the trivial one, that is, each coalition is formed by a unique player. To formalize it, we introduce the concept of coalitional values for singletons.

We say that a coalitional value g on $\mathcal{G}^{cs}$ is a *coalitional f–value for singletons*, where f is a value on $\mathcal{G}$, if $g(N,w,P^n) = f(N,w)$ for all $(N,w) \in \mathcal{G}$. It is known that the Owen value Φ is a coalitional Shapley value for singletons ($\Phi(N,w,P^n) = \varphi(N,w)$) and the Banzhaf-Owen value Ψ is a coalitional Banzhaf value for singletons ($\Psi(N,w,P^n) = \beta(N,w)$).

3 Characterization results

This section contains the main results of the work and several remarks.

3.1 Axioms

Below we introduce several properties for a coalitional value g.

A1. (*2-Efficiency within unions*). For all $(N,w,P) \in \mathcal{G}^{cs}$, any $P_k \in P$, and all $i,j \in P_k$ with $i \neq j$,

$$g_i(N,w,P) + g_j(N,w,P) = g_p\left(N^{\{i,j\}}, w^{\{i,j\}}, P^{\{i,j\}}\right).$$

A2. (*Efficiency*). For all $(N,w,P) \in \mathcal{G}^{cs}$,

$$\sum_{i \in N} g_i(N,w,P) = w(N).$$

A3. (*Balanced contributions within unions*). For all $(N,w,P) \in \mathcal{G}^{cs}$, any $P_k \in P$, and all $i,j \in P_k \in P$ with $i \neq j$,

$$g_i(N,w,P) - g_i(N,w,P_{-j}) = g_j(N,w,P) - g_j(N,w,P_{-i}).$$

A4. (*Neutrality for the reduced game*). For all $(N,w,P) \in \mathcal{G}^{cs}$, any $P_k \in P$, all $i,j \in P_k$ with $i \neq j$, and all $l \in N \backslash P_k$,

$$g_l(N,w,P) = g_l\left(N^{\{i,j\}}, w^{\{i,j\}}, P^{\{i,j\}}\right).$$

A property with the same savour than A1 was used by Nowak [23] to give an axiomatization of the Banzhaf value without the so-called additivity axiom. According to Nowak, A1 assumes a "reduction" property with an easy interpretation. It was originally discussed in Lehrer [19]. A similar property was also applied in Alonso-Meijide et al. ([3] and [5]) to characterize the Banzhaf-Owen coalitional value.

The property of efficiency is standard in the literature and it is usual to find it in the characterizations of the Shapley and Owen value.

One important axiom in the literature is the balanced contributions axiom. This property is based on a rule of reciprocity, as introduced by Myerson [22], and it is often used in the literature on the Shapley value. Myerson's property of balanced contributions asserts that for any two players the gain or loss to each player when the other "leaves" the game should be equal. Property A3 follows this principle but taking into account the role of the unions. Thus, A3 states that the loss (or gain) of a player $i \in P_k$ when a player $j \in P_k$ decides to leave the union and remain alone is the same as the loss (or gain) of player j when player i decides to leave the union. This property was introduced in Vázquez-Brage et al. [27] and it was used by Alonso–Meijide et al. [4] in the axiomatic characterization of the symmetric coalitional binomial semivalues. On the other hand, adaptations of this property were used by Álvarez-Mozos and Tejada [7] to characterize extensions of coalitional values to the model of games with levels structure of cooperation. Moreover, the principle of balanced contributions has also been used in other contexts, e.g., Lorenzo-Freire et al. [21] in generalized bankruptcy situations.

A similar property to A4, called delegation neutrality, was defined by Amer et al. [9][1]. It says that the merger of two players of the same union does not affect the payoffs of the players outside this union.

3.2 Main results

Now, we present a new axiomatic characterization for the Owen value.

Theorem 3.1 *The Owen value is the only coalitional Shapley value for singletons that satisfies A2, A3, and A4.*

Proof.
(a)Existence.

In Vázquez-Brage et al. [27] it is proved that the Owen value is a coalitional Shapley value and it satisfies the property of balanced contributions within unions (A3). Besides, in Owen [24] it is shown that the Owen value satisfies efficiency (A2).

Since it is straightforward to prove that the Owen value satisfies the property of neutrality for the reduced game (A4), we omit the proof.

(b)Uniqueness.

