CROSSED PRODUCTS FOR WEAK HOPF ALGEBRAS

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Abstract. Weak crossed products by a weak bialgebra are defined. The resulting structure is not that of a unital algebra but an associative algebra with preunit. A general formula of such a product is given in terms of a weak 2-cocycle and a weak measuring. The relation with weak cleft extensions is studied. Equivalences of weak crossed products are defined and are related to gauge transformations. A relation between cleaving maps is described in terms of gauge transformations.

1. Introduction

Blattner,Cohen and Montgomery in [6] and independently Doi and Takeuchi in [19] introduced the notion of a crossed product of an algebra B by a Hopf algebra H. A crossed product consists of an algebra structure defined on $B \otimes H$ given in terms of two maps, a measuring and a cocycle, and its unit is $1_B \otimes 1_H$. Crossed products became an important tool in the study of Galois extensions, and in particular of cleft extensions, as a cleft extension can be identified with a crossed product with convolution invertible cocycle, as Blattner and Montgomery prove in [7] and Doi and Takeuchi in [19].

The context of Hopf algebras experimented several generalizations during the last ten years. In [13] Brzeziński and Majid defined entwining structures, that unified the study of different categories of modules over Hopf algebras. Following different ideas, Böhm, Nill and Szlachaányi generalized Hopf algebras to weak Hopf algebras [9] just by weakening the properties of the unit and the counit. Now the properties of the unit and the counit are summarized in four idempotent maps Π^R, Π^L, $\bar{\Pi}^R$ and $\bar{\Pi}^L$. Weak Hopf algebras and entwining structures where unified by Caeneepel and De Groot using weak entwining structures [16]. In this general context of weak entwining structures Alonso Álvarez *et al.* obtained the concept of weak cleft extension [2] (that we will denote W-cleft) as the classical one was not suitable in this new weak context. Of course, weak cleft extensions for weak entwining structures can be easily particularized to the case of weak Hopf algebras, or in general to weak bialgebras.

The more recent Hopf-type object is a Hopf algebroid (see for example [20]). A Hopf algebroid consists of an object endowed with two ring structures and two coring structures over two different algebras. To generalize a weak Hopf algebra to a Hopf algebroid we need to consider the images of the idempotent morphisms Π^R and Π^L. In this context a generalization of cleft extensions was given by Böhm and Brzeziński in [11]. They also find a crossed product of a Hopf algebroid by an R-ring (where the algebra R is one of the base algebras related to the Hopf algebroid). Moreover they obtain a generalization of Blattner and Montgomery and Doi and Takeuchi result, so they identify cleft extensions with crossed products with an invertible cocycle. Finally they give a definition for a weak cleft extension of Hopf algebroids, that we will denote $\boldsymbol{W}$-cleft, and identify it with a crossed product that has a left invertible cocycle. If we apply this general theory to the case of weak Hopf algebras we obtain a definition of a $\boldsymbol{W}$-cleft extension for weak Hopf algebras. It seems natural to ask about the relation

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of W-cleft extensions and ***W***-cleft entensions. It is not difficult to prove that ***W***-cleft extensions are W-cleft. Unfortunately the converse is not true in general. The main reason is that when weak Hopf algebras are interpreted as Hopf algebroids we need information from the images of the four idempotent morphisms $\Pi^R, \Pi^L, \bar{\Pi}^R$ and $\bar{\Pi}^L$. Hence ***W***-cleft extensions for Hopf algebroids also use all these information. When we deal with W-cleft extensions we just need information from Π^R and $\bar{\Pi}^R$ so, roughly speaking, we do not have enough information to obtain the converse. Hence the theory of crossed products given in [11] is not suitable to deal with W-cleft extensions.

A first approach to obtain an appropriate definition of crossed products for weak Hopf algebras was given by Alonso Álvarez *et al.* in [4], where they define an associative multiplication on $B \otimes H$ with preunit $1_B \otimes 1_H$, for a weak Hopf algebra H and an algebra B. This multiplication is given in terms of two morphisms $H \otimes B \to B \otimes H$ and $H \otimes H \to B \otimes H$ that generalize the measuring and the cocycle respectively. To generalize Blattner and Montgomery and Doi and Takeuchi theorem using this crossed product it is necessary to impose the condition $f(1_H) = 1_B$ on the cleaving morphism f, as the preunit has to be $1_B \otimes 1_H$. Recall that in the Hopf algebra case this property does not suppose any restriction, as if f is a cleaving map then $\bar{f}(h) = f^{-1}(1_H)f(h)$ is also a cleaving map that moreover is unit-preserving. In the weak Hopf algebra case it is not possible to prove this assertion in general, and hence it seems necessary to change the definition of a weak crossed product to obtain one whose preunit is not $1_B \otimes 1_H$ necessarily.

In the present paper we define a weak crossed product structure of an algebra B by a weak balgebra H that consists of an associative multiplication on $B \otimes H$ equipped with a preunit. This multiplication is given in terms of a weak weak measuring and a cocycle. These maps must satisfy certain necessary and sufficient conditions, that generalize the ones in [6, 19], for product to be associative and to have a preunit. An idempotent morphism $\Omega : B \otimes H \to B \otimes H$ associated to the preunit arises naturally, and its image is an algebra associated to the weak crossed product. In the bialgebra case this idempotent is the identity and we recover crossed product algebras given in [6, 19]. When we study weak crossed products in relation with W-cleft extensions, we prove that a W-cleft extension induces a weak crossed product with a left invertible cocycle, for the definition of invertible cocycle given in [4, 11]. Conversely, the existence of a left invertible cocycle implies that the algebra associated to the weak crossed product is a weak cleft extension of its coinvariants. In this case we cannot proceed as in the classical one to prove that these coinvariants are the algebra B. Therefore we give necessary and sufficient conditions for this to occur, kindly communicated to us by Gabriella Böhm. Finally we define equivalence of weak crossed products and we obtain that any two equivalent crossed products are related by a gauge transformation. As a corollary we obtain that different weak cleaving maps associated to the same weak cleft extension are related by a gauge transformation, following the ideas due to Doi [18] in the context of Hopf algebras and to Böhm and Brzeziński [11] in the context of Hopf algebroids.

Notation. Throughout the paper we work over an associative commutative unital ring k. By an algebra we understand an associative algebra A with unit 1_A and a coalgebra means a coassociative coalgebra C with counit ε_C. All the modules and comodules are supposed to be unitary or counitary unless otherwise specified. The set of homomorphisms of right (resp. left) A-modules will be denoted by $Hom_A(V, W)$ (resp. ${}_AHom(V, W)$), and the homomorphisms of right (resp. left) C comodules by $Hom^C(V, W)$ (resp. ${}^CHom(V, W)$). We use Sweedler-like notation, so $v_{[0]} \otimes v_{[1]}$ denotes the right coaction of a right comodule V, and $c_{(1)} \otimes c_{(2)}$ the comultiplication of a coalgebra.

