

The Witten deformation of the Dolbeault complex

J. Álvarez López* and P. Gilkey

Abstract. We introduce a Witten-Novikov type perturbation $\bar{\partial}_\omega$ of the Dolbeault complex of any complex Kähler manifold, defined by a form ω of type $(1,0)$ with $\partial\omega = 0$. We give an explicit description of the associated index density which shows that it exhibits a nontrivial dependence on ω . The heat invariants of lower order are shown to be zero.

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1. Introduction

1.1. Historical Summary and motivation

Let M be a closed manifold of dimension m , and let h be a smooth function on M . Witten [25] introduced a perturbation of the de Rham differential of the form $d_h = d + \text{ext}(dh)$, using exterior multiplication by dh . Since d_h is gauge equivalent to d , the Betti numbers are unchanged. Given a Riemannian metric g on M , the perturbed de Rham codifferential is $\delta_h = \delta + \text{int}(dh)$ where int denotes interior multiplication. In general, the perturbed Laplacian, $\Delta_h = d_h\delta_h + \delta_h d_h$, is not gauge equivalent to Δ and thus can have a different spectrum. In fact, when h is a Morse function, Witten gave a beautiful analytical proof of the Morse inequalities by analyzing the spectrum of Δ_{sh} as $s \rightarrow \infty$. This program of Witten was continued by Helffer and Sjöstrand [14], and by Bismut and Zhang [4].

With more generality, Novikov [18, 19] defined similar perturbed operators d_ι , δ_ι and Δ_ι , replacing dh with a real closed 1-form ι on M . He used Witten's procedure to estimate the zeros of ι if ι is of Morse type. Since

*Corresponding author.

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d_ι need not be gauge equivalent to d , the new twisted Betti numbers can be different. However, one can show that the twisted Betti numbers of $d_{s\iota}$ ($s \in \mathbb{R}$) are constant except for a finite number of values of s , where the dimensions may jump. Those ground values of twisted Betti numbers are called the Novikov numbers of the cohomology class $[\iota]$; they are used in the Novikov version of the Morse inequalities. We refer to related work of Braverman and Farber [5] and of Pazhitnov [22], and, more recently, to the work of many other authors [6, 7, 13, 16, 17].

In previous work [1], we used methods of invariance theory to prove that the local index density for the Witten-Novikov Laplacian Δ_ι is the Euler form if m is even and, in particular, does not depend on ι . If m is odd, the local index density vanishes. The heat trace invariants of smaller order are also trivial, but the heat trace invariants of higher order exhibit a nontrivial dependence on ι . A different proof of the invariance of the twisted index density was also given by the first author, Kordyukov, and Leichtnam [2], where it was applied to study certain trace formulas for foliated flows (our original motivation). In [1], we also extended the invariance of the twisted index density to the setting of manifolds with boundary, and gave an equivariant version of that invariance for maps. The situation in the complex setting is quite different. We proved that the local index density for a Witten-Novikov type perturbation of the Dolbeault complex exhibits non-trivial dependence on the twisting 1-form in the case of Riemann surfaces.

In the present paper, we extend the study of the Witten-Novikov type perturbation of the Dolbeault complex to the case of an arbitrary complex Kähler manifold (M, g, J) of dimension $m = 2m$. We consider the Dolbeault complex $\bar{\partial}$ with coefficients in an auxiliary holomorphic vector bundle E over M equipped with a Hermitian metric h . The Hirzebruch-Riemann-Roch Theorem states that its index is given by the integral on M of $\{\text{Td}(M, g, J) \wedge \text{ch}(E, h)\}_m$ (the homogeneous component of degree m of the product of the Todd genus of (M, g, J) and the Chern character of (E, h)). This theorem was refined by the second author [9, 10], and by Atiyah, Bott, and Patodi [3], showing that $\{\text{Td}(M, g, J) \wedge \text{ch}(E, h)\}_m$ is indeed the index density that shows up in the asymptotic expansion of the heat kernel.

In this complex setting, the Witten-Novikov deformation of the Dolbeault complex is $\bar{\partial}_{\bar{\omega}} = \bar{\partial} + \text{ext}(\bar{\omega})$, where ω is a form of type $(1, 0)$ on M with $\partial\omega = 0$. We use methods of invariance theory to give an explicit description of its index density, which is a perturbation of $\{\text{Td}(M, g, J) \wedge \text{ch}(E, h)\}_m$ with a non trivial dependence on ω . The other heat invariants of lower order are shown to be zero. As a possible application, this description might be a step in a version for the leafwise Dolbeault complex of the trace formula for foliated flows given in [2].

1.2. Operators of Laplace Type

Henceforth, let dvol be the measure defined by a Riemannian metric g on a closed manifold M of dimension m and let h be a Hermitian fiber metric on a vector bundle E over M . A second order partial differential operator D

on $C^\infty(E)$ is said to be of *Laplace type* if the leading symbol is given by the metric tensor, i.e. if

$$D = - \left\{ \sum_{i,j=1}^m g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \text{id} + \sum_{k=1}^m A^k \frac{\partial}{\partial x^k} + B \right\}$$

relative to a system of local coordinates (x^1, \dots, x^m) for M and relative to a local frame for E where $g^{ij} = g(dx^i, dx^j)$ and where A^k and B are endomorphisms of E . The following result follows from work of Seeley [23] and others.

Theorem 1.1. *Let D be an operator of Laplace type.*

1. *There exists a smooth kernel $K(t, x, y, D)$ for $t > 0$ so that*

$$\{e^{-tD}\phi\}(x) = \int_M K(t, x, y, D)\phi(y) \text{dvol}(y).$$

2. *There exist local invariants $a_{m,2n}(D)(x)$ so that as $t \downarrow 0$,*

$$\begin{aligned} \text{Tr}_{E_x} K(t, x, x, D) &\sim \sum_{n=0}^{\infty} t^{(2n-m)/2} a_{m,2n}(D)(x), \\ \text{Tr}_{L^2} \{e^{-tD}\} &\sim \sum_{n=0}^{\infty} t^{(2n-m)/2} \int_M a_{m,2n}(D)(x) \text{dvol}(x). \end{aligned}$$

1.3. The local index density

Let $\mathcal{E} = \{d_i : C^\infty(E_i) \rightarrow C^\infty(E_{i+1})\}$ where (E_i, h_i) is a finite collection of Hermitian vector bundles and where the d_i are first order partial differential operators. We shall say that \mathcal{E} is an *elliptic complex of Dirac type* if $d_{i+1}d_i = 0$ and if the associated self-adjoint second order operators $D_i := d_i^*d_i + d_{i-1}d_{i-1}^*$ are of Laplace type. The cohomology groups of \mathcal{E} are given by

$$H^i(\mathcal{E}) := \frac{\text{kernel}(d_i : C^\infty(E_i) \rightarrow C^\infty(E_{i+1}))}{\text{image}(d_{i-1} : C^\infty(E_{i-1}) \rightarrow C^\infty(E_i))}.$$

The Hodge Decomposition Theorem permits us to identify $H^i(\mathcal{E})$ with $\ker(D_i)$; these groups are finite dimensional and we take the super-trace to define

$$a_{m,2n}(\mathcal{E}) := \sum_i (-1)^i a_{m,2n}(D_i) \text{ and } \text{index}(\mathcal{E}) := \sum_i (-1)^i \dim H^i(\mathcal{E}).$$

If m is odd, then $\text{index}(\mathcal{E}) = 0$ so we assume m even henceforth. The invariant $a_{m,2n}(\mathcal{E})$ for $2n = m$ is called the *local index density* as a cancellation argument due to Bott shows that

$$\int_M a_{m,2n}(\mathcal{E})(x) \text{dvol}(x) = \begin{cases} \text{index}(\mathcal{E}) & \text{if } 2n = m \\ 0 & \text{if } 2n \neq m \end{cases}.$$

1.4. The de Rham complex

Let $\Lambda^i(M)$ be the vector bundle of i -forms and let

$$d : C^\infty(\Lambda^i(M)) \rightarrow C^\infty(\Lambda^{i+1}(M)) \text{ for } 0 \leq i \leq m-1,$$

be exterior differentiation. This defines an elliptic complex of Dirac type we shall denote by $\mathcal{E}_{\text{deR}}(M, g)$. Let $\chi(M)$ be the Euler-Poincaré characteristic of M . The Hodge-de Rham theorem permits us to identify $H^p(\mathcal{E}_{\text{deR}}(M, g))$ with the topological cohomology groups $H^p(M; \mathbb{C})$ and shows that $\text{index}(\mathcal{E}_{\text{deR}}(M, g)) = \chi(M)$.