To prove uniqueness, let us suppose that there exists other coalitional value g in the conditions established in this theorem. We should then prove that $g = \Phi$. To this aim, we distinguish two cases:

- If $P = P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$, since both values are coalitional Shapley values for singletons, we know that for all TU game (N, w) :

$$g(N, w, P) = \varphi(N, w) = \Phi(N, w, P).$$

- Suppose now that $|P| < n$. In this case, we define $q = \max_{h \in M} |P_h|$ (note that $2 \leq q \leq n$) and $M_p = \{h \in M : |P_h| = p\}$, with $1 \leq p \leq q$. Then, the proof

[1] In the section of Final Remarks, we discuss in more detail the properties of delegation neutrality and neutrality for the reduced game, as well as 2-efficiency within unions and the so-called delegation transfer.

goes by induction on the number p. We use the recursive procedure indicated below:

– We start with $p = 1$. If $M_1 = \emptyset$, then go to the next stage. On the contrary, if $M_1 \neq \emptyset$, we can consider $k \in M_1$. Suppose then that $P_k = \{i\}$ and let us choose a union P_h with $h \neq k$, such that $|P_h| > 1$.

  Then, we consider the steps described below:

  * *Step 1.* Let us take two different players $j_1, j_2 \in P_h$ and consider $(N^{\{j_1,j_2\}}, w^{\{j_1,j_2\}}, P^{\{j_1,j_2\}})$ as the $\{j_1, j_2\}$-reduced game of (N, w, P). Then, by A4, we have that $g_i(N, w, P) = g_i(N^{\{j_1,j_2\}}, w^{\{j_1,j_2\}}, P^{\{j_1,j_2\}})$ and $\Phi_i(N, w, P) = \Phi_i(N^{\{j_1,j_2\}}, w^{\{j_1,j_2\}}, P^{\{j_1,j_2\}})$.
    We define $(N^1, w^1, P^1) = (N^{\{j_1,j_2\}}, w^{\{j_1,j_2\}}, P^{\{j_1,j_2\}})$. If $\left|P_h^1\right| = 1$, go to Step 2. Otherwise, since $\left|P_h^1\right| > 1$, we choose other two different players $j_3, j_4 \in P_h^1$ and repeat the procedure given above, obtaining the $\{j_3, j_4\}$-reduced game of (N^1, w^1, P^1). We denote this coalitional game by (N^2, w^2, P^2).
    Finally, after $|P_h|-1$ iterations, we get a new coalitional game denoted by $(N^{|P_h|-1}, w^{|P_h|-1}, P^{|P_h|-1})$, where $|P_h^{|P_h|-1}| = 1$ and such that, by A4, $g_i(N, w, P) = g_i(N^{|P_h|-1}, w^{|P_h|-1}, P^{|P_h|-1})$ and $\Phi_i(N, w, P) = \Phi_i(N^{|P_h|-1}, w^{|P_h|-1}, P^{|P_h|-1})$.
  * *Step 2.* Let us take the game $(N^{|P_h|-1}, w^{|P_h|-1}, P^{|P_h|-1})$. If we carry out the same procedure to that described in Step 1 for all the unions $P_{h'}$ with $h' \in M \setminus (M_1 \cup \{k, h\})$, we obtain a coalitional game denoted[2] by $(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)})$. This coalitional game satisfies that $i \in N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}$, $P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}$ is a coalitional structure with only one player in each union and, moreover, by A4,

$$g_i(N, w, P) = g_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)})$$

    and

$$\Phi_i(N, w, P) = \Phi_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}).$$

  Then, taking into account that g and Φ are coalitional Shapley values for singletons,

$$\begin{aligned} & g_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}) \\ = \; & \varphi_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}) \\ = \; & \Phi_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}) \end{aligned}$$

  and we conclude that $g_i(N, w, P) = \Phi_i(N, w, P)$.

[2] This notation corresponds to the number of iterations used to obtain a coalitional game with all the unions formed by isolated players.