2. Preliminaries

2.1. **Weak bialgebras and weak Hopf algebras.** Recall that a *weak bialgebra* is an algebra and a coalgebra H such that the coproduct is a multiplicative map, $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')}$, and for all $h, k \in H$ $\varepsilon_H(hkl) = \varepsilon_H(hk_{(1)})\varepsilon_H(k_{(2)}l) = \varepsilon_H(hk_{(2)})\varepsilon_H(k_{(1)}l)$. The behavior of the unit with respect to the comultiplication and of the counit with respect to the multiplication are sumarized in [16] (propositions 4.3 to 4.8) using the projections [9]:

$$\Pi^L(h) = \varepsilon_H(1_{(1)}h)1_{(2)} \qquad \bar{\Pi}^L(h) = 1_{(1)}\varepsilon_H(1_{(2)}h)$$
$$\Pi^R(h) = 1_{(1)}\varepsilon_H(h1_{(2)}) \qquad \bar{\Pi}^R(h) = \varepsilon_H(h1_{(1)})1_{(2)}.$$

If there also exists a linear map $S : H \to H$ such that for all $h \in H$ $h_{(1)}S(h_{(2)}) = \varepsilon_H(1_{(1)}h)1_{(2)}$, $S(h_{(1)})h_{(2)} = 1_{(1)}\varepsilon_H(h1_{(2)})$ and $S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$ then H is called a *weak Hopf algebra* and the map S is called the *antipode*. Note that in this case $\Pi^L(h) = h_{(1)}S(h_{(2)})$ and $\Pi^R(h) = S(h_{(1)})h_{(2)}$. Other equalities relating the idempotent maps and the antipode are sumarized in [14] (proposition 36.11) or in [2].

Right H-comodule algebras over a weak Hopf algebra H, or in general over weak bialgebras [16], were defined by Böhm in [8]. A is a right H-comodule algebra if it is an H-comodule and an algebra such that $a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]} = (ab)_{[0]}(ab)_{[1]}$ for all $a, b \in A$ and satisfies the following proposition, given by Caenepeel and De Groot in [16]:

Proposition 2.1. *Let H be a weak bialgebra and A a right H-comodule wich is also an algebra that verifies*

$$a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]} = (ab)_{[0]}(ab)_{[1]}$$

for all $a, b \in A$. Then the following statements are equivalent:

(1) $1_{[0]} \otimes 1_{[1]} \otimes 1_{[2]} = 1_{[0]} \otimes 1_{[1]}1_{(1)} \otimes 1_{(2)}$
(2) $1_{[0]} \otimes 1_{[1]} \otimes 1_{[2]} = 1_{[0]} \otimes 1_{(1)}1_{[1]} \otimes 1_{(2)}$
(3) *For all $a \in A$ $a_{[0]} \otimes \bar{\Pi}^R(a_{[1]}) = a1_{[0]} \otimes 1_{[1]}$*
(4) *For all $a \in A$ $a_{[0]} \otimes \Pi^L(a_{[1]}) = 1_{[0]}a \otimes 1_{[1]}$*
(5) $1_{[0]} \otimes \bar{\Pi}^R(1_{[1]}) = 1_{[0]} \otimes 1_{[1]}$
(6) $1_{[0]} \otimes \bar{\Pi}^R(1_{[1]}) = 1_{[0]} \otimes 1_{[1]}$

If any of these properties hold we say that A is a weak H-comodule algebra.

This proposition also implies that $1_{[0]}\varepsilon_H(h_{(1)}1_{[1]}) \otimes h_{(2)} = 1_{[0]} \otimes h1_{[1]}$ for all $h \in H$ and $1_{[0]}a\varepsilon_H(h1_{[1]}) = a_{[0]}\varepsilon_H(ha_{[1]})$ for all $a \in A$. Furthermore, the map $\psi(h \otimes a) = a_{[0]} \otimes ha_{[1]}$ gives a weak entwining structure (A, H, ψ) (see [16]). This fact permits us to use the general theory for weak cleft extensions introduced in [2].

For a weak bialgebra H and a weak H-comodule algebra A, we define the subalgebra of coinvariants as $B = A^{coH} = \{b \in A \,|\, b_{[0]} \otimes b_{[1]} = b1_{[0]} \otimes 1_{[1]}\}$. Note that due to the properties of weak H-comodule algebras it also can be defined as $B = A^{coH} = \{b \in A \,|\, b_{[0]} \otimes b_{[1]} = b_{[0]} \otimes \bar{\Pi}^R(b_{[1]})\}$. Furthermore, as $\bar{\Pi}^R \circ \Pi^L = \Pi^L$ it can be shown that $B = A^{coH} = \{b \in A \,|\, b_{[0]} \otimes b_{[1]} = b_{[0]} \otimes \Pi^L(b_{[1]})\}$. It is not difficult to see that, when specialized to the non-weak bialgebra case, we obtain the usual subalgebra of coinvariants.

2.2. **Weak H-cleft extensions.** In this subsection we will recall the definition of a weak entwining structure for weak bialgebras given in [2]:

Definition 2.2. Suppose that H is a weak bialgebra, A a weak H-comodule algebra and B the subalgebra of coinvariants. We say that $B \hookrightarrow A$ is a *weak cleft extension* if there exist a right H-comodule map $f : H \to A$, called a *weak cleaving map*, and a map $\hat{f} : H \to A$ that verifies the following conditions for all $h \in H$:

(i) $\hat{f}(h_{(1)})f(h_{(2)}) = 1_{[0]}\varepsilon_H(h1_{[1]})$ for all $h \in H$,

(ii) $\hat{f}(h_{(2)})_{[0]} \otimes h_{(1)}\hat{f}(h_{(2)})_{[1]} = \hat{f}(h)1_{[0]} \otimes 1_{[1]}$.

If f is a weak cleaving map (in fact if it is just a map of H-comodules) then for all $h \in H$ $f(h_{(1)})1_{[0]}\varepsilon_H(h_{(2)}1_{[1]}) = f(h)$. Furthermore as a consequence of (ii) $\hat{f}$ satisfies $1_{[0]}\varepsilon_H(h_{(1)}1_{[1]})\hat{f}(h_{(2)}) = \hat{f}(h)$, and hence $1_{[0]}\hat{f}(h1_{[1]}) = \hat{f}(h)$.