1.5. The Dolbeault complex

Let J be an integrable almost complex structure on a smooth manifold M of complex dimension \mathfrak{m} and corresponding real dimension $m = 2\mathfrak{m}$. Let g be a J invariant Riemannian metric on M ; (M, g, J) is a Hermitian holomorphic manifold. Let $\Omega(X, Y) := g(X, JY)$ be the Kähler form; we say (M, g, J) is *Kähler* if $d\Omega = 0$. Let E be an auxiliary holomorphic vector bundle over M equipped with a Hermitian metric h . Let $\sqrt{2} \bar{\partial}$ be the normalized Dolbeault operator; the normalizing constant of $\sqrt{2}$ is present to ensure that this is of Dirac type. The Dolbeault complex $\mathcal{E}_{\text{Dol}}(M, g, J, E, h)$ is defined by

$$\sqrt{2} \bar{\partial} : C^\infty(\Lambda^{0,i}(M) \otimes E) \rightarrow C^\infty(\Lambda^{0,i+1}(M) \otimes E) \text{ for } 0 \leq i \leq \mathfrak{m} - 1.$$

Let $H^p(M; \mathcal{O}(E))$ be the cohomology groups of M with coefficients in the sheaf of holomorphic sections to E . Identify $H^p(\mathcal{E}_{\text{Dol}}(M, g, J, E, h))$ with $H^p(M; \mathcal{O}(E))$; if E is the trivial line bundle, then $\text{index}\{\mathcal{E}_{\text{Dol}}(M, g, J, E, h)\}$ is the arithmetic genus of M .

1.6. The Chern-Gauss-Bonnet and Hirzebruch-Riemann-Roch Theorems

Let $m = 2\mathfrak{m}$. Let R_{ijkl} denote the components of the curvature tensor relative to a local orthonormal frame for the tangent bundle of M . We follow the discussion in Chern [8] and define the Pfaffian or Euler form by setting:

$$\text{Pf}_m(x, g) := \sum_{i_1, \dots, i_m, j_1, \dots, j_m=1}^m \frac{(-1)^{\mathfrak{m}}}{8^{\mathfrak{m}} \pi^{\mathfrak{m}} \mathfrak{m}!} g(e^{i_1} \wedge \dots \wedge e^{i_m}, e^{j_1} \wedge \dots \wedge e^{j_m}) \\ R_{i_1 i_2 j_1 j_2} \dots R_{i_{\mathfrak{m}-1} i_{\mathfrak{m}} j_{\mathfrak{m}-1} j_{\mathfrak{m}}}(x).$$

In the complex setting, let $\text{Td}(M, g, J)$ be the total Todd genus of the complex tangent bundle of (M, g, J) and let $\text{ch}(E, h)$ be the total Chern character; we refer to Hirzebruch [15] for details. We set

$$\{\text{Td}(M, g, J) \wedge \text{ch}(E, h)\}_m = \sum_{i+j=m} \text{Td}_i(M, g, J) \wedge \text{ch}_j(E, h).$$

We use the Hodge \star operator to identify top dimensional forms with scalar functions. We refer to Chern [8] for the proof of Assertion (1) and to Hirzebruch [15] for the proof of Assertion (2) in the following result.

Theorem 1.2.

1. $\text{index}(\mathcal{E}_{\text{deR}}(M, g)) = \int_M \text{Pf}_m(M, g) \text{dvol}.$
2. $\text{index}(\mathcal{E}_{\text{Dol}}(M, g, J, E, h)) = \int_M \star\{\text{Td}(M, g, J) \wedge \text{ch}(E, h)\}_m \text{dvol}.$

By identifying the local index densities of the de Rham and Dolbeault complexes with the integrands of Theorem 1.2, Patodi [20, 21] gave a heat equation proof of Theorem 1.2 (1) in the real setting and of Theorem 1.2 (2) in the complex Kähler setting by showing:

Theorem 1.3.

1. $a_{m,2n}(\mathcal{E}_{\text{deR}}(M, g)) = \begin{cases} 0 & \text{if } 2n < m \\ \text{Pf}_m(M, g) & \text{if } 2n = m \end{cases}.$
2. *If (M, g, J) is Kähler, then $a_{m,2n}(\mathcal{E}_{\text{Dol}}(M, g, J, E, h))$*

$$= \begin{cases} 0 & \text{if } 2n < m \\ \star\{\text{Td}(M, g, J) \wedge \text{ch}(E, h)\}_m & \text{if } 2n = m \end{cases}.$$

Shortly thereafter, other proofs of Theorem 1.3 were given. Gilkey [9, 10] used invariance theory directly and Atiyah, Bott, and Patodi [3] combined invariance theory with a study of the twisted signature complex and the twisted spin-c complex to prove Theorem 1.3. The subject has an extensive history and we refer to [11] for further details. As noted, the twisted signature complex and twisted spin complex can be treated using heat equation methods [3, 9] and a heat equation proof of the full Atiyah-Singer index theorem given thereby. We note that the local index density for the Dolbeault complex does not agree in general with the Hirzebruch-Riemann-Roch integrand of Theorem 1.2 (2) in the non-Kähler setting as was shown in later work by Gilkey, Nikčević, and Pohjanpelto [12]; Theorem 1.3 (2) can fail if (M, g, J) is not assumed Kähler.

1.7. The Witten deformation

If ι is a closed 1-form on a Riemannian manifold (M, g) , one can define the deformed de Rham complex $\mathcal{E}_{\text{deR}}(M, g, \iota)$ by setting

$$d_\iota := d + \text{ext}(\iota) : C^\infty(\Lambda^i M) \rightarrow C^\infty(\Lambda^{i+1}(M)).$$

The assumption that ι is closed ensures that $d_\iota^2 = 0$; since ι introduces a 0th perturbation, the leading symbol of the associated second order operators is unchanged so $\mathcal{E}_{\text{deR}}(M, g, \iota)$ is an elliptic complex of Dirac type. The authors [1] showed previously that the deformation ι does not enter in the index density in this setting

$$a_{m,2n}(\mathcal{E}_{\text{deR}}(M, g, \iota)) = \begin{cases} 0 & \text{if } 2n < m \\ \text{Pf}_m(M, g) & \text{if } 2n = m \end{cases}.$$

This result is sharp; if $2n > m$, then $a_{m,2n}(\mathcal{E}_{\text{deR}}(M, g, \iota))$ does depend upon ι in general.

In the complex setting, let ω be a form of type $(1, 0)$ on M with $\partial\omega = 0$ and let $\mathcal{M} := (M, g, J, \omega, E, h)$. Set $\bar{\partial}_\omega := \bar{\partial} + \text{ext}(\bar{\omega})$. The assumption $\partial\omega = 0$

implies $\bar{\partial}\bar{\omega} = 0$ and ensures $\bar{\partial}\bar{\omega}^2 = 0$. Let $\mathcal{E}_{\text{Dol}}(\mathcal{M})$ be the Witten perturbation of the Dolbeault complex with coefficients in E defined by taking

$$\sqrt{2} \bar{\partial}_{\bar{\omega}} : C^\infty(\Lambda^{(0,i)} \otimes E) \rightarrow C^\infty(\Lambda^{(0,i+1)} \otimes E) \text{ for } 0 \leq i \leq \mathfrak{m} - 1.$$

This is an elliptic complex of Dirac type. Let $\Im(\omega) = \frac{1}{2\sqrt{-1}}(\omega - \bar{\omega})$ be the imaginary part of ω . Let

$$\Theta := \sum_k \frac{1}{k! \pi^k} \{d\Im(\omega)\}^k,$$

$$\{\text{Td} \wedge \text{ch} \wedge \Theta\}_m := \sum_{i+j+k=m} \text{Td}(M, g, J)_i \wedge \text{ch}(E, h)_j \wedge \Theta_k.$$

The following is the main new result of this paper.