– For $2 \leq p \leq q$, suppose that the payoffs are determined for every i in a union $P_h \in P$ such that $\mid P_h \mid < p$. We will determine the payoffs for players in a union P_h with $\mid P_h \mid = p$. Remember that $M_p = \{k \in M : |P_k| = p\}$. If $M_p = \emptyset$, then go to the next stage. Otherwise, fix $k \in M_p$. If we choose $i \in P_k$, by A3 we know that for all $j \in P_k \setminus \{i\}$,

$$g_i(N, w, P) - g_j(N, w, P) = g_i(N, w, P_{-j}) - g_j(N, w, P_{-i}),$$

$$\Phi_i(N, w, P) - \Phi_j(N, w, P) = \Phi_i(N, w, P_{-j}) - \Phi_j(N, w, P_{-i}),$$

and, since $|(P_{-j})_k| = |(P_{-i})_k| = p - 1$, by applying stage $p - 1$ we deduce that $g_i(N, w, P_{-j}) = \Phi_i(N, w, P_{-j})$ and $g_j(N, w, P_{-i}) = \Phi_j(N, w, P_{-i})$. It means that, for all $j \in P_k \setminus \{i\}$,

$$g_i(N, w, P) - g_j(N, w, P) = \Phi_i(N, w, P) - \Phi_j(N, w, P). \quad (1)$$

Let us now consider the other unions P_h, $h \neq k$. There are two cases:

  * *First case.* There does not exist any union P_h, $h \neq k$ such that $|P_h| \neq 1$.

    By the property A2, we have that,

    $$\sum_{j \in P_k} g_j(N, w, P) = w(N) - \sum_{j \in N \setminus P_k} g_j(N, w, P) \text{ and}$$

    $$\sum_{j \in P_k} \Phi_j(N, w, P) = w(N) - \sum_{j \in N \setminus P_k} \Phi_j(N, w, P).$$

    Since $|P_h| = 1$ for all $h \neq k$, by Stage 1 we deduce that for all $j \in N \setminus P_k$,

    $$g_j(N, w, P) = \Phi_j(N, w, P).$$

    Taking into account the last two equalities, we obtain that:

    $$\sum_{j \in P_k} g_j(N, w, P) = \sum_{j \in P_k} \Phi_j(N, w, P). \quad (2)$$

    Finally, by equations (1) and (2), we conclude that for all $i \in P_k$, $g_i(N, w, P) = \Phi_i(N, w, P)$.

  * *Second case.* Suppose that there exists a union P_h with $h \neq k$, such that $|P_h| > 1$. In this case we proceed as in Steps 1-2, obtaining the game $(N^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}, w^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}, P^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)})$, such that $P_k^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)} = P_k$ and where $|P_h^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}| = 1$ for all $h \neq k$.

    Applying A4 and taking into account that this new coalitional game is in the conditions of First case, we conclude that for all $i \in P_k$,

    $$\begin{aligned} & g_i(N, w, P) \\ = \; & g_i(N^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}, w^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}, P^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}) \\ = \; & \Phi_i(N^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}, w^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}, P^{\sum_{h' \in M \setminus \{k\}} (|P_{h'}|-1)}) \\ = \; & \Phi_i(N, w, P). \quad \square \end{aligned}$$

Remark 3.2 *Independence of the properties of Theorem 3.1.*

a. The coalitional value given by $g_i(N, w, P) = \dfrac{w(N)/m}{\mid P_k \mid}$ for all $(N, w, P) \in \mathcal{G}^{cs}$ and all $i \in N$, where $P_k \in P$ is the union such that $i \in P_k$, satisfies A2, A3, and A4, but it is not a coalitional Shapley value for singletons.

b. The coalitional value Γ given in Alonso-Meijide et al. [5], is a coalitional Shapley value for singletons and satisfies A3 and A4, but not A2.

c. The coalitional value given by $g_i(N, w, P) = \varphi_k(M, w^P)/\mid P_k \mid$ for all $(N, w, P) \in \mathcal{G}^{cs}$ and all $i \in N$, where $P_k \in P$ is the union such that $i \in P_k$, is a coalitional Shapley value for singletons and satisfies A2 and A4, but not A3.

d. The coalitional value given by $g(N, w, P) = \varphi(N, w)$ for all $(N, w, P) \in \mathcal{G}^{cs}$ is a coalitional Shapley value for singletons and satisfies A2 and A3, but not A4.

A parallel axiomatic characterization for the Banzhaf-Owen value can be given just replacing the property efficiency with 2-efficiency within unions and taking into account that the Banzhaf-Owen value is a coalitional Banzhaf value.

Theorem 3.3 *The Banzhaf-Owen value is the only coalitional Banzhaf value for singletons that satisfies A1, A3, and A4.*

Proof.
(a)Existence.