3. Weak crossed products by weak bialgebras

In this section we will discuss a general theory of weak crossed products by weak bialgebras that includes the classical one for Hopf algebras given in [6] and [19] and the recent results on this subject obtained by Böhm and Brzeziński in [11]. First of all we will briefly recall the definition of a preunit [16] and some basic properties that will be useful in the development of further results:

Definition 3.1. Let A be an algebra without unit. An element $e \in A$ is a *preunit* if

$$ea = ae = ae^2$$

for all $a \in A$.

Associated to a preunit consider the idempotent multiplicative map

$$\Omega : A \to A,\ a \mapsto \Omega(a) = ae$$

whose image is an algebra with unit e. Conversely, if A is an algebra without unit but with a multiplicative idempotent morphism $\Omega : A \to A$ such that $Im\Omega$ is an algebra with unit, then the unit $e \in Im\Omega$ is a preunit in A, $\Omega(a) = ae$ and $e^2 = e$. Note that even if we require $e^2 = e$ in general it would not be a restriction, because if e is a preunit in A then e^2 also is. Moreover, the associated idempotent morphism to e is the same as the one associated to e^2. In what follows we will assume $e^2 = e$.

Our next task is to give the definition of a weak measuring (see [21] for Hopf algebras) for weak bialgebras:

Definition 3.2. We say that a weak bialgebra H *measures* the algebra B if there exists a k-linear map $H \otimes B \to B$, $h \otimes b \mapsto h \cdot b$ such that for all $h, k \in H$ and $b, b' \in B$:

$$h \cdot (bb') = (h_{(1)} \cdot b)(h_{(2)} \cdot b').$$

The map $b \otimes h \mapsto b \cdot h$ is called a *weak measuring*.

Definition 3.3. Let H be a weak bialgebra and B an algebra measured by H. And let $\sigma : H \otimes H \to B$ be a map that satisfies for all $h, k, m \in H$:

$$\sigma(h\Pi^R(k), m) = \sigma(h, \Pi^R(k)m). \tag{1}$$

If for all $h, k \in H$

$$\begin{aligned} \sigma(h,k) &= (h_{(1)} \cdot 1_B)\sigma(h_{(2)}, k) \\ \sigma(h,k) &= \sigma(h_{(1)}, k_{(1)})(h_{(2)}k_{(2)} \cdot 1_B) \end{aligned} \tag{2}$$

and the *cocycle condition* is satisfied for all $h, k, m \in H$:

$$(h_{(1)} \cdot \sigma(k_{(1)}, m_{(1)}))\sigma(h_{(2)}, k_{(2)}m_{(2)}) = \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, m) \tag{3}$$

we call σ a *weak 2-cocycle*.

Definition 3.4. Let B be an algebra measured by the weak bialgebra H with a 2-cocycle $\sigma : H \otimes H \to B$. Assume also that there exists $e = e_{<0>} \otimes e_{<1>} \in B \otimes H$ such that:

$$e_{<0>} \otimes e_{<1>} \otimes e_{<2>} = e_{<0>} \otimes 1_{(1)}e_{<1>} \otimes 1_{(2)} \tag{4}$$

where $e_{<0>}\otimes e_{<1>}\otimes e_{<2>} = e_{<0>}\otimes e_{<1>_{(1)}}\otimes e_{<1>_{(2)}}$. Then B is called a *weak σ-twisted left H-module* provided that, for all $h,k\in H$:

$$(h_{(1)}\cdot(k_{(1)}\cdot b))\sigma(h_{(2)},k_{(2)}) = \sigma(h_{(1)},k_{(1)})(h_{(2)}k_{(2)}\cdot b) \tag{5}$$

$$\begin{aligned} h\cdot 1_B &= (h_{(1)}\cdot e_{<0>})\sigma(h_{(2)},e_{<1>}) \\ h\cdot 1_B &= e_{<0>}\sigma(e_{<1>},h) \end{aligned} \tag{6}$$

Observe that while $h\cdot 1_B = \varepsilon_H(h)1_B$ for a measuring in the non-weak case, this equality does not hold if the measuring is weak. Instead of it conditions (2) and (6) are required. Condition (2) also replaces the normalization property of a 2-cocycle defined in [6].

Remark 3.5. Since $\bar{\Pi}^R\circ\Pi^L = \Pi^L$, $1_{(1)}h\otimes 1_{(2)} = h_{(1)}\otimes\Pi^L(h_{(2)})$ and $h1_{(1)}\otimes 1_{(2)} = h_{(1)}\otimes\bar{\Pi}^R(h_{(2)})$, equation (4) is equivalent to

$$e_{<0>}\otimes e_{<1>}\otimes e_{<2>} = e_{<0>}\otimes e_{<1>}1_{(1)}\otimes 1_{(2)}$$

Proposition 3.6. *Let H be a weak bialgebra and B an algebra weak measured by H. Consider also a map $\sigma: H\otimes H\to B$ satisfying (1) and (2), and an element $e\in B\otimes H$ fulfilling (4).*

Define the left B-linear right H-colinear product $B\otimes H\otimes B\otimes H\to B\otimes H$ by the formula:

$$(b\otimes h)(b'\otimes k) = b(h_{(1)}\cdot b')\sigma(h_{(2)},k_{(1)})\otimes h_{(3)}k_{(2)}. \tag{7}$$

Endowed with this product $B\otimes H$ is an associative algebra with preunit e and associated idempotent morphism $\Omega(b\otimes h) = bh_{(1)}\cdot 1_B\otimes h_{(2)}$ if and only if σ is a 2-cocycle, B is a weak σ-twisted left H-module and the following equality holds:

$$e_{<0>}(e_{<1>}\cdot b)\otimes e_{<2>} = be_{<0>}\otimes e_{<1>}. \tag{8}$$

Proof:
Suppose that (7) is an associative product with preunit e. Then we have

$$\begin{aligned} \Omega(1_B\otimes h) &= (1_B\otimes h)e \\ &= (h_{(1)}\cdot e_{<0>})\sigma(h_{(2)},e_{<1>})\otimes h_{(3)} \end{aligned}$$

using (4), the properties of weak Hopf algebras and (1) and (2). On the other hand, as e is a preunit, we obtain

$$\begin{aligned} \Omega(1_B\otimes h) &= e(1_B\otimes h) \\ &= e_{<0>}\sigma(e_{<1>},h_{(1)})\otimes h_{(2)} \end{aligned}$$

by similar computations. But we have that $\Omega(1_B\otimes h) = h_{(1)}\cdot 1_B\otimes h_{(2)}$, so

$$(B\otimes\varepsilon_H)(\Omega(1_B\otimes h)) = h\cdot 1_B = (h_{(1)}\cdot e_{<0>})\sigma(h_{(2)},e_{<1>}) = e_{<0>}\sigma(e_{<1>},h)$$

and hence we obtain (6).