Theorem 1.4. $a_{m,2n}(\mathcal{E}_{\text{Dol}}(\mathcal{M})) = \left\{ \begin{array}{ll} 0 & \text{if } 2n < m \\ \star \{\text{Td} \wedge \text{ch} \wedge \Theta\}_m & \text{if } 2n = m \end{array} \right\}.$

1.8. The signature and spin complexes

Let M be an oriented manifold of dimension $4k$. Let $d + \delta : C^\infty(\Lambda^\pm(M)) \rightarrow C^\infty(\Lambda^\mp(M))$ be the Hirzebruch signature complex. We then have $d_\iota + \delta_\iota = d + \delta + (\text{ext} + \text{int})(\iota)$. Now $(\text{ext} - \text{int})(\iota) : \Lambda^\pm \rightarrow \Lambda^\mp$ but $(\text{ext} + \text{int})(\iota)$ does not have this property if $\iota \neq 0$. So $d_\iota + \delta_\iota$ does not induce a map on the signature complex; it is not possible to deform the signature complex in this fashion. Similarly the spin complex can not be deformed in this fashion. The de Rham and Dolbeault complexes are \mathbb{Z} graded and this seems to be crucial in studying the Witten deformation; the signature and spin complexes, on the other hand, are \mathbb{Z}_2 graded and this makes all the difference. For this reason, we shall not follow the approach of Atiyah, Bott, and Patodi [3] to study the Witten deformation of the Dolbeault complex by passing to the spin-c complex. Instead, we shall return to the original treatment of Gilkey [10] and apply invariance theory directly.

1.9. Brief guide to the paper

In Section 2, we discuss product formulas for the heat trace asymptotics. In Section 3, we normalize the systems of coordinates and vector bundle frames to be considered up to arbitrarily high, but finite, order; this in effect reduces the structure group to the unitary group. In Section 4, we introduce the requisite spaces of invariants; the precise notion of what is meant by a “local invariant” or a “local formula” is crucial to our study. In Section 5, we discuss the restriction map. In Section 6, we use invariance under the action of the unitary group $U(\mathfrak{m})$ to establish certain technical results. In Section 7, we complete the proof of Theorem 1.4.

2. Product formulas

The following observations are well known – see, for example, the discussion in Gilkey [11]. Let $M = M_1 \times M_2$ where M_i are closed manifolds of dimension

m_i . Let $\pi_i : M \rightarrow M_i$ be projection on the i^{th} factor. Let g_i be Riemannian metrics on M_i and let $g = \pi_1^* g_1 + \pi_2^* g_2$ be the associated Riemannian metric on M . Let (E_i, h_i) be Hermitian vector bundles over M_i and let $E := \pi_1^* E_1 \otimes \pi_2^* E_2$ and $h := \pi_1^* h_1 \otimes \pi_2^* h_2$ define the associated vector bundle and Hermitian inner product over M . Let $D = D_1 \otimes \text{id} + \text{id} \otimes D_2$ be an operator of Laplace type on $C^\infty(E)$ over M where D_i are operators of Laplace type on $C^\infty(E_i)$ over M_i . We have $e^{-tD} = e^{-tD_1} \otimes e^{-tD_2}$ and the associated kernel function is given by

$$K(t, (x_1, x_2), (y_1, y_2), D) = K(t, x_1, y_1, D_1) \otimes K(t, x_2, y_2, D_2).$$

We multiply the resulting asymptotic expansions for the heat kernels and equate coefficients of t to obtain corresponding local expressions

$$a_{m,2n}(D)(x_1, x_2) = \sum_{n_1+n_2=n} a_{m_1,2n_1}(D_1)(x_1) \cdot a_{m_2,2n_2}(D_2)(x_2). \quad (2.a)$$

Let \mathcal{E}_1 and \mathcal{E}_2 be elliptic complexes of Dirac type over M_1 and M_2 , respectively. The elliptic complex of Dirac type $\mathcal{E} := \mathcal{E}_1 \otimes \mathcal{E}_2$ over M is defined by setting

$$\begin{aligned} E_k &= \oplus_{i+j=k} \pi_1^*(E_{1,i}) \otimes \pi_2^*(E_{2,j}), \\ d_k &= \oplus_{i+j=k} d_{1,i} \otimes \text{id} + (-1)^i \text{id} \otimes d_{2,j}; \end{aligned}$$

the factor of $(-1)^i$ is present to ensure $d^2 = 0$. The associated operators of Laplace type then take the form $D_k = \oplus_{i+j=k} D_{1,i} \otimes \text{id} + \text{id} \otimes D_{2,j}$ and consequently taking the super trace and applying Equation (2.a) yields

$$a_{m,2n}(\mathcal{E})(x_1, x_2) = \sum_{n_1+n_2=n} a_{m_1,2n_1}(\mathcal{E}_1)(x_1) \cdot a_{m_2,2n_2}(\mathcal{E}_2)(x_2). \quad (2.b)$$

We now turn to the Dolbeault complex. Let $\mathcal{M}_1 = (M_1, g_1, J_1, E_1, h_1, \omega_1)$ and $\mathcal{M}_2 = (M_2, g_2, J_2, E_2, h_2, \omega_2)$ be given. Let

$$M := M_1 \times M_2, \quad g := \pi_1^* g_1 + \pi_2^* g_2, \quad E := \pi_1^* E_1 \otimes \pi_2^* E_2, \quad h := \pi_1^* h_1 \otimes \pi_2^* h_2$$

be as given above. Let $\omega := \pi_1^* \omega_1 + \pi_2^* \omega_2$. We have that

$$J := \pi_1^* J_1 \oplus \pi_2^* J_2 \text{ on } T(M) = \pi_1^* T(M_1) \oplus \pi_2^* T(M_2)$$

is an integrable almost complex structure on M ; the auxiliary bundle E is then holomorphic. We set $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 = (M, g, J, E, h, \omega)$ and obtain that

$$\mathcal{E}_{\text{Dol}}(\mathcal{M}) = \mathcal{E}_{\text{Dol}}(\mathcal{M}_1) \otimes \mathcal{E}_{\text{Dol}}(\mathcal{M}_2).$$

Equation (2.b) then yields a corresponding decomposition of the local heat trace invariants

$$a_{m,2n}(\mathcal{E}_{\text{Dol}}(\mathcal{M})) = \sum_{n_1+n_2=n} a_{m_1,2n_1}(\mathcal{E}_{\text{Dol}}(\mathcal{M}_1)) a_{m_2,2n_2}(\mathcal{E}_{\text{Dol}}(\mathcal{M}_2)) \quad (2.c)$$

3. Normalizing the coordinates and the local frame

Let $\bar{z} = (z^1, \dots, z^m)$ where $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$ is a system of local holomorphic coordinates on M . Let

$$\frac{\partial}{\partial z^\alpha} := \frac{1}{2} \left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1} \frac{\partial}{\partial y^\alpha} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^\alpha} := \frac{1}{2} \left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1} \frac{\partial}{\partial y^\alpha} \right).$$