In Alonso-Meijide et al. [3] it is shown that the Banzhaf-Owen value is a coalitional Banzhaf value for singletons and it satisfies the property of balanced contributions within unions (A3).

It is straightforward to prove that the Banzhaf-Owen value satisfies the property of 2-efficiency within unions (A1) and the property of the neutrality for the reduced game (A4). So, we omit the proof.

(b)Uniqueness.

The proof of uniqueness follows similar lines to that of the uniqueness in Theorem 3.1. Thus, let us suppose that there exists other coalitional value g in the conditions established in this theorem. We should then prove that $g = \Psi$. To this aim, we distinguish two cases:

- If $P = P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$, since both values are coalitional Banzhaf values for singletons, we know that for all TU game (N, w):

$$g(N, w, P) = \beta(N, w) = \Psi(N, w, P).$$

- Suppose now that $|P| < n$. It means that we can find $P_k \in P$ such that $p_k = |P_k| \geq 2$.

  We define $q = \max_{k \in M} |P_k|$ (note that $2 \leq q \leq n$) and $M_p = \{k \in M : |P_k| = p\}$, with $1 \leq p \leq q$. Then, the proof goes by induction on the number p. We use the recursive procedure indicated below:

– We start with $p = 1$. If $M_1 = \emptyset$, then go to the next stage. On the contrary, if $M_1 \neq \emptyset$, we can consider $k \in M_1$. Suppose then that $P_k = \{i\}$ and let us choose a union P_h with $h \neq k$, such that $|P_h| > 1$.

  At this stage, we proceed as in Steps 1-2 of Theorem 3.1, obtaining the coalitional game $(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)})$. By A4 and taking into account that g and Ψ are coalitional Banzhaf values for singletons, we have that:

$$
\begin{aligned}
& g_i(N, w, P) \\
= \;& g_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}) \\
= \;& \beta_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}) \\
= \;& \Psi_i(N^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, w^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}, P^{\sum_{h' \in M\setminus\{k\}}(|P_{h'}|-1)}) \\
= \;& \Psi_i(N, w, P).
\end{aligned}
$$

– For $2 \leq p \leq q$, suppose that the payoffs are determined for every i in a union $P_h \in P$ such that $\mid P_h \mid < p$. We will determine the payoffs for players in a union P_h with $\mid P_h \mid = p$. If $M_p = \emptyset$, then go to the next stage. Otherwise, fix $k \in M_p$. If we choose $i \in P_k$, by A3 we know that for all $j \in P_k \setminus \{i\}$,

$$g_i(N, w, P) - g_j(N, w, P) = g_i(N, w, P_{-j}) - g_j(N, w, P_{-i}),$$

$$\Psi_i(N, w, P) - \Psi_j(N, w, P) = \Psi_i(N, w, P_{-j}) - \Psi_j(N, w, P_{-i}),$$

  and, since $|(P_{-j})_k| = |(P_{-i})_k| = p - 1$, by applying stage $p - 1$ we deduce that $g_i(N, w, P_{-j}) = \Psi_i(N, w, P_{-j})$ and $g_j(N, w, P_{-i}) = \Psi_j(N, w, P_{-i})$. It means that, for all $j \in P_k \setminus \{i\}$,

$$g_i(N, w, P) - g_j(N, w, P) = \Psi_i(N, w, P) - \Psi_j(N, w, P). \tag{3}$$

  Furthermore, by A1 we have that, for all $i, j \in P_k$,

$$g_i(N, w, P) + g_j(N, w, P) = g_p(N^{\{i,j\}}, w^{\{i,j\}}, P^{\{i,j\}}) \text{ and}$$

$$\Psi_i(N, w, P) + \Psi_j(N, w, P) = \Psi_p(N^{\{i,j\}}, w^{\{i,j\}}, P^{\{i,j\}}).$$

  Since $|P_k^{\{i,j\}}| = p - 1$, by applying again stage $p - 1$, we obtain that

$$g_p(N^{\{i,j\}}, w^{\{i,j\}}, P^{\{i,j\}}) = \Psi_p(N^{\{i,j\}}, w^{\{i,j\}}, P^{\{i,j\}}).$$

  Thus,

$$g_i(N, w, P) + g_j(N, w, P) = \Psi_i(N, w, P) + \Psi_j(N, w, P). \tag{4}$$

  Therefore, by equations (3) and (4), we obtain that $g_i(N, w, P) = \Psi_i(N, w, P)$ and $g_j(N, w, P) = \Psi_j(N, w, P)$. $\square$

