Research Article

Existence of Solutions for Fractional Differential Inclusions with Antiperiodic Boundary Conditions

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We study the existence of solutions for a class of fractional differential inclusions with anti-periodic boundary conditions. The main result of the paper is based on Bohnenblust-Karlins fixed point theorem. Some applications of the main result are also discussed.

1. Introduction

In some cases and real world problems, fractional-order models are found to be more adequate than integer-order models as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, and so forth, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details and examples, see [1–14] and the references therein.

Antiperiodic boundary value problems have recently received considerable attention as antiperiodic boundary conditions appear in numerous situations, for instance, see [15–22].

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, and so forth. and are widely studied by many authors, see [23–27] and the references therein. For some recent development on differential inclusions, we refer the reader to the references [28–32].

Chang and Nieto [33] discussed the existence of solutions for the fractional boundary value problem:

$$
{}^c_0 D^\delta_t y(t) \in F(t, y(t)), \quad t \in [0, 1], \ \delta \in (1, 2),
$$
$$
y(0) = \alpha, \quad y(1) = \beta, \quad \alpha, \beta \neq 0.
$$

(1.1)

In this paper, we consider the following fractional differential inclusions with antiperiodic boundary conditions

$$
{}^c D^q x(t) \in F(t, x(t)), \quad t \in [0, T], \ T > 0, \ 1 < q \leq 2,
$$
$$
x(0) = -x(T), \quad x'(0) = -x'(T),
$$

(1.2)

where ${}^c D^q$ denotes the Caputo fractional derivative of order q, $F : [0, T] \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\varnothing\}$. Bohnenblust-Karlin fixed point theorem is applied to prove the existence of solutions of (1.2).

2. Preliminaries

Let $C([0, T])$ denote a Banach space of continuous functions from $[0, T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in [0,T]} \{|x(t)|\}$. Let $L^1([0, T], \mathbb{R})$ be the Banach space of functions $x : [0, T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L_1} = \int_0^T |x(t)| dt$.

Now we recall some basic definitions on multivalued maps [34, 35].

Let $(X, \| \cdot \|)$ be a Banach space. Then a multivalued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{x \in \mathbb{B}} G(x)$ is bounded in X for any bounded set $\mathbb{B}$ of X (i.e., $\sup_{x \in \mathbb{B}} \{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set $\mathbb{B}$ of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}$ of x_0 such that $G(\mathcal{N}) \subseteq \mathbb{B}$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of X. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, that is, $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$. In the following study, $BCC(X)$ denotes the set of all nonempty bounded, closed, and convex subset of X. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

Let us record some definitions on fractional calculus [8, 11, 13].

Definition 2.1. For a function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order $q > 0$ is defined as

$$
{}^c D^q g(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n-q-1} g^{(n)}(s) ds, \quad n - 1 < q < n, \ n = [q] + 1,
$$

(2.1)

where $[q]$ denotes the integer part of the real number q and Γ denotes the gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order $q > 0$ for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \tag{2.2}$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.3. The Riemann-Liouville fractional derivative of order $q > 0$ for a function g is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q] + 1, \tag{2.3}$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

In passing, we remark that the Caputo derivative becomes the conventional nth derivative of the function as $q \to n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer derivatives. On the other hand, the Riemann-Liouville fractional derivative could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations (the same applies to the boundary value problems of fractional differential equations). Moreover, the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero. For more details, see [13].

For the forthcoming analysis, we need the following assumptions:

(A_1) let $F : [0,T] \times \mathbb{R} \to BCC(\mathbb{R}); (t,x) \to f(t,x)$ be measurable with respect to t for each $x \in \mathbb{R}$, u.s.c. with respect to x for a.e. $t \in [0,T]$, and for each fixed $x \in \mathbb{R}$, the set $S_{F,y} := \{f \in L^1([0,T],\mathbb{R}) : f(t) \in F(t,x) \text{ for a.e. } t \in [0,T]\}$ is nonempty,

(A_2) for each $r > 0$, there exists a function $m_r \in L^1([0,T],\mathbb{R}_+)$ such that $\|F(t,x)\| = \sup\{|v| : v(t) \in F(t,x)\} \le m_r(t)$ for each $(t,x) \in [0,T] \times \mathbb{R}$ with $|x| \le r$, and

$$\liminf_{r \to +\infty} \left(\frac{\int_0^T m_r(t)dt}{r}\right) = \gamma < \infty, \tag{2.4}$$

where m_r depends on r. For example, for $F(t,x) = x$, we have $m_r(t) = r$ and hence $\gamma = T$. If $F(t,x) = x^2$, then γ is not finite.