We extend g to a symmetric bilinear form on the complex tangent bundle; the condition that g is J invariant, then yields

$$g \left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) = g \left(\frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial z^\beta} \right) = 0 \quad \text{so} \quad g_{\alpha\bar{\beta}} := g \left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right)$$

defines a positive definite Hermitian form. Introduce formal variables

$$g_{\alpha_0 \bar{\beta}_0 / \alpha_1 \dots \alpha_j \bar{\beta}_1 \dots \bar{\beta}_k} := \frac{\partial}{\partial z^{\alpha_1}} \dots \frac{\partial}{\partial z^{\alpha_j}} \frac{\partial}{\partial \bar{z}^{\beta_1}} \dots \frac{\partial}{\partial \bar{z}^{\beta_k}} g \left(\frac{\partial}{\partial z^{\alpha_0}}, \frac{\partial}{\partial \bar{z}^{\beta_0}} \right)$$

for the holomorphic and anti-holomorphic derivatives of the components of g where there are no holomorphic derivatives if $j = 0$ and no anti-holomorphic derivatives if $k = 0$. We may express the Kähler form $\Omega(X, Y) = g(X, JY)$ as

$$\Omega = \frac{1}{2\sqrt{-1}} \sum_{\alpha_0=1}^m \sum_{\beta_0=1}^m g_{\alpha_0 \bar{\beta}_0} dz^{\alpha_0} \wedge d\bar{z}^{\beta_0}.$$

We say (M, g, J) is *Kähler* if $d\Omega = 0$ and we impose this condition henceforth. This condition is equivalent to the symmetries:

$$g_{\alpha_0 \bar{\beta}_0 / \alpha_1} = g_{\alpha_1 \bar{\beta}_0 / \alpha_0} \quad \text{and} \quad g_{\alpha_0 \bar{\beta}_0 / \bar{\beta}_1} = g_{\alpha_0 \bar{\beta}_1 / \bar{\beta}_0}.$$

We can differentiate these relations to see the variables $g_{\alpha_0 \bar{\beta}_0 / \alpha_1 \dots \alpha_j \bar{\beta}_1 \dots \bar{\beta}_k}$ are symmetric in $\{\alpha_0 \dots \alpha_j\}$ and in $\{\beta_0 \dots \beta_k\}$. Set

$$g_{(\alpha_0 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)} := g_{\alpha_0 \bar{\beta}_0 / \alpha_1 \dots \alpha_j \bar{\beta}_1 \dots \bar{\beta}_k}. \quad (3.a)$$

If σ_1 and σ_2 are permutations of $j+1$ and $k+1$ indices, respectively, then

$$g_{(\alpha_0 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)} = g_{(\alpha_{\sigma_1(0)} \dots \alpha_{\sigma_1(j)}; \bar{\beta}_{\sigma_2(0)} \dots \bar{\beta}_{\sigma_2(k)})}. \quad (3.b)$$

Similarly introduce formal variables

$$h_{(p\bar{q}; \alpha_1 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)} := h_{p\bar{q} / \alpha_1 \dots \alpha_j \bar{\beta}_1 \dots \bar{\beta}_k} \quad (3.c)$$

for the derivatives of the components of the Hermitian metric on E . If σ_3 and σ_4 are permutations of j and k indices, respectively, then

$$h_{(p\bar{q}; \alpha_1 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)} = h_{(p\bar{q}; \alpha_{\sigma_3(1)} \dots \alpha_{\sigma_3(j)}; \bar{\beta}_{\sigma_4(1)} \dots \bar{\beta}_{\sigma_4(k)})}. \quad (3.d)$$

Since $\partial\omega = 0$, $\omega_{\alpha_0 / \alpha_1} = \omega_{\alpha_1 / \alpha_0}$ and similarly we set

$$\begin{aligned} \omega_{(\alpha_0 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)} &= \omega_{\alpha_0 / \alpha_1 \dots \alpha_j \bar{\beta}_1 \dots \bar{\beta}_k}, \\ \bar{\omega}_{(\alpha_1 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)} &= \bar{\omega}_{\bar{\beta}_0 / \alpha_1 \dots \alpha_j \bar{\beta}_1 \dots \bar{\beta}_k}. \end{aligned} \quad (3.e)$$

If σ_5 is a permutation of $j + 1$ indices, σ_6 is a permutation of k indices, σ_7 is a permutation of j indices, and σ_8 is a permutation of $k + 1$ indices, then

$$\begin{aligned}\omega_{(\alpha_0 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)} &= \omega_{(\alpha_{\sigma_5(0)} \dots \alpha_{\sigma_5(j)}; \bar{\beta}_{\sigma_6(1)} \dots \bar{\beta}_{\sigma_6(k)}), \\ \bar{\omega}_{(\alpha_1 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)} &= \bar{\omega}_{(\alpha_{\sigma_7(1)} \dots \alpha_{\sigma_7(j)}; \bar{\beta}_{\sigma_8(0)} \dots \bar{\beta}_{\sigma_8(k)}).\end{aligned}\tag{3.f}$$

Lemma 2 of [10] yields the following result.

Lemma 3.1. *Let $\mathcal{M} = (M, g, J, \omega, E, h)$. Fix a point $z_0 \in M$ and a positive integer N . There is a holomorphic coordinate system $\vec{z} = (z^1, \dots, z^m)$ centered at z_0 and a holomorphic frame \vec{e} for E defined near z_0 so that*

1. $g_{\alpha\bar{\beta}}(z_0) = \delta_{\alpha\beta}$ and $h_{p\bar{q}}(z_0) = \delta_{p,q}$.
2. $g_{(\alpha_0 \dots \alpha_j; \bar{\beta}_0)}(z_0) = g_{(\alpha_0; \bar{\beta}_0 \dots \bar{\beta}_k)}(z_0) = 0$ for $1 \leq j, k \leq N$.
3. $h_{(p\bar{q}; \alpha_1 \dots \alpha_j)}(z_0) = h_{(p\bar{q}; \bar{\beta}_1 \dots \bar{\beta}_k)}(z_0) = 0$ for $1 \leq j, k \leq N$.

In other words, these variables vanish at the basepoint if either there are no holomorphic or there are no anti-holomorphic derivatives. With these normalizations, the variables $g_{(\cdot, \cdot)}$, $h_{(p\bar{q}; \cdot, \cdot)}$, $\omega_{(\cdot, \cdot)}$, and $\bar{\omega}_{(\cdot, \cdot)}$ are tensorial; we have reduced the structure group to the unitary groups $U(\mathfrak{m})$ and $U(\dim(E))$ modulo transformations of order $O(|z|^{N+1})$.

4. Spaces of invariants

We must be rather precise in what is meant by a ‘‘local invariant’’ or a ‘‘local formula’’. We do this as follows.

4.1. The algebra \mathfrak{A}

We introduce the polynomial algebra \mathfrak{A}_m in the variables of Equations (3.a), (3.c), and (3.e) and impose the relations of Equations (3.b), (3.d), (3.f), and Lemma 3.1:

$$\begin{aligned}\mathfrak{A}_m &:= \mathbb{C}[g_{(\alpha_0 \dots \alpha_{j_1}; \bar{\beta}_0 \dots \bar{\beta}_{k_1})}, h_{(p\bar{q}; \alpha_1 \dots \alpha_{j_2}; \bar{\beta}_1 \dots \bar{\beta}_{k_2})}, \omega_{(\alpha_0 \dots \alpha_{j_3}; \bar{\beta}_1 \dots \bar{\beta}_{k_3})}, \\ &\quad \bar{\omega}_{(\alpha_1 \dots \alpha_{j_4}; \bar{\beta}_0 \dots \bar{\beta}_{k_4})}] \text{ for } j_1 \geq 1, k_1 \geq 1, j_2 \geq 1, k_2 \geq 1, \\ &\quad j_3 \geq 0, k_3 \geq 0, j_4 \geq 0, \text{ and } k_4 \geq 0.\end{aligned}$$

If $k_3 = 0$, there are no anti-holomorphic derivatives of ω and if $j_4 = 0$, there are no anti-holomorphic derivatives of $\bar{\omega}$. We introduce the complex dimension m into the notation as it plays an important role; we suppress the fiber dimension of E in the interests of notational simplicity.