Remark 3.4 *Independence of the properties of Theorem 3.3.*

a. The coalitional value given by $g_i(N, w, P) = 0$ for all $i \in N$ and for all $(N, w, P) \in \mathcal{G}^{cs}$ satisfies A1, A3, and A4, but it is not a coalitional Banzhaf value for singletons.

b. The symmetric coalitional Banzhaf value π defined in Alonso-Meijide and Fiestras-Janeiro [6], is a coalitional Banzhaf value for singletons and satisfies A3 and A4, but not A1.

c. The coalitional value given by $g_i(N, w, P) = \dfrac{\beta_k(M, w^P)}{2^{|P_k|-1}}$ for all $(N, w, P) \in \mathcal{G}^{cs}$ and all $i \in N$, where $P_k \in P$ is the union such that $i \in P_k$, is a coalitional Banzhaf value for singletons and satisfies A1 and A4, but not A3.

d. The coalitional value given by $g(N, w, P) = \beta(N, w)$ for all $(N, w, P) \in \mathcal{G}^{cs}$ is a coalitional Banzhaf value for singletons and satisfies A1 and A3, but not A4.

3.3 Further discussion

In this paper, we provide two parallel characterizations of the Owen and Banzhaf-Owen coalitional values in the sense that we show that these coalitional values are similar except, mainly, that the first satisfies efficiency and the second one 2-efficiency within unions. Moreover, they are coalitional Shapley and Banzhaf values, respectively. So, in addition to axioms A1-A4, we need in Theorem 3.1 also other properties of the Shapley value, for the class of games (N, v, P^n). In a similar way, in Theorem 3.3 we require also the corresponding properties of the Banzhaf value, for the class of games (N, v, P^n). We explain this in detail. First, we introduce other standard properties for a coalitional value g.

A5. (*2-Efficiency for singletons*). For all $(N, w) \in \mathcal{G}$ and all $i, j \in N$ with $i \neq j$,

$$g_i(N, w, P^n) + g_j(N, w, P^n) = g_p(N^{\{i,j\}}, w^{\{i,j\}}, P^{n^{\{i,j\}}}).$$

Notice that, when $P = P^n$, the property of 2-efficiency is a collusion property between unions: it means that if two unions $\{i\}, \{j\} \in P^n$ collude to one union $\{p\} \in (P^n \backslash \{\{i\}, \{j\}\}) \cup \{p\}$, then the payoff to union $\{p\}$ in the new game is equal to the sum of the payoffs of the unions $\{i\}$ and $\{j\}$ in the old game. Note that this property is different from the property of 2-efficiency, since in 2-efficiency only the players who belong to the same union can be represented by other new player. In this case, the isolated unions are substituted by other isolated union.

A6. (*Efficiency for singletons*). For all $(N, w) \in \mathcal{G}$, $\sum_{i \in N} g_i(N, w, P^n) = w(N)$.

A7. (*Symmetry for singletons*). For all $(N, w) \in \mathcal{G}$ and all $i, j \in N$ that are symmetric players in (N, w), then $g_i(N, w, P^n) = g_j(N, w, P^n)$.[3]

[3] We say that two players $i, j \in N$ are symmetric in a TU game (N, w) if $w(S \cup \{i\}) = w(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

A8. (*Null player property for singletons*). For all $(N, w) \in \mathcal{G}$ and all $i \in N$ that is a null player in (N, w), then $g_i(N, w, P^n) = 0$.[4]

A9. (*Additivity for singletons*). For all $(N, w_1), (N, w_2) \in \mathcal{G}$, $g(N, w_1 + w_2, P^n) = g(N, w_1, P^n) + g(N, w_2, P^n)$.

Now, we establish two well-known propositions without proof.

Proposition 3.5

a. *A coalitional value g satisfies $A6$, $A7$, $A8$, and $A9$ if, and only if, it is a coalitional Shapley value, i.e. $g(N, w, P^n) = \varphi(N, w)$ for all $(N, w) \in \mathcal{G}$.*

b. *A coalitional value g satisfies $A5$, $A7$, $A8$, and $A9$ if, and only if, it is a coalitional Banzhaf value, i.e. $g(N, w, P^n) = \beta(N, w)$ for all $(N, w) \in \mathcal{G}$.*

Proposition 3.6

a. *The Owen value satisfies $A6$, $A7$, $A8$, and $A9$.*

b. *The Banzhaf-Owen value satisfies $A5$, $A7$, $A8$, and $A9$.*

As a consequence, we set up equivalent formulations of Theorems 3.1 and 3.3.