Note that the map $B\to B\otimes H$, $b\mapsto be$, that it is an algebra homomorphism between B and $Im\Omega$, gives the inclusion of B in $Im\Omega$ and hence in $B\otimes H$. Then:

$$\begin{aligned} be &= ebe \\ &= e_{<0>}(e_{<1>}\cdot b)\otimes e_{<2>} \end{aligned}$$

by elementary calculations and (6) proved previously, so we obtain (8).

To prove (5) we just have to use the inclusion of B in $B\otimes H$ and the fact that the multiplication (7) is associative. On the one hand:

$$\begin{aligned}(B\otimes\varepsilon_H)(((1_B\otimes h)(1_B\otimes k))be) &= (B\otimes\varepsilon_H)((\sigma(h_{(1)}k_{(1)})\otimes h_{(2)}k_{(2)})be)\\ &= \sigma(h_{(1)},k_{(1)})(h_{(2)}k_{(2)}\cdot be_{<0>})\\ &\quad \sigma(h_{(3)}k_{(3)},e_{<1>})\varepsilon_H(h_{(4)}k_{(4)}e_{<2>})\\ &= \sigma(h_{(1)},k_{(1)})(h_{(2)}k_{(2)}\cdot be_{<0>})\\ &\quad \sigma(h_{(3)}k_{(3)},\Pi^R(k_{(4)})e_{<1>})\\ &= \sigma(h_{(1)},k_{(1)})(h_{(2)}k_{(2)}\cdot b)(h_{(3)}k_{(3)}\cdot 1_B)\\ &= \sigma(h_{(1)},k_{(1)})(h_{(2)}k_{(2)}\cdot b),\end{aligned}$$

where we used (2), the properties of weak bialgebras, the weak measuring property given in Definition 3.2 and (1). On the other hand

$$\begin{aligned}(B\otimes\varepsilon_H)((1_B\otimes h)((1_B\otimes k)be)) &= (B\otimes\varepsilon_H)((1_B\otimes h)(k_{(1)}\cdot b\otimes k_{(2)}))\\ &= h_{(1)}\cdot(k_{(1)}\cdot b)\sigma(h_{(2)},\Pi^R(h_{(3)})k_{(2)})\\ &= (h_{(1)}\cdot(k_{(1)}\cdot b))\sigma(h_{(2)},k_{(2)}),\end{aligned}$$

by the properties of weak bialgebras, the equality in Definition 3.2, and equations (1), (4), (2) and the already proven condition (6), so we obtain (5). The proof that σ be a weak cocycle is similar.

The converse follows from standard arguments and we leave the details to the reader. □

In light of Proposition 3.6 we define:

Definition 3.7. Let H be a weak bialgebra and B an algebra measured by H. Suppose also that there exists $\sigma: H\otimes H\to B$ satisfying (1) and (2) and an element $e\in B\otimes H$ that verifies (4).

Under these circumstances, if $B\otimes H$ endowed with the multiplication given by (7) is an associative algebra with preunit e and associated idempotent $\Omega(b\otimes h)=bh_{([)}1]\cdot 1_B\otimes h_{([)}2]$, we say that it is a *weak crossed product* and we denote it by $B\sharp_\sigma B$.

In the next theorem we describe the condition that a left B-linear right H-colinear multiplication on $B\otimes H$ must satisfy to be a weak crossed product. But first we need:

Lemma 3.8. *For a weak bialgebra H and an algebra B consider a left B-linear associative product on $B\otimes H$ with a preunit e satisfying (4) and such that for all $b,b'\in B$, $h,k\in H$ $(b\otimes h)(b'\otimes k)\in Im\Omega$, where $\Omega: B\otimes H\to B\otimes H$ is the idempotent map related to e. Take the map $H\otimes B\to B$:*

$$b\cdot h=(B\otimes\varepsilon_H)((1_B\otimes h)be)$$

and the morphism $\sigma: H\otimes H\to B$:

$$\sigma(h,k)=(B\otimes\varepsilon_H)((1_B\otimes h)(1_B\otimes k)).$$

Then the multiplication on $B\otimes H$ is right H-colinear if and only if following assertions hold:

(a) $h_{(1)}\cdot b\otimes h_{(2)}=(1_B\otimes h)be$, $\forall h\in H, b\in B$.
(b) $(1_B\otimes h)(1_B\otimes k)=\sigma(h_{(1)},k_{(1)})\otimes h_{(2)}k_{(2)}$, $\forall h,k\in H$.

In any of these cases, the map σ satisfies (1) and the multiplication holds

$$(b\otimes h\Pi^R(m))(b'\otimes k)=(b\otimes h)(b'\otimes\Pi^R(m)k) \tag{9}$$

for all $b,b'\in B$, $h,k,m\in H$.

Proof:
Suppose that the multiplication on $B\otimes H$ is left H-colinear. To prove (9) compute on the one hand:

$$(b\otimes(h\Pi^R(m))_{(1)})(b'\otimes k_{(1)})\otimes(h\Pi^R(m))_{(2)}k_{(2)}=(b\otimes h_{(1)})(b'\otimes k_{(1)})\otimes h_{(2)}\Pi^R(m)k_{(2)}$$

as a consequence of $(\Pi^R(m))_{(1)} \otimes (\Pi^R(m))_{(2)} = 1_{(1)} \otimes 1_{(2)}\Pi^R(m)$. On the other hand equality $(\Pi^R(m)k)_{(1)} \otimes (\Pi^R(m)k)_{(2)} = k_{(1)} \otimes \Pi^R(m)k_{(2)}$ yields

$$(b \otimes h_{(1)})(b' \otimes (\Pi^R(m)k)_{(1)}) \otimes h_{(2)}(\Pi^R(m)k)_{(2)} = (b \otimes h_{(1)})(b' \otimes k_{(1)}) \otimes h_{(2)}\Pi^R(m)k_{(2)}.$$

Then:

$$\begin{aligned} &(b \otimes h\Pi^R(m))(b' \otimes k) = (b \otimes h)(b' \otimes \Pi^R(m)k) \\ &= (B \otimes \varepsilon_H)((b \otimes h_{(1)})(b' \otimes k_{(1)}) \otimes h_{(2)}\Pi^R(m)k_{(2)}) \end{aligned}$$

for all $h, k \in H$, $b, b' \in B$ because the multiplication is right H-colinear.

Now recall that the Hom-tensor relations lead to the isomorphism:

$$Hom^H(M, N \otimes H) \to Hom(M, N),\ \alpha \mapsto (M \otimes \varepsilon_H) \circ \alpha \tag{10}$$

for all M, N right H-comodules, and whose inverse is given by $\beta \mapsto [m \in M \mapsto \beta(m_{[0]}) \otimes m_{[1]}]$.