If $P \in \mathfrak{A}_m$ and if A is a monomial, we let $c(A, P)$ be the coefficient of A in P and express $P = \sum_A c(A, P)A$. We say A is a monomial of P or that A appears in P if $c(A, P) \neq 0$. The maps $P \rightarrow c(A, P)$ are linear maps from \mathfrak{A}_m to \mathbb{C} .

4.2. The weight

To count the number of derivatives of g and h , we set:

$$\text{weight}\{g_{(\alpha_0\dots\alpha_j;\bar{\beta}_0\dots\bar{\beta}_k)}\} = \text{weight}\{h_{(p\bar{q};\alpha_1\dots\alpha_j;\bar{\beta}_1\dots\bar{\beta}_k)}\} := j + k.$$

Since the 1-forms ω and $\bar{\omega}$ appear as 0th perturbations of $\bar{\partial}$ and the adjoint $\bar{\partial}^*$, we give ω and $\bar{\omega}$ weight 1 and define

$$\text{weight}\{\omega_{(\alpha_0\dots\alpha_j;\bar{\beta}_1\dots\bar{\beta}_k)}\} = \text{weight}\{\bar{\omega}_{(\alpha_1\dots\alpha_j;\bar{\beta}_0\dots\bar{\beta}_k)}\} := 1 + j + k.$$

Distinguish the variables of weight 1 and set

$$\Xi_{(\alpha_1\dots\alpha_e;\bar{\beta}_1\dots\bar{\beta}_f)} := \omega_{\alpha_1} \dots \omega_{\alpha_e} \bar{\omega}_{\bar{\beta}_1} \dots \bar{\omega}_{\bar{\beta}_f}$$

where there are no holomorphic indices if $e = 0$, no anti-holomorphic indices if $f = 0$, and $\Xi = 1$ if $e = f = 0$; we set $\text{weight}\{\Xi_{(\alpha_1\dots\alpha_e;\bar{\beta}_1\dots\bar{\beta}_f)}\} := e + f$.

4.3. Monomials

Let U_* and \bar{V}_* be (possibly empty) collections of holomorphic and anti-holomorphic indices, respectively. If A is a monomial, we express

$$\begin{aligned} A &= g_{(U_1;\bar{V}_1)} \cdots g_{(U_a;\bar{V}_a)} h_{(p_1\bar{q}_1;U_{a+1};\bar{V}_{a+1})} \cdots h_{(p_b\bar{q}_b;U_{a+b};\bar{V}_{a+b})} \\ &\quad \cdot \omega_{(U_{a+b+1};\bar{V}_{a+b+1})} \cdots \omega_{(U_{a+b+c};\bar{V}_{a+b+c})} \bar{\omega}_{(U_{a+b+c+1};\bar{V}_{a+b+c+1})} \cdots \\ &\quad \cdot \bar{\omega}_{(U_{a+b+c+d};\bar{V}_{a+b+c+d})} \Xi_{(U_{a+b+c+d+1};\bar{V}_{a+b+c+d+1})}. \end{aligned} \quad (4.a)$$

In this expression, the variables in g_* , h_* , ω_* , and $\bar{\omega}_*$ have weight at least 2; we distinguish the variables of weight 1 separately in Ξ ; since we have imposed the relations of Lemma 3.1, all the variables defining \mathfrak{A}_m have positive weight.

We extend the notion of weight to be the sum of the weights of the variables comprising A . If A has the form given in Equation (4.a), then

$$\begin{aligned} \text{weight}(A) &= \sum_{i=1}^a \text{weight}\{g_{(U_i;\bar{V}_i)}\} + \sum_{i=a+1}^{a+b} \text{weight}\{h_{(p_i\bar{q}_i;U_{a+i};\bar{V}_{a+i})}\} \\ &\quad + \sum_{i=a+b+1}^{a+b+c} \text{weight}\{\omega_{(U_i;\bar{V}_i)}\} + \sum_{i=a+b+c+1}^{a+b+c+d} \text{weight}\{\bar{\omega}_{(U_i;\bar{V}_i)}\} + e + f. \end{aligned}$$

We say that a polynomial P is *homogeneous of weight $2n$* if all the monomials of P have weight $2n$.

4.4. The length

If A has the form given in Equation (4.a), we define the length $\ell(A)$ by setting

$$\ell(A) := \left\{ \begin{array}{ll} a + b + c + d & \text{if } \Xi = 1 \\ a + b + c + d + 1 & \text{if } \Xi \neq 1 \end{array} \right\}.$$

4.5. Local invariants

If \vec{z} is a normalized coordinate system on M , if \vec{s} is a normalized local holomorphic frame for E , and if $P \in \mathfrak{A}_m$, we evaluate $P(\mathcal{M})(z_0)(\vec{z}, \vec{s})$ in the obvious fashion. If $P(\mathcal{M})(z_0) := P(\mathcal{M})(z_0)(\vec{z}, \vec{s})$ is independent of the particular normalized coordinate system \vec{z} and normalized frame \vec{s} for any \mathcal{M} and any z_0 , then we shall say that P is *invariant*. The scalar curvature τ and the heat trace asymptotics $a_{m,2n}(\mathcal{E}_{\text{Dol}})$ are invariant.

Definition 4.1. Let $\mathcal{P}_{m,2n}$ be the subspace of \mathfrak{A}_m of invariant polynomials which are homogeneous of weight $2n$; there are no invariant polynomials of odd weight.

Example 4.2. The scalar curvature τ is an element of $\mathcal{P}_{m,2}$ since τ is linear in the 2-jets of the metric and quadratic in the 1-jets of the metric with coefficients which are smooth functions of the metric tensor.

The following observation follows from the explicit combinatorial algorithm given by Seeley [23] for computing the heat trace invariants.

Lemma 4.3. $a_{m,2n}(\mathcal{E}_{\text{Dol}}) \in \mathcal{P}_{m,2n}$.

5. The restriction map

5.1. The degree

Let \deg_α and $\deg_{\bar{\beta}}$ be the total number of times the index α or $\bar{\beta}$ appears in one of the variables comprising \mathfrak{A} :

$$\begin{aligned} \deg_\alpha \{g_{(\alpha_0 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)}\} &= \sum_{\nu=0}^j \delta_{\alpha \alpha_\nu}, & \deg_{\bar{\beta}} \{g_{(\alpha_0 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)}\} &= \sum_{\nu=0}^k \delta_{\bar{\beta} \bar{\beta}_\nu}, \\ \deg_\alpha \{h_{(p\bar{q}; \alpha_1 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)}\} &= \sum_{\nu=1}^j \delta_{\alpha \alpha_\nu}, & \deg_{\bar{\beta}} \{h_{(p\bar{q}; \alpha_1 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)}\} &= \sum_{\nu=1}^k \delta_{\bar{\beta} \bar{\beta}_\nu}, \\ \deg_\alpha \{\omega_{(\alpha_0 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)}\} &= \sum_{\nu=0}^j \delta_{\alpha \alpha_\nu}, & \deg_{\bar{\beta}} \{\omega_{(\alpha_0 \dots \alpha_j; \bar{\beta}_1 \dots \bar{\beta}_k)}\} &= \sum_{\nu=1}^k \delta_{\bar{\beta} \bar{\beta}_\nu}, \\ \deg_\alpha \{\bar{\omega}_{(\alpha_1 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)}\} &= \sum_{\nu=1}^j \delta_{\alpha \alpha_\nu}, & \deg_{\bar{\beta}} \{\bar{\omega}_{(\alpha_1 \dots \alpha_j; \bar{\beta}_0 \dots \bar{\beta}_k)}\} &= \sum_{\nu=0}^k \delta_{\bar{\beta} \bar{\beta}_\nu}, \\ \deg_\alpha \{\Xi_{(\alpha_1 \dots \alpha_e; \bar{\beta}_1 \dots \bar{\beta}_f)}\} &= \sum_{\nu=1}^e \delta_{\alpha \alpha_\nu}, & \deg_{\bar{\beta}} \{\Xi_{(\alpha_1 \dots \alpha_e; \bar{\beta}_1 \dots \bar{\beta}_f)}\} &= \sum_{\nu=1}^f \delta_{\bar{\beta} \bar{\beta}_\nu}. \end{aligned}$$