Theorem 3.7 *The Owen value is the only coalitional value that satisfies $A2$, $A3$, $A4$, $A7$, $A8$, and $A9$.*

In this result, the properties $A6$ (that is implied by $A2$), $A7$, $A8$, and $A9$ could be replaced by any other system of axioms that characterizes the Shapley value, by pointing out that we consider trivial structure of coalitions, as we have done.

Theorem 3.8 *The Banzhaf-Owen value is the only coalitional value that satisfies $A1$, $A3$, $A4$, $A5$, $A7$, $A8$, and $A9$.*

Analogously, in this another result, the properties $A5$, $A7$, $A8$, and $A9$ could be replaced by any other system of axioms that characterizes the Banzhaf value, by adding the nuance that we consider trivial structure of coalitions, as we have done.

4 Final remarks

To better appreciate the scope of the results of the paper, we consider two close articles on the same subject. Firstly, we introduce auxiliary notation and axioms. Let $(N, w) \in \mathcal{G}$ be a cooperative game and consider two players $i, j \in N$ with $i \neq j$. The so-called *delegation game* $(N, w^{i \triangleleft j})$ is defined, for all $S \subseteq N$, by:

[4] We say that a player $i \in N$ is a null player in a TU game (N, w) if $w(S \cup \{i\}) = w(S)$ for all $S \subseteq N$.

$$w^{i\lhd j}(S) = \begin{cases} w(S\cup\{j\}) & \text{if } i\in S \\ \\ w(S\backslash\{j\}) & \text{if } i\notin S. \end{cases}$$

Note that the game $(N, w^{i\lhd j})$ allocates to every coalition containing player i the utility that could be obtained by joining player j, whereas the remaining coalitions lose the marginal contribution of player j. This implies that j is a null player in the game $(N, w^{i\lhd j})$. Amer et al. [9] and Calvo et al. [13] introduce some properties for a coalitional value g.

A1'. (*Delegation transfer*) For all $(N, w, P) \in \mathcal{G}^{cs}$, any $P_k \in P$, and all $i, j \in P_k$ with $i \neq j$, $g_i(N, w, P) + g_j(N, w, P) = g_i(N, w^{i\lhd j}, P)$.

A3'. (*Intracoalitional balanced contributions*) For all $(N, w, P) \in \mathcal{G}^{cs}$, any $P_k \in P$, and all $i, j \in P_k$ with $i \neq j$, $g_i(N, w, P) - g_i(N\backslash\{j\}, w_{N\backslash\{j\}}, P_{N\backslash\{j\}}) = g_j(N, w, P) - g_j(N\backslash\{i\}, w_{N\backslash\{i\}}, P_{N\backslash\{i\}})$.[5]

A4'. (*Delegation neutrality*) For all $(N, w, P) \in \mathcal{G}^{cs}$, any $P_k \in P$, all $i, j \in P_k$ with $i \neq j$, and all $l \in N\backslash P_k$, $g_l(N, w, P) = g_l(N, w^{i\lhd j}, P)$.

Calvo et al. [13] use the axiom of coalitional balanced contributions among unions joint with intracoalitional balanced contributions (A3') and efficiency (A2) in order to characterize the Owen value. A3 and A3' have the same flavor, the main difference being that A3 works on a fixed player set whereas A3' works on a variable player set, that is, when a player separates from an a priori union, he leaves the game.

Amer et al. [9] make use of the axioms of delegation transfer (A1') and delegation neutrality (A4'), among others, to characterize the Banzhaf-Owen value. Both axioms, A1' and A4', breathe the same spirit of axioms A1 and A4, respectively. However, A1' and A4' work on a fixed set player, whereas A1 and A4 allow variable population. In fact, A1' and A4' are centered on the delegation game and A1 and A4 are related to the neutrality game.

We explore whether A1' and A4' can be used instead of A1 and A4, and A3' can be replaced with A3, in order to obtain new comparable axiomatizations of the Owen and Banzhaf-Owen values. We have that the Owen value is the only coalitional Shapley value for singletons that satisfies A2, A3', and A4 and the Banzhaf-Owen value is the only coalitional Banzhaf value for singletons that satisfies A1, A3', and A4. These results are obtained in Lorenzo-Freire [20].