If the multiplication is of right H-comodules, we can use this isomorphism to define σ, and therefore for all $h, k \in H$ we have

$$(1_B \otimes h)(1_B \otimes k) = \sigma(h_{(1)}, k_{(1)}) \otimes h_{(2)}k_{(2)}$$

and we obtain *(b)*. Now by a proof similar to the one for (9) we obtain that σ satisfies equality (1). Condition *(a)* follows by straightforward calculations using equality (9).

Conversely suppose that *(a)* and *(b)* are satisfied. Then as Ω is idempotent, left B-linear and $(b \otimes h)(b' \otimes k) \in Im\Omega$ for all $h, k \in H$, $b, b' \in B$:

$$\begin{aligned} (b \otimes h)(b' \otimes k) &= b\Omega(1_B \otimes h)b'\Omega(1_B \otimes k) \\ &= b(1_B \otimes h)b'e(1_B \otimes k) \\ &= b(h_{(1)} \cdot b' \otimes h_{(2)})(1_B \otimes k) \\ &= b(h_{(1)} \cdot b')\sigma(h_{(2)}, k_{(1)}) \otimes h_{(3)}k_{(2)}. \end{aligned}$$

Therefore the multiplication on $B \otimes H$ is right H-colinear. □

Theorem 3.9. *Given a left B-linear right H-colinear multiplication in $B \otimes H$, where B is an algebra and H is a weak bialgebra, and such that there exists $e \in B \otimes H$ satisfying (4), the following assertions are equivalent:*

1. *The multiplication in $B \otimes H$ is associative with preunit e and $(b \otimes h)(b' \otimes k) \in Im\Omega$ for all $b, b' \in B$, $h, k \in H$.*
2. *$B\sharp_\sigma H$ is a weak crossed product with preunit e.*

Proof:

Suppose that we are under the conditions given in *1.* and define

$$b \cdot h = (B \otimes \varepsilon_H)((1_B \otimes h)be)$$

and

$$\sigma(h, k) = (B \otimes \varepsilon_H)((1_B \otimes h)(1_B \otimes k)).$$

By Lemma 3.8 we have that σ satisfies (1) and that

$$(1_B \otimes h)be = h_{(1)} \cdot b \otimes h_{(2)}.$$

As a first consequence the left B-linear right H-colinear idempotent morphism Ω associated to e is given by the formula $\Omega(b \otimes h) = bh_{(1)} \cdot 1_B \otimes h_{(2)}$. Recall that Ω satisfies:

$$(b \otimes h)(b' \otimes k) = \Omega((b \otimes h)(b' \otimes k)) = \Omega(b \otimes h)\Omega(b \otimes k).$$

To show that $h \cdot b$ is a weak action observe that the inclusion of B in $B \otimes H$ is given by $b \mapsto be$, and then use the associativity of the product and the properties of Ω and e to obtain:

$$\begin{aligned}(h_{(1)}\cdot b)(h_{(2)}\cdot b') &= (B\otimes\varepsilon_H)(((1_B\otimes h)be)b'e)\\ &= (B\otimes\varepsilon_H)((1_B\otimes h)(beb'e))\\ &= h\cdot(bb').\end{aligned}$$

The proof for (3) and (5) is similar. Equalities (6), (8) and (2) are easily shown using the properties of the preunit and Ω.

Finally we just have to see that the product can be expressed as in (7), but this is done in the proof of Lemma 3.8.

To demonstrate the converse first check that $\Omega((b\otimes h)(b'\otimes k)) = (b\otimes h)(b'\otimes k)$ for all $b, b' \in B$ and $h, k \in H$ using (2) and equality $\Omega(b\otimes h) = bh_{(1)}\cdot 1_B\otimes h_{(2)}$. Proposition 3.6 and Lemma 3.8 complete the proof. □

Remark 3.10. If H is a bialgebra and B is an algebra, it is easy to show that the weak action and the normal twisted 2-cocycle given in [6, 19] satisfy the required conditions for $B\sharp_\sigma H$ be a weak crossed product. Observe that in this case the idempotent morphism $\Omega = id_{B\otimes H}$.

4. Crossed products and weak cleft extensions

In this section we relate weak crossed products given in the previous section with weak cleft extensions (in the sense of [2]) of weak Hopf algebras. Our aim is to obtain a relation between weak cleft extensions and crossed products with an invertible cocycle, following the ideas in [6, 19]. As we are working in a weak context the usual notion for an invertible cocycle is not suitable, because of the behavior of the unit and the counit. Hence we need a weaker version of an invertible cocycle given by Alonso Álvarez *et al.* in [4] and independently by Böhm and Brzeziński in [11]:

Definition 4.1. Let H be a weak Hopf algebra and B an algebra measured by H. We say that the 2-cocycle σ has a left inverse if there exists a map $\hat{\sigma}: H\otimes H \to B$ such that for all $h, k \in H$:

$$\hat{\sigma}(h_{(1)}, k_{(1)})\sigma(h_{(2)}, k_{(2)}) = (hk)\cdot 1_B. \tag{11}$$

Given a weak cleft extension we can define a crossed product with left invertible cocycle, as the following proposition establishes:

Proposition 4.2. *Let $B \hookrightarrow A$ be a weak cleft H-extension with a cleaving map f and its inverse $\hat{f}$. Then B is measured by H with a weak measuring*

$$H\otimes B \to B, \qquad h\cdot b = f(h_{(1)})b\hat{f}(h_{(2)}).$$

Furthermore the map

$$\sigma: H\otimes H \to B, \qquad \sigma(h,k) = f(h_{(1)})f(k_{(1)})\hat{f}(h_{(2)}k_{(2)})$$

is a left invertible weak 2-cocycle. Consequently, $B\sharp_\sigma H$ is a weak crossed product with preunit $e = \hat{f}(1_{(1)})\otimes 1_{(2)}$ and $A \simeq Im\Omega$ as left B-modules, right H-comodules and algebras.

Proof:
The map $h\cdot b = f(h_{(1)})b\hat{f}(h_{(2)})$ satisfies $h\cdot b = f(h_{(1)})b\hat{f}(h_{(2)}) \in B$ because of the properties of $\hat{f}$. To prove that it is indeed a weak measuring (this is, that verifies Definition 3.2) we just have to use that $b\in B$, the properties of f and $\hat{f}$ and the fact that A is a weak H-comodule algebra.

Using the properties mentioned above and the usual calculations for weak Hopf algebras it is easy (but tedious) to show that σ is a well defined weak 2-cocycle and that B is a weak σ twisted H module, where $e = \hat{f}(1_{(1)})\otimes 1_{(2)}$. It is also not difficult to demonstrate (8) by using the properties of e related to σ and of the weak measuing.