As with the weight, we extend the degree by summing over the variables comprising A . If A has the form given in Equation (4.a), then

$$\begin{aligned} \deg_\star(A) &= \sum_{i=1}^a \deg_\star\{g_{(U_i; \bar{V}_i)}\} + \sum_{i=a+1}^{a+b} \deg_\star\{h_{(p_i; \bar{q}_i; U_{a+b}; \bar{V}_{a+b})}\} \\ &+ \sum_{i=a+b+1}^{a+b+c} \deg_\star\{\omega_{(U_i; \bar{V}_i)}\} + \sum_{i=a+b+c+1}^{a+b+c+d} \deg_\star\{\bar{\omega}_{(U_i; \bar{V}_i)}\} \\ &+ \deg_\star\{\Xi_{(U_{a+b+c+d+1}; \bar{V}_{a+b+c+d+1})}\}. \end{aligned}$$

If an index does not appear in a monomial, we set the degree to zero.

5.2. Product with a flat torus

Let \mathbb{T} be the flat 2-dimensional torus with E trivial and $\omega = 0$. If \mathcal{N} has complex dimension $\mathfrak{m}-1$, we set $\mathcal{M} = \mathcal{N} \times \mathbb{T}$. If P is an invariant local formula in complex dimension \mathfrak{m} , then the natural association $\mathcal{N} \rightarrow \mathcal{M}$ defines dually an invariant local formula $r(P)$ in dimension $\mathfrak{m}-1$ so that

$$r(P)(\mathcal{N})(z_1) = P(\mathcal{N} \times \mathbb{T})(z_1, z_2);$$

the point $z_2 \in \mathbb{T}$ that is chosen is irrelevant since \mathbb{T} is homogeneous. Restriction defines a linear map

$$r : \mathcal{P}_{\mathfrak{m}, 2n} \rightarrow \mathcal{P}_{\mathfrak{m}-1, 2n}.$$

Example 5.1. The scalar curvature in dimension $2\mathfrak{m}$ is defined by summing over repeated indices relative to a local orthonormal frame

$$\tau = \sum_{i,j=1}^{2\mathfrak{m}} R_{ijji}.$$

The restriction $r(\tau)$ is defined by restricting the range of summation to lie over $1 \leq i, j \leq 2\mathfrak{m}-2$.

5.3. Algebraic formulation

The restriction map can be defined algebraically. Let $P \in \mathcal{P}_{\mathfrak{m}, 2n}$. It is immediate from the definition that $r(P) = 0$ if and only if $\deg_{\mathfrak{m}}(A) \geq 1$ for every monomial A of P . Since we can permute the indices, if P is invariant, we have that:

Lemma 5.2. *If $P \in \mathcal{P}_{\mathfrak{m}, 2n}$, then $r(P) = 0$ if and only if $\deg_\alpha(A) > 0$ for every monomial A of P and every index $1 \leq \alpha \leq \mathfrak{m}$.*

It is immediate that the heat trace invariants of the Dolbeault complex vanish on \mathbb{T} . Consequently, Equation (2.c) yields

Lemma 5.3. $r(a_{\mathfrak{m}, 2n}(\mathcal{E}_{\text{Dol}})) = 0$.

6. Invariance Theory

The coordinate and frame normalizations of Lemma 3.1 are invariant under the action of the unitary groups $U(\mathfrak{m})$ and $U(\dim(E))$; although invariance under $U(\dim(E))$ will play no direct role in our analysis, it was central to the proof of Theorem 1.3. We exploit unitary invariance to show

Lemma 6.1. *Let $0 \neq P \in \mathcal{P}_{\mathfrak{m}, 2n}$. If A is a monomial of P , then*

$$\deg_\alpha(A) = \deg_{\bar{\alpha}}(A) \text{ for any } \alpha.$$

Proof. Since P is invariant, we can permute the indices. Thus we may suppose that $\alpha = 1$. Make a unitary change of coordinates to define a new holomorphic coordinate system \vec{w} so that

$$\frac{\partial}{\partial w^\alpha} = \begin{cases} e^{\sqrt{-1}\theta} \frac{\partial}{\partial z^1} & \text{if } \alpha = 1 \\ \frac{\partial}{\partial z^\alpha} & \text{if } \alpha > 1 \end{cases} \quad \text{and} \quad \frac{\partial}{\partial \bar{w}^\beta} = \begin{cases} e^{-\sqrt{-1}\theta} \frac{\partial}{\partial \bar{z}^1} & \text{if } \beta = 1 \\ \frac{\partial}{\partial \bar{z}^\beta} & \text{if } \beta > 1 \end{cases}.$$

To compute P^w , we formally replace the index 1 by $e^{\sqrt{-1}\theta} \cdot 1$ and $\bar{1}$ by $e^{-\sqrt{-1}\theta} \cdot \bar{1}$, and we leave the remaining indices unchanged to expand each monomial of P multi-linearly. Thus $A^w = e^{\sqrt{-1}(\deg_1 A - \deg_{\bar{1}} A)\theta} A$ so

$$P = \sum_A c(A, P)A \text{ and } P^w = \sum_A e^{\sqrt{-1}(\deg_1(A) - \deg_{\bar{1}}(A))\theta} c(A, P)A.$$

Since P is invariant, $P^w = P$ and thus $\deg_1(A) = \deg_{\bar{1}}(A)$ if $c(A, P) \neq 0$. \square

Definition 6.2. Let $|\xi|^2 + |\eta|^2 = 1$. Make a unitary change of coordinates to define a new holomorphic coordinate system \vec{w} so that

$$\frac{\partial}{\partial w^\alpha} = \begin{cases} \xi \frac{\partial}{\partial z^1} + \eta \frac{\partial}{\partial z^2} & \text{if } \alpha = 1 \\ -\bar{\eta} \frac{\partial}{\partial z^1} + \bar{\xi} \frac{\partial}{\partial z^2} & \text{if } \alpha = 2 \\ \frac{\partial}{\partial z^\alpha} & \text{if } \alpha > 2 \end{cases},$$

$$\frac{\partial}{\partial \bar{w}^\beta} = \begin{cases} \bar{\xi} \frac{\partial}{\partial \bar{z}^1} + \bar{\eta} \frac{\partial}{\partial \bar{z}^2} & \text{if } \beta = 1 \\ -\eta \frac{\partial}{\partial \bar{z}^1} + \xi \frac{\partial}{\partial \bar{z}^2} & \text{if } \beta = 2 \\ \frac{\partial}{\partial \bar{z}^\beta} & \text{if } \beta > 2 \end{cases}.$$

If P is a polynomial, let P^w be the expression of P in this new coordinate system. We formally replace each index

$$1 \rightarrow \xi \cdot 1 + \eta \cdot 2, \quad 2 \rightarrow -\bar{\eta} \cdot 1 + \bar{\xi} \cdot 2, \quad \bar{1} \rightarrow \bar{\xi} \cdot \bar{1} + \bar{\eta} \cdot \bar{2}, \quad \bar{2} \rightarrow -\eta \bar{1} + \xi \cdot \bar{2}$$

and leave the remaining indices unchanged. We then expand multilinearly to compute P^w . Of course, the use of the indices ‘1’ and ‘2’ is intended to be illustrative only, any pair of distinct indices would suffice.