Definition 2.4 ([16, 33]). A function $x \in C([0,T])$ is a solution of the problem (1.2) if there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,T]$ and

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds$$

$$+ \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds, \tag{2.5}$$

which, in terms of Green's function $G(t, s)$, can be expressed as

$$x(t) = \int_0^T G(t, s) f(s) ds, \tag{2.6}$$

where

$$G(t, s) = \begin{cases} -\dfrac{(T - s)^{q-1}}{2\Gamma(q)} + \dfrac{(T - 2t)(T - s)^{q-2}}{4\Gamma(q - 1)}, & 0 < t < s < T, \\ \dfrac{(t - s)^{q-1} - \dfrac{1}{2}(T - s)^{q-1}}{\Gamma(q)} + \dfrac{(T - 2t)(T - s)^{q-2}}{4\Gamma(q - 1)}, & 0 < s < t < T. \end{cases} \tag{2.7}$$

Here, we remark that the Green's function $G(t, s)$ for $q = 2$ takes the form (see [22])

$$G(t, s) = \begin{cases} \dfrac{1}{4}(-T - 2t + 2s), & 0 < t < s < T, \\ \dfrac{1}{4}(-T + 2t - 2s), & 0 < s < t < T. \end{cases} \tag{2.8}$$

Now we state the following lemmas which are necessary to establish the main result of the paper.

Lemma 2.5 (Bohnenblust-Karlin [36]). *Let D be a nonempty subset of a Banach space X, which is bounded, closed, and convex. Suppose that $G : D \rightarrow 2^X \setminus \{0\}$ is u.s.c. with closed, convex values such that $G(D) \subset D$ and $\overline{G(D)}$ is compact. Then G has a fixed point.*

Lemma 2.6 ([37]). *Let I be a compact real interval. Let F be a multivalued map satisfying (A_1) and let Θ be linear continuous from $L^1(I, \mathbb{R}) \rightarrow C(I)$, then the operator $\Theta \circ S_F : C(I) \rightarrow BCC(C(I)), x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(I) \times C(I)$.*

3. Main Result

Theorem 3.1. *Suppose that the assumptions (A_1) and (A_2) are satisfied, and*

$$\gamma < \frac{4\Gamma(q)}{(5 + q)T^{q-1}}. \tag{3.1}$$

Then the antiperiodic problem (1.2) has at least one solution on $[0, T]$.

Proof. To transform the problem (1.2) into a fixed point problem, we define a multivalued map $\Omega : C([0,T]) \rightarrow 2^{C([0,T])}$ as

$$
\Omega(x) = \left\{ h \in C([0,T]) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \right.
$$
$$
\left. + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds, \ f \in S_{F,x} \right\}.
$$

(3.2)

Now we prove that Ω satisfies all the assumptions of Lemma 2.6, and thus Ω has a fixed point which is a solution of the problem (1.2). As a first step, we show that $\Omega(x)$ is convex for each $x \in C([0,T])$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$
h_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)ds
$$
$$
+ \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)ds, \quad i = 1,2.
$$

(3.3)

Let $0 \leq \lambda \leq 1$. Then, for each $t \in J$, we have

$$
[\lambda h_1 + (1-\lambda)h_2](t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1-\lambda)f_2(s)]ds
$$
$$
- \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1-\lambda)f_2(s)]ds
$$
$$
+ \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} [\lambda f_1(s) + (1-\lambda)f_2(s)]ds.
$$

(3.4)

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\lambda h_1 + (1-\lambda)h_2 \in \Omega(x)$.