However, other combinations of the above properties do not provide new parallel axiomatic characterizations for the Banzhaf-Owen and Owen values. Vidal–Puga [28] introduces a value that, as well as the Owen value, is also a coalitional Shapley value for singletons that satisfies A2, A3', and A4'. Finally, we have defined a value different from the Owen value that is also a coalitional Shapley value for singletons that satisfies

[5]If $i \in N$, $w_{N\backslash\{i\}}$ is the TU-game with set of players $N\backslash\{i\}$ such that $w_{N\backslash\{i\}}(S) = w(S)$ for each $S \subseteq N\backslash\{i\}$ and $P_{N\backslash\{i\}}$ is the coalition structure over $N\backslash\{i\}$ such that ${P_{N\backslash\{i\}}}_h = P_h$ if $i \notin P_h$ and ${P_{N\backslash\{i\}}}_h = P_h\backslash\{i\}$ if $i \in P_h$ for each $h \in M$.

A2, A3, and A4' and a value different from the Banzhaf-Owen value that is also a coalitional Banzhaf value for singletons that satisfies A1', A3, and A4'. Below we show these two values.

Example 4.1 Let g be the coalitional value defined by:

$$g_i(N,w,P) = \begin{cases} \varphi_i(N,w) & \text{if } n=3,\ P=P^n \\ \frac{2}{3}\varphi_i(N,w) + \frac{1}{6}\sum_{j\neq i}\varphi_j(N,w) & \text{if } n=3,\ P=P^N \\ 0 & \text{if } n=3,\ P=\{\{i\},N\backslash\{i\}\} \\ \varphi_i(N,w)+\frac{1}{2}\varphi_j(N,w) & \text{if } n=3,\ P=\{\{j\},N\backslash\{j\}\},\ j\neq i \\ \Phi_i(N,w,P) & \text{if } n\neq 3, \end{cases}$$

for all $(N,w,P) \in \mathcal{G}_N^{cs}$ and $i \in N$.

It is easy to see, and we omit the demonstration, that both g and the Owen value are coalitional Shapley values that satisfy efficiency (A2), balanced contributions within unions (A3), and delegation neutrality (A4').

Example 4.2 Let g be the coalitional value defined by:

$$g_i(N,w,P) = \begin{cases} 0 & \text{if } n=3,\ P=\{\{i\},N\backslash\{i\}\} \\ \beta_i(N,w) & \text{if } n=3,\ P\neq\{\{i\},N\backslash\{i\}\} \\ \Psi_i(N,w,P) & \text{if } n\neq 3, \end{cases}$$

for all $(N,w,P) \in \mathcal{G}_N^{cs}$ and $i \in N$.

It is easy to see, and we also omit the proof, that both g and the Banzhaf-Owen value are coalitional Banzhaf values that satisfy delegation transfer (A1'), balanced contributions within unions (A3), and delegation neutrality (A4').

Finally, we put two separate tables including properties for Owen (Table 1) and Banzhaf-Owen (Table 2) values where it is indicated which combinations of properties give the uniqueness for the value (this is noted with ✓) and what combinations do not provide this uniqueness (this is noted with $\times$).

				Owen value
Coalitional Shapley value	*Efficiency*	*Balanced contributions within unions*	*Neutrality for the reduced game*	✓
Coalitional Shapley value	*Efficiency*	*Intracoalitional balanced contributions*	*Neutrality for the reduced game*	✓
Coalitional Shapley value	*Efficiency*	*Balanced contributions within unions*	*Delegation neutrality*	×
Coalitional Shapley value	*Efficiency*	*Intracoalitional balanced contributions*	*Delegation neutrality*	×

Table 1: Systems of axioms that satisfies the Owen value.

				Banzhaf Owen value
Coalitional Banzhaf value	$2-Efficiency$ *within unions*	*Balanced contributions within unions*	*Neutrality for the reduced game*	✓
Coalitional Banzhaf value	$2-Efficiency$ *within unions*	*Intracoalitional balanced contributions*	*Neutrality for the reduced game*	✓
Coalitional Banzhaf value	*Delegation transfer*	*Balanced contributions within unions*	*Delegation neutrality*	×

Table 2: Systems of axioms that satisfies the Banzhaf-Owen value.

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