Definition 6.3. If B is a monomial, let $\mathcal{B}(B)$ be the set of all monomials A so that changing a single index $1 \rightarrow 2$ or $\bar{2} \rightarrow \bar{1}$ in A yields B ; alternatively, so that A arises by changing a single index $2 \rightarrow 1$ or $\bar{1} \rightarrow \bar{2}$ in B . Let P^w

be the expression of a polynomial P in the new coordinate system given in Definition 6.2 by taking $\xi = \cos(\phi)$ and $\eta = \sin(\phi)e^{\sqrt{-1}\theta}$.

$$\begin{aligned} P^w &= c_0(B, P) \cos(\phi)^{u-1} \sin(\phi) e^{\sqrt{-1}\theta} B + \text{other terms where} \\ u &:= \deg_1(B) + \deg_2(B) + \deg_{\bar{1}}(B) + \deg_{\bar{2}}(B) \text{ and} \\ c_0(B, P) &= \sum_{A \in \mathcal{B}(B)} \nu(A) c(A, P) \text{ where } \nu(A) \neq 0. \end{aligned} \quad (6.a)$$

Example 6.4. If $B = g_{(12; \bar{1}\bar{2})} g_{(12; \bar{1}\bar{1})}$, then $\mathcal{B}(B) = \{A_1, A_2, A_3, A_4\}$ where we have marked with \star the index $1 \rightarrow 2$ or $\bar{2} \rightarrow \bar{1}$ that was changed in A to create B ; we apply the symmetries of Equation (3.b) to obtain:

$$\begin{aligned} A_1 &= g_{(1^*1; \bar{1}\bar{2})} g_{(12; \bar{1}\bar{1})} \Rightarrow B = g_{(2^*1; \bar{1}\bar{2})} g_{(12; \bar{1}\bar{1})} \text{ by } 1^* \rightarrow 2^*, \\ A_1 &= g_{(11^*; \bar{1}\bar{2})} g_{(12; \bar{1}\bar{1})} \Rightarrow B = g_{(12^*; \bar{1}\bar{2})} g_{(12; \bar{1}\bar{1})} \text{ by } 1^* \rightarrow 2^*, \nu(A_1) = 2, \\ A_2 &= g_{(12; \bar{1}\bar{2})} g_{(1^*1; \bar{1}\bar{1})} \Rightarrow B = g_{(12; \bar{1}\bar{2})} g_{(2^*1; \bar{1}\bar{1})} \text{ by } 1^* \rightarrow 2^*, \\ A_2 &= g_{(12; \bar{1}\bar{2})} g_{(11^*; \bar{1}\bar{1})} \Rightarrow B = g_{(12; \bar{1}\bar{2})} g_{(12^*; \bar{1}\bar{1})} \text{ by } 1^* \rightarrow 2^*, \nu(A_2) = 2, \\ A_3 &= g_{(12; \bar{2}^*2)} g_{(12; \bar{1}\bar{1})} \Rightarrow B = g_{(12; \bar{1}^*\bar{2})} g_{(12; \bar{1}\bar{1})} \text{ by } \bar{2}^* \rightarrow \bar{1}^*, \\ A_3 &= g_{(12; \bar{2}\bar{2}^*)} g_{(12; \bar{1}\bar{1})} \Rightarrow B = g_{(12; \bar{2}\bar{1}^*)} g_{(12; \bar{1}\bar{1})} \text{ by } \bar{2}^* \rightarrow \bar{1}^*, \nu(A_3) = -2, \\ A_4 &= g_{(12; \bar{1}\bar{2})} g_{(12; \bar{1}\bar{2}^*)} \Rightarrow B = g_{(12; \bar{1}\bar{2})} g_{(12; \bar{1}\bar{1}^*)} \text{ by } \bar{2}^* \rightarrow \bar{1}^*, \nu(A_4) = -1, \\ u &= 8 \text{ and } c_0(B, P) = 2c(A_1, P) + 2c(A_2, P) - 2c(A_3, P) - c(A_4, P). \end{aligned}$$

Lemma 6.5. *Let $0 \neq P \in \mathcal{P}_{\mathfrak{m}, 2n}$. If B is any monomial, then either no monomial of $\mathcal{B}(B)$ appears in P or at least two monomials of $\mathcal{B}(B)$ appear in P .*

Proof. If $A \in \mathcal{B}(B)$, then $\deg_1(B) - \deg_{\bar{1}}(B) = \deg_1(A) - \deg_{\bar{1}}(A) - 1$. Thus if B is a monomial of P , no monomial of $\mathcal{B}(B)$ is a monomial of P by Lemma 6.1 and Lemma 6.5 follows. We therefore assume B is not a monomial of P and thus $c_0(B, P) = 0$. We use Equation (6.a). Since the multiplicities $\nu(A)$ are non-zero integers, if $c(A_1, P) \neq 0$ for some A_1 , there must exist at least another monomial $A_2 \in \mathcal{B}(B)$ to cancel off $\nu(A_1)c(A_1, P)$ in Equation (6.a). \square

We use Lemma 6.5 as to prove the following result.

Lemma 6.6. *Let $0 \neq P \in \ker(r) \cap \mathcal{P}_{\mathfrak{m}, 2n}$ where $n \leq \mathfrak{m}$. There exists a monomial A of P of the form given in Equation (4.a) so that $U_\nu = (\nu, \dots, \nu)$ for $\nu \leq n$. Furthermore, $\ell(A) \geq \mathfrak{m}$, and if $\ell(A) = \mathfrak{m}$, then none of the U_ν is empty.*

Proof. We shall apply Lemma 6.1 and Lemma 6.5.

Step 1: We concentrate on the collections $(U_\nu; \bar{V}_\nu)$ and suppress the particular variables $g, h, \omega, \bar{\omega}$ or Ξ in which they appear. Choose a monomial $A = (U_1; \bar{V}_1)A_0$ of P such that $\deg_1(U_1)$ is maximal. If $U_1 = (1 \dots 1)$, we proceed to Step 2. Let $U_1 = (1 \dots 1\alpha^* \dots)$ for $\alpha \neq 1$. By permuting the indices we may assume $\alpha = 2$. Set $B = (1 \dots 11^* \dots; \bar{V}_1)A_0$. Use Lemma 6.5 to choose a monomial $A_1 \in \mathcal{B}(B)$ of P different from A . Since $A_1 \neq A$, A_1 does not transform to B by changing an index of U_1 and thus A_1 has an index

collection \tilde{U}_1 with one more occurrence of the index ‘1’ which contradicts the maximality of A . This contradiction shows $U_1 = (1 \dots 1)$.

Step 2: Choose $A = (1 \dots 1; \bar{V}_1)(U_2; \bar{V}_2)A_0$ so the number of occurrences of the index 2 in U_2 is maximal. If $U_2 = (2 \dots 2)$ proceed to Step 3. Otherwise assume $U_2 = (2 \dots 2\alpha^* \dots)$ for $\alpha \neq 2$. Let $B = (1 \dots 1; \bar{V}_1)(2 \dots 22^* \dots; \bar{V}_2)A_0$ be obtained by changing the index α to the index 2. By Lemma 6.5 (where we replace the indices $(1, 2)$ by $(2, \alpha)$), we can choose $A_1 \in \mathcal{B}(B)$ to be a monomial of P different from A . Changing the index $\alpha \rightarrow 2$ does not affect $U_1 = (1 \dots 1)$. Since $A_1 \neq A$, it has an index collection $\tilde{U}_2 = (2 \dots 22 \dots)$ which contradicts the maximality of A . Thus we can choose $A = (1 \dots 1; \bar{V}_1)(2 \dots 2; \bar{V}_2)A_0$.