In order to show that $\Omega(x)$ is closed for each $x \in C([0,T])$, let $\{u_n\}_{n\geq 0} \in \Omega(x)$ be such that $u_n \rightarrow u(n \rightarrow \infty)$ in $C([0,T])$. Then $u \in C([0,T])$ and there exists a $v_n \in S_{F,x}$ such that

$$
u_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)ds
$$
$$
+ \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_n(s)ds.
$$

(3.5)

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T], \mathbb{R}_+)$. Thus, $v \in S_{F,x}$ and

$$u_n(t) \longrightarrow u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)ds$$
$$+ \frac{1}{4}(T - 2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(s)ds. \tag{3.6}$$

Hence $u \in \Omega(x)$.

Next we show that there exists a positive number r such that $\Omega(B_r) \subseteq B_r$, where $B_r = \{x \in C([0,T]) : \|x\| \le r\}$. Clearly B_r is a bounded closed convex set in $C([0,T])$ for each positive constant r. If it is not true, then for each positive number r, there exists a function $x_r \in B_r, h_r \in \Omega(x_r)$ with $\|\Omega(x_r)\| > r$, and

$$h_r(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_r(s)ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_r(s)ds$$
$$+ \frac{1}{4}(T - 2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_r(s)ds, \quad \text{for some } f_r \in S_{F,x_r}. \tag{3.7}$$

On the other hand, in view of (A_2), we have

$$r < \|\Omega(x_r)\|$$
$$\le \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |f_r(s)|ds + \frac{1}{2} \int_0^T \frac{|T-s|^{q-1}}{\Gamma(q)} |f_r(s)|ds$$
$$+ \frac{1}{4}|T - 2t| \int_0^T \frac{|T-s|^{q-2}}{\Gamma(q-1)} |f_r(s)|ds$$
$$\le \frac{T^{q-1}}{\Gamma(q)} \int_0^T m_r(s)ds + \frac{T^{q-1}}{2\Gamma(q)} \int_0^T m_r(s)ds + \frac{T^{q-1}}{4\Gamma(q-1)} \int_0^T m_r(s)ds$$
$$= \frac{T^{q-1}(5+q)}{4\Gamma(q)} \int_0^T m_r(s)ds, \tag{3.8}$$

where we have used the fact that

$$\left|(T-2t)(T-s)^{q-2}\right| \le \left|(T-t)(T-s)^{q-2}\right| \le |(T-t)|^{q-1}, \quad \text{for } t < s,$$
$$\left|(T-2t)(T-s)^{q-2}\right| \le \left|(T-t)(T-s)^{q-2}\right| \le |(T-s)|^{q-1}, \quad \text{for } t \ge s. \tag{3.9}$$

Dividing both sides of (3.8) by r and taking the lower limit as $r \to \infty$, we find that $\gamma \ge 4\Gamma(q)/(5+q)T^{q-1}$, which contradicts (3.1). Hence there exists a positive number r' such that $\Omega(B_{r'}) \subseteq B_{r'}$.

Now we show that $\Omega(B_{r'})$ is equicontinuous. Let $t', t'' \in [0, T]$ with $t' < t''$. Let $x \in B_{r'}$ and $h \in \Omega(x)$, then there exists $f \in S_{F,x}$ such that for each $t \in [0, T]$, we have

$$h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s) ds + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s) ds. \quad (3.10)$$

Using (3.8), we obtain

$$
\begin{aligned}
\left| h(t'') - h(t') \right| &= \left| \int_0^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s) ds + \frac{1}{4}(T-2t'') \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s) ds \right. \\
&\quad \left. - \int_0^{t'} \frac{(t'-s)^{q-1}}{\Gamma(q)} f(s) ds - \frac{1}{4}(T-2t') \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s) ds \right| \\
&\leq \left| \int_0^{t'} \frac{\left[(t''-s)^{q-1} - (t'-s)^{q-1}\right]}{\Gamma(q)} f(s) ds \right| + \left| \int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s) ds \right| \\
&\quad + \left| -\frac{(t''-t')}{2} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s) ds \right| \\
&\leq \frac{1}{\Gamma(q)} \int_0^{t'} \left| (t''-s)^{q-1} - (t'-s)^{q-1} \right| m_{r'}(s) ds + \frac{T^{q-1}}{\Gamma(q)} \int_{t'}^{t''} m_{r'}(s) ds \\
&\quad + \frac{(t''-t')T^{q-2}}{2\Gamma(q-1)} \int_0^T m_{r'}(s) ds.
\end{aligned}
\quad (3.11)
$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' \to t'$. Thus, Ω is equicontinuous.