Now defining $\hat{\sigma} = f(h_{(1)}k_{(1)})\hat{f}(k_{(2)})\hat{f}(h_{(2)})$ we obtain that $\hat{\sigma}$ is a left weak inverse for σ.

To complete the proof we just have to show that $A \simeq Im\Omega$. If we take

$$\omega : A \to B \otimes H,\ a \mapsto a_{[0]}\hat{f}(a_{[1]}) \otimes a_{[2]}$$

this gives a well defined multiplicative map of left B-modules and of right H-comodules. Now giving

$$\omega' : B \otimes H \to A,\ b \otimes h \mapsto bf(h)$$

we obtain, on the one hand, that $\omega' \circ \omega = id_A$, and on the other hand $\omega \circ \omega' = \Omega$, then $\omega \circ \omega'|_{Im\Omega} = id_{Im\Omega}$. This equality also implies that $\omega(A) \subset Im\Omega$, therefore ω and $\omega'|_{Im\Omega}$ are inverse multiplicative isomorphisms. In view of equality $1_{[0]} \otimes 1_{[1]} \otimes 1_{[2]} = 1_{[0]} \otimes 1_{(1)}1_{[1]} \otimes 1_{(2)}$ (A is an H-comodule algebra [16]) and Definition 2.2, $\omega(1_A) = e$, therefore ω is an isomorphism of algebras. □

The following proposition is a partial converse to Proposition 4.2:

Proposition 4.3. *Let H be a weak Hopf algebra and B an algebra such that $B\sharp_\sigma H$ is a weak crossed product with left invertible cocycle. Then $(Im\Omega)^{coH} \hookrightarrow Im\Omega$ is a weak H-cleft extension.*

Proof:
As the multiplication in $Im\Omega$ is left B-linear and right H-colinear it is clear that $Im\Omega$ is a weak H-comodule algebra. Observe that $1_{[0]} \otimes 1_{[1]} = e_{<0>} \otimes e_{<1>} \otimes e_{<2>}$, so $1_{[0]}\varepsilon_H(h1_{[1]}) = e_{<0>} \otimes e_{<1>}\Pi^R(h)$, hence the first condition we need to show for a (possible) cleaving map $f : H \to Im\Omega$ is that there exists a map $\hat{f} : H \to Im\Omega$ such that $\hat{f}(h_{(1)})f(h_{(2)}) = e_{<0>} \otimes e_{<1>}\Pi^R(h)$ and that satisfies (ii) of Definition 2.2. Consider the right H-colinear map

$$f(h) = \Omega(1_B \otimes h), \tag{12}$$

and define

$$\hat{f}(h) = \Omega(e_{<0>}\hat{\sigma}(e_{<1>}S(h_{(2)}), h_{(3)}) \otimes S(h_{(1)})). \tag{13}$$

Using the H-colinearity of Ω and the equalities $\Pi^L = \bar{\Pi}^R \circ S$ [14] and $a_{[0]} \otimes \bar{\Pi}^R(a_{[1]}) = a1_{[0]} \otimes 1_{[1]}$ (A is an H-comodule algebra [16]) one easily finds that $\hat{f}$ satisfies condition (ii) of Definition 2.2. Finally, using that $(Im\Omega)^{coH} = Im(\Omega \circ (B \otimes \Pi^L))$ and equalities $\bar{\Pi}^R \circ \Pi^L = \Pi^L$ and $\bar{\Pi}^R(m)k = \varepsilon_H(mk_{(1)})k_{(2)}$ compute:

$$\begin{aligned}
\hat{f}(h_{(1)})f(h_{(2)}) &= e_{<0>}\varepsilon_H(e_{<1>}S(h_{(4)}))\hat{\sigma}(S(h_{(3)}), h_{(5)})\sigma(S(h_{(2)}), h_{(6)}) \otimes S(h_{(1)})h_{(7)} \\
&= e_{<0>}\varepsilon_H(e_{<1>}S(h_{(3)}))((S(h_{(2)})h_{(4)}) \cdot 1_B) \otimes S(h_{(1)})h_{(5)} \\
&= e_{<0>}(e_{<1>} \cdot 1_B) \otimes e_{<2>}\Pi^R(h) \\
&= e_{<0>} \otimes e_{<1>}\Pi^R(h)
\end{aligned}$$

and hence f is a cleaving map. □

In light of this last Proposition, if we want to obtain the equivalence between crossed products with an invertible cocycle and weak cleft extenisions we just have to give the conditions for obtaining an isomorphism $B \simeq (Im\Omega)^{coH}$. The next Lemma, kindly communicated to us by Gabriella Böhm, sets some necessary and sufficient conditions for this to occur:

Lemma 4.4. *Given a weak crossed product $B\sharp_\sigma H$ with a preunit e, for a weak Hopf algebra H and an H-comodule algebra B, the map*

$$\beta : B \to (Im\Omega)^{coH},\ \ b \mapsto be \tag{14}$$

is an isomorphism if and only if there exists a morphism $g : Im\Pi^L \to B$ that satisfies

1. $e_{<0>}g(e_{<1>}) = 1_B$
2. $\beta(((h^L 1_{(1)}) \cdot 1_B)g(1_{(2)})) = \Omega(1_B \otimes h^L)$ *for all* $h^L \in Im\Pi^L$.

Proof:
Let $\beta : B \to (Im\Omega)^{coH}$ be an isomorphism and define

$$g : h^L \in Im\Pi^L \mapsto g(h^L) = \beta^{-1}(\Omega(1_B \otimes h^L)).$$

As β and Ω are both left B-linear maps then $\beta(e_{<0>}\beta^{-1}(\Omega(1_B \otimes e_{<1>})) = e$ and then condition *1.* holds. Now considering that β is a map of left B-modules, equalities $\bar{\Pi}^R \circ \Pi^L = \Pi^L$ and $\bar{\Pi}^R(h)k = \varepsilon_H(hk_{(1)})k_{(2)}$ and the condition of the weak measuring given in Definition 3.2 compute:

$$\begin{aligned} \beta((h^L 1_{(1)} \cdot 1_B)\beta^{-1}(\Omega(1_B \otimes 1_{(2)}))) &= (h^L 1_{(1)} \cdot 1_B)(1_{(2)} \cdot 1_B \otimes 1_{(3)}) \\ &= \varepsilon_H(h^L 1_{(1)})1_{(2)} \cdot 1_B \otimes 1_{(3)} \\ &= \Omega(1_B \otimes h^L) \end{aligned}$$

and we obtain condition *2.*.