As Ω satisfies the above assumptions, therefore it follows by Ascoli-Arzela theorem that Ω is a compact multivalued map.

Finally, we show that Ω has a closed graph. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. We will show that $h_* \in \Omega(x_*)$. By the relation $h_n \in \Omega(x_n)$, we mean that there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0, T]$,

$$
\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_n(s) ds \\
&\quad + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_n(s) ds.
\end{aligned}
\quad (3.12)
$$

Thus we need to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0, T]$,

$$
\begin{aligned}
h_*(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s) ds \\
& + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s) ds.
\end{aligned}
\tag{3.13}
$$

Let us consider the continuous linear operator $\Theta : L^1([0,T], \mathbb{R}) \to C([0,T])$ so that

$$
\begin{aligned}
f \longmapsto \Theta(f)(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s) ds \\
& + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s) ds.
\end{aligned}
\tag{3.14}
$$

Observe that

$$
\begin{aligned}
\|h_n(t) - h_*(t)\| = {} & \left\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s)) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s)) ds \right. \\
& \left. + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} (f_n(s) - f_*(s)) ds \right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
\tag{3.15}
$$

Thus, it follows by Lemma 2.6 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, Lemma 2.6 yields

$$
\begin{aligned}
h_*(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s) ds \\
& + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s) ds, \quad \text{for some } f_* \in S_{F,x_*}.
\end{aligned}
\tag{3.16}
$$

Hence, we conclude that Ω is a compact multivalued map, u.s.c. with convex closed values. Thus, all the assumptions of Lemma 2.6 are satisfied and so by the conclusion of Lemma 2.6, Ω has a fixed point x which is a solution of the problem (1.2). □

Remark 3.2. If we take $F(t, y) = \{f(t, y)\}$, where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, then our results correspond to a single-valued problem (a new result).

Applications

As an application of Theorem 3.1, we discuss two cases in relation to the nonlinearity F in (1.2), namely, F has (a) sublinear growth in its second variable (b) linear growth in its second variable (state variable). In case of sublinear growth, there exist functions $\eta(t), \rho(t) \in L^1([0,T], \mathbb{R}_+), \mu \in [0,1)$ such that $\|F(t,x)\| \le \eta(t)|x|^\mu + \rho(t)$ for each $(t,x) \in [0,T] \times \mathbb{R}$.

In this case, $m_r(t) = \eta(t)r^\mu + \rho(t)$. For the linear growth, the nonlinearity F satisfies the relation $\|F(t,x)\| \leq \eta(t)|x| + \rho(t)$ for each $(t,x) \in [0,T] \times \mathbb{R}$. In this case $m_r(t) = \eta(t)r + \rho(t)$ and the condition (3.1) modifies to $\|\eta\|_{L^1} < 4\Gamma(q)/(5+q)T^{q-1}$. In both the cases, the antiperiodic problem (1.2) has at least one solution on $[0,T]$.

Examples

(a) We consider $\|F(t,x)\| \leq \eta(t)|x|^{1/3} + \rho(t)$ and $T = 1$ in (1.2). Here, $\eta(t), \rho(t) \in L^1([0,1], \mathbb{R}_+)$. Clearly $F(t,x)$ satisfies the assumptions of Theorem 3.1 with $0 < 4\Gamma(q)/(5+q)$ (condition (3.1)). Thus, by the conclusion of Theorem 3.1, the antiperiodic problem (1.2) has at least one solution on $[0,1]$.

   (b) As a second example, let $F(t,x)$ be such that $\|F(t,x)\| \leq (1/(1+t)^2)|x| + e^{-t}$ and $T = 1$ in (1.2). In this case, (3.1) takes the form $1/2 < 4\Gamma(q)/(5+q)$. As all the assumptions of Theorem 3.1 are satisfied, the antiperiodic problem (1.2) has at least one solution on $[0,1]$.

Acknowledgments

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