Conversely by straightforward computations we obtain that

$$\beta^{-1} : (Im\Omega)^{coH} \to B,\ \beta^{-1}(b \otimes h) = bh1_{(1)} \cdot 1_B g(1_{(2)})$$

is the inverse of β. □

Corollary 4.5. *The following properties of a weak crossed product* $B\sharp_\sigma H$ *with a left invertible cocycle and preunit* e *are equivalent:*

1. $B \to Im\Omega,\ b \mapsto be$ *is a weak cleft extension.*
2. $\beta : B \to (Im\Omega)^{coH},\ b \mapsto be$ *is an isomorphism.*
3. *There exists a map* $g : Im\Pi^L \to B$ *satisfying conditions given in Lemma 4.4.*

5. Equivalent crossed products

In this section we give the definition of equivalent crossed products and study how equivalent crossed products are related by a suitable gauge transformation.

Definition 5.1. Let H be a weak bialgebra, B an algebra and $B\sharp_\sigma H$ and $B\sharp_{\sigma'} H$ two weak crossed products with associated idempotents Ω and Ω' respectively. We say that $B\sharp_\sigma H$ and $B\sharp_{\sigma'} H$ are *equivalent* if there exist multiplicative maps $\phi : B\sharp_{\sigma'} H \to B\sharp_\sigma H$ and $\hat{\phi} : B\sharp_\sigma H \to B\sharp_{\sigma'} H$ of left B-modules and right H-comodules such that

$$\phi \circ \hat{\phi} = \Omega, \qquad \hat{\phi} \circ \phi = \Omega', \tag{15}$$

$$\hat{\phi} \circ \phi \circ \hat{\phi} = \hat{\phi}, \qquad \phi \circ \hat{\phi} \circ \phi = \phi. \tag{16}$$

Note that under these conditions ϕ and $\hat{\phi}$ preserve the preunits and also induce isomorphisms between the related algebras $Im\Omega$ and $Im\Omega'$.

Proposition 5.2. *Suppose that* H *is a weak bialgebra and* B *an algebra such that* $B\sharp_\sigma H$ *is a crossed product with weak measuring* $b \cdot h$. *Suppose also that there exist maps* $\chi, \hat{\chi} \in Hom(H, B)$ *such that:*

$$\hat{\chi}(h_{(1)})\chi(h_{(2)}) = h \cdot 1_B \tag{17}$$

$$\begin{aligned} \hat{\chi}(h_{(1)})\chi(h_{(2)})\hat{\chi}(h_{(3)}) &= \hat{\chi}(h) \\ \chi(h_{(1)})\hat{\chi}(h_{(2)})\chi(h_{(3)}) &= \chi(h). \end{aligned} \tag{18}$$

Then $h \cdot^\chi b = \chi(h_{(1)})(h_{(2)} \cdot b)\hat{\chi}(h_{(3)})$ *is a weak measuring,* $\sigma^\chi(h, k) = \chi(h_{(1)})(h_{(2)} \cdot \chi(k_{(1)}))$ $\sigma(h_{(3)}, k_{(2)})\hat{\chi}(h_{(4)}k_{(3)})$ *is a weak cocycle and* B *is a* σ^χ*-twisted module.*

Proof:
Define left B-module right H-comodule maps $\phi, \hat{\phi} : B \otimes H \to B \otimes H$ by $\phi(b \otimes h) = b\chi(h_{(1)}) \otimes h_{(2)}$ and $\hat{\phi}(b \otimes h) = b\hat{\chi}(h_{(1)}) \otimes h_{(2)}$. Observe that $\phi \circ \hat{\phi} = \Omega$ and that conditions (16) are satisfied, hence we can define an associative product in $B \otimes H$ by

$$(b \otimes h)(b' \otimes k) = \hat{\phi}(\phi(b \otimes h)\sharp_\sigma \phi(b' \otimes k)), \tag{19}$$

where $(b \otimes h)\sharp_\sigma(b' \otimes k)$ denotes the multiplication in $B\sharp_\sigma H$. It is easy to prove that $\hat{\phi}$ preserves this product. Moreover if e is the preunit of $B\sharp_\sigma H$, then $e' = \hat{\phi}(e)$ is a preunit for (19) which furthermore satisfies equations (4). Note also that if Ω' is the idempotent map associated to this new product we easily obtain that $\Omega'(b \otimes h) = (b \otimes h)e' = (\hat{\phi} \circ \phi)(b \otimes h)$. As the new product given by (19) is left B-linear and right H-colinear, we can use isomorphism (10) to define

$$\sigma^\chi(h, k) = (B \otimes \varepsilon_H)(\hat{\phi}(\phi(b \otimes h)\phi(b' \otimes k))),$$

and after some easy calculations we obtain

$$\sigma^\chi(h, k) = \chi(h_{(1)})(h_{(2)} \cdot \chi(k_{(1)}))\sigma(h_{(3)}, k_{(2)})\hat{\chi}(h_{(4)}k_{(3)}).$$

Equalities

$$(\Pi^R(h)k)_{(1)} \otimes (\Pi^R(h)k)_{(2)} \otimes (\Pi^R(h)k)_{(3)} = k_{(1)} \otimes k_{(2)} \otimes \Pi^L(k_{(3)})\Pi^R(h)k_{(4)}$$

and

$$(h\bar{\Pi}^L(k))_{(1)} \otimes (h\bar{\Pi}^L(k))_{(2)} \otimes (h\bar{\Pi}^L(k))_{(3)} \otimes (h\bar{\Pi}^L(k))_{(4)} = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \otimes h_{(4)}\bar{\Pi}^L(k)$$

allow us to prove that σ^χ satisfies conditions (1) of Definition 3.3. Hence we can apply Theorem 3.9 and obtain that (19) induces a weak crossed product structure on $B \otimes H$ with weak measuring

$$h \cdot^\chi b = (B \otimes \varepsilon_H)(\hat{\phi}(\phi(1_B \otimes h)b\phi(e'))),$$

and straightforward calculations yield

$$h \cdot^\chi b = \chi(h_{(1)})(h_{(2)} \cdot b)\chi(h_{(3)}).$$

□

This last proposition leads to the definition of a gauge transformation:

Definition 5.3. A pair of morphisms $\chi, \hat{\chi} \in Hom(H, B)$ that satisfies (17) and (18) is called a *gauge transformation*.

Note that from the proof of Proposition 5.2 we can conclude that a gauge transformation induces an equivalence of weak crossed products. In the next theorem we see that the converse is also true.

Theorem 5.4. *Let H be a weak bialgebra and B an algebra. The crossed products $B\sharp_\sigma H$ and $B\sharp_{\sigma'} H$ are equivalent if and only if they are related by a gauge transformation.*

Proof:
Suppose that there is an equivalence of crossed products given by $\phi : B\sharp_{\sigma'} H \to B\sharp_\sigma H$ and $\hat{\phi} : B\sharp_\sigma H \to B\sharp_{\sigma'} H$. The Hom-tensor relations give the isomorphism

$$\gamma :_B Hom^H(B \otimes H, B \otimes H) \to Hom(B, H), \phi \mapsto \gamma(\phi(h)) = (B \otimes \varepsilon_H)(\phi(1_B \otimes h)).$$

The map γ satisfies $\gamma(\phi \circ \varphi)(h) = \gamma(\varphi)(h_{(1)})\gamma(\phi)(h_{(2)})$ for all $\phi, \varphi \in Hom_B^H(B \otimes H, B \otimes H)$. Since $\gamma(\Omega(1_B \otimes h)) = h \cdot b$ and due to the properties of the equivalence between crossed products we obtain that $(\gamma(\phi), \gamma(\hat{\phi}))$ is a gauge transformation.

It is clear that $(b\otimes h)\sharp_{\sigma'}(b'\otimes k)=\hat{\phi}(\phi(b\otimes h)\sharp_{\sigma}(b'\otimes k))$, and by analogous techniques as the ones used to demonstrate Proposition 5.2 we conclude the proof. □

In the next corollary we use gauge transformations to relate different weak cleaving morphisms:

Corollary 5.5. *Suppose that $B\hookrightarrow A$ is a weak cleft extension, where A is a comodule algebra over the weak bialgebra H and B is its subalgebra of coinvariants, and let f be a weak cleaving map. Then $f':H\to A$ is also a cleaving map if and only if there exists a gauge transformation $(\chi,\hat{\chi})$ such that, for all $h\in H$, $f'(h)=\chi(h_{(1)})f(h_{(2)})$ and $\hat{f}'=\hat{f}(h_{(1)})\hat{\chi}(h_{(2)})$.*

Proof:
Suppose that f and f' are cleaving maps. Then by Proposition 4.2 they both induce weak crossed product structures $B\sharp_{\sigma}H$ and $B\sharp_{\sigma'}H$ respectively. Recall that the maps $\omega:A\to B\otimes H$ and $\omega':B\otimes H\to A$ given by $\omega(a)=a_{[0]}\hat{f}(a_{[1]})\otimes a_{[2]}$ and $\omega'(b\otimes h)=bf(h)$ are not only isomorphisms between A and $Im\Omega$ but also $\omega\circ\omega'=\Omega$. Let ν and ν' be the correspondent maps $\nu(a)=a_{[0]}\hat{f}'(a_{[1]})\otimes a_{[2]}$ and $\nu'(b\otimes h)=bf'(h)$, and Ω' the associated idempotent map. If we define $\hat{\phi}=\nu'\circ\omega$ and $\phi=\omega'\circ\nu$ we obtain an equivalence of weak crossed products between $B\sharp_{\sigma}H$ and $B\sharp_{\sigma'}H$, so they are related by a gauge transformation that is given explicitly by $\chi(h)=f'(h_{(1)})\hat{f}(h_{(2)})$ and $\hat{\chi}(h)=f(h_{(1)})\hat{f}'(h_{(2)})$, which clearly leads us to the required relation.

The converse follows by straightforward calculations. □

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References

[1] J. N. Alonso Álvarez and R. González Rodríguez: *Crossed products for weak Hopf algebras with coalgebra splitting*, J. Algebra **281** (2004), 731-752.
[2] J. N. Alonso Álvarez, R. González Rodríguez, J. M. Fernández Vilaboa and A. B. Rodríguez Raposo: *Weak C-cleft extensions, weak entwining structures and weak Hopf algebras*, J. Algebra **284** (2005) 679-704.
[3] J. N. Alonso Álvarez, R. González Rodríguez, J. M. Fernández Vilaboa and A. B. Rodríguez Raposo: *Crossed products in weak contexts*, preprint.
[4] J. N. Alonso Álvarez, R. González Rodríguez, J. M. Fernández Vilaboa and A. B. Rodríguez Raposo: *Crossed products and cleft extensions of weak Hopf algebras*, preprint.
[5] Y. Bespalov and B. Drabant: *Cross product bialgebras I*, J. Algebra 219(2) (1999) 466-505.
[6] R. J. Blattner, M. Cohen and S. Montgomery: *Crossed products and inner actions of Hopf algebras*, Trans. Amer. Math. Soc. **298** (1986), 671-711.
[7] R. J. Blattner and S. Montgomery *Crossed products and Galois extensions of Hopf algebras*, Pacific J. Math. **137** (1) (1989), 37-54.
[8] G. Böhm: *Doi-Hopf modules over weak Hopf algebras*, Comm. Algebra **28** (2000) 4687-4698.
[9] G. Böhm, F. Nill and K. Szlachányi: *Weak Hopf algebras I. Integral theory and C^*-structure*, J. Algebra **221** (1999), 385-438.
[10] G. Böhm, K. Szlachányi: *Hopf algebroids with bijective antipodes: axioms, integrals and duals*, J. Algebra **274** (2004) 708-750.

[11] G. Böhm and T. Brzeziński: *Cleft extensions of Hopf algebroids*, Appl. Categ. Structures 14(5-6) (2006) 411-419.
[12] T. Brzeziński: *Crossed products by a coalgebra*, Comm. Algebra **25** (11) (1997) 3551-3575.
[13] T. Brzeziński, S. Majid: *Coalgebra bundles*, Comm. Math. Phys. **191** (1998) 467-294.
[14] T. Brzeziński and R. Wisbauer: Corings and comodules. Cambridge University Press, Cambridge, 2003.
[15] T. Brzeziński, S. Caenepeel and G. Militaru: *Doi-Koppinen modules for quantum grupoids*, J. Pure Appl. Algebra **175** (2002) 45-62.
[16] S. Caenepeel and E. De Groot: *Modules over weak entwining structures*, Contemp. Math. **267** (2000) 31-54.
[17] Y. Doi: *Cleft comodule algebras and Hopf modules*, Comm. Alg. **12** (1984), 1115-1169.
[18] Y. Doi: *Equivalent crossed products for a Hopf algebra*, Comm. Alg. **17** (1989), 3053-3085.
[19] Y. Doi and M. Takeuchi: *Cleft comodule algebras for a bialgebra*, Comm. Alg. **14** (1986), 801-817.
[20] J. H. Lu: *Hopf algeboids and quantum grupoids*, Int. J. Math. **7** (1996), 47-70.
[21] M. E. Sweedler: *Hopf algebras*, Benjamin, New York (1969).

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