Step 3: We continue in this fashion to construct A with the desired form. The process stops when $\nu = \mathbf{m}$ or when $\nu = \ell(A)$. Since $\deg_\nu(A) \neq 0$ for $1 \leq \nu \leq \mathbf{m}$ by Lemma 5.2, we have $\ell(A) \geq \mathbf{m}$. If $\ell(A) = \mathbf{m}$, none of the U_ν could be empty or the index ν would not appear in A . \square

7. The proof of Theorem 1.4

The subalgebra of variables of weight 2 will play a distinguished role and we set

$$\mathfrak{B}_m = \mathbb{C}[g_{(\alpha_0\alpha_1; \bar{\beta}_0\bar{\beta}_1)}, h_{(p\bar{q}; \alpha_1; \bar{\beta}_1)}, \omega_{(\alpha_0; \bar{\beta}_1)}, \bar{\omega}_{(\alpha_1; \bar{\beta}_0)}].$$

By Lemma 5.3, $a_{m, 2n}(\mathcal{E}_{\text{Dol}}) \in \ker(r)$. Consequently, the fact that $a_{m, 2n}(\mathcal{E}_{\text{Dol}}) = 0$ for $2n < m$ will follow from the following result.

Lemma 7.1. *Let $2n \leq m$.*

1. $\ker(r) \cap \mathcal{P}_{m, 2n} = \{0\}$ if $2n < m$.
2. $\ker(r) \cap \mathcal{P}_{m, m} \subset \mathfrak{B}_m \oplus \bigoplus_{\alpha_1 \bar{\beta}_1} \omega_{\alpha_1} \bar{\omega}_{\beta_1} \mathfrak{B}_m$.

Proof. Let $0 \neq P \in \ker(r) \cap \mathcal{P}_{m, 2n}$ where $2n \leq m$. Choose a monomial A of P satisfying the conclusions of Lemma 6.6. We then have $\ell(A) \geq \mathbf{m}$ and, by Lemma 5.2, $\deg_\alpha(A) \neq 0$ for all α . We examine $\Xi = \Xi_{(U; \bar{V})}$ where $|U| = e$ and $|V| = f$; the role of the variables of weight 1 is crucial.

Case 1. Suppose $e = 0$. Then $\deg_\alpha(A) = 0$ for $\alpha > a + b + c + d$. Since $r(P) = 0$, $\deg_{\mathbf{m}}(A) \neq 0$. Thus $a + b + c + d \geq \mathbf{m}$. The normalizations of Lemma 3.1 show

$$m \geq 2n = \text{weight}(A) \geq 2a + 2b + 2e + 2d + f \geq 2a + 2b + 2c + 2d \geq 2\mathbf{m} = m.$$

Thus equality holds. This implies $2n = m$, $f = 0$, and P is a polynomial in the variables of weight 2.

Case 2. Suppose $f = 0$. A similar argument using the anti-holomorphic indices shows $2n = m$, $e = 0$, and P is a polynomial in the variables of weight 2.

Case 3: Suppose $e > 0$ and $f > 0$. If $a + b + c + d + 1 < \mathbf{m}$, we could choose A so that $\deg_{\mathbf{m}} A = 0$ which is false. Consequently, $a + b + c + d + 1 \geq \mathbf{m}$. We

estimate

$$\begin{aligned} m &\geq 2n = \text{weight}(A) \geq 2a + 2b + 2c + 2d + e + f \\ &\geq 2a + 2b + 2c + 2d + 2 \geq 2\mathbf{m} = m. \end{aligned}$$

Thus all the inequalities are in fact equalities. This implies $2n = m$, $e = f = 1$, and the remaining variables comprising A all have weight 2.

If a $\bar{\omega}_*$ variable of weight 2 does not contain a holomorphic index, Lemma 6.6 shows $\deg_{\bar{\nu}}(A) = 0$ for some holomorphic index which is false since $r(A) = 0$. Since Lemma 6.6 also holds for the anti-holomorphic indices, if any of the ω_* variables of weight 2 does not contain an anti-holomorphic index, then $\deg_{\bar{\nu}}(A) = 0$ for some anti-holomorphic index which is false. Thus all the variables of weight 2 which divide A belong to \mathfrak{B}_m and Assertion (2) holds. \square

Let $\Re(\cdot)$ and $\Im(\cdot)$ be the real and imaginary parts of a 1-form.

Lemma 7.2. *If \mathcal{M} is a Riemann surface, then $a_{2,2}(\mathcal{M}) = \star \left(\text{Todd}_2(\mathcal{M}) + \frac{d\Im\omega}{\pi} \right)$.*

Proof. Álvarez López and Gilkey [1] showed that $a_{2,2}(\mathcal{M}) = \frac{\tau}{8\pi} - \frac{1}{\pi}\delta(\Re(\omega))$. We have $\frac{\tau}{8\pi} = \star \text{Todd}_2(\mathcal{M})$ by Hirzebruch [15]. Let

$$\begin{aligned} \omega &= (u + \sqrt{-1}v)(dx + \sqrt{-1}dy) = (udx - vdy) + \sqrt{-1}(udy + vdx); \\ -\delta(\Re(\omega)) &= -\delta(udx - vdy) = u_x - v_y, \text{ and} \\ d(\Im(\omega)) &= d(udy + vdx) = (u_x - v_y)dx \wedge dy. \end{aligned} \quad \square$$

We must improve Lemma 6.6.

Lemma 7.3. *Let $0 \neq P \in \ker(r) \cap \mathcal{P}_{\mathbf{m},m}$. There exists a monomial A of P so*

$$\begin{aligned} A &= g_{(11; \bar{\beta}_1 \bar{\beta}_2)} \cdots g_{(aa; \bar{\beta}_{2a-1} \bar{\beta}_{2a})} h_{(p_{a+1} \bar{q}_{a+1}; a+1; \overline{a+1})} \cdots h_{(p_b \bar{q}_b; a+b; \overline{a+b})} \\ &\quad \omega_{(a+b+1; \overline{a+b+1})} \cdots \omega_{(a+b+c; \overline{a+b+c})} \bar{\omega}_{(a+b+c+1; \overline{a+b+c+1})} \\ &\quad \cdots \bar{\omega}_{(a+b+c+d; \overline{a+b+c+d})} \Xi \\ &\text{where } \Xi = 1 \text{ or } \Xi = \Xi_{(\mathbf{m}; \bar{\mathbf{m}})} \text{ and } \beta_\nu \leq a \text{ for } 1 \leq \nu \leq 2a. \end{aligned}$$

Proof. Apply Lemma 6.6 and Lemma 7.1 to choose a monomial A of P of the form

$$\begin{aligned} A &= g_{(11; \bar{\beta}_1 \bar{\beta}_2)} \cdots g_{(aa; \bar{\beta}_{2a-1} \bar{\beta}_{2a})} h_{(p_{a+1} \bar{q}_{a+1}; a+1; \bar{\beta}_{a+1})} \cdots h_{(p_b \bar{q}_b; a+b; \bar{\beta}_{a+b})} \\ &\quad \omega_{(a+b+1; \bar{\beta}_{a+b+1})} \cdots \omega_{(a+b+c; \bar{\beta}_{a+b+c})} \bar{\omega}_{(a+b+c+1; \bar{\beta}_{a+b+c+1})} \\ &\quad \cdots \bar{\omega}_{(a+b+c+d; \bar{\beta}_{a+b+c+d})} \Xi \text{ where } \Xi = 1 \text{ or } \Xi_{(\mathbf{m}; \bar{\beta}_m)}. \end{aligned} \quad (7.a)$$

We use Lemma 6.1 and Equation (7.a) to see

$$a < \beta \iff \deg_\beta(A) = 1 \iff \deg_{\bar{\beta}}(A) = 1.$$

We say that an anti-holomorphic index $\bar{\beta}$ touches an anti-holomorphic index $\bar{\gamma}$ in A if A is divisible by $g_{(\alpha\alpha; \bar{\beta}\bar{\gamma})}$ for some α ; we say that $\bar{\beta}$ touches itself in A if we can take $\bar{\beta} = \bar{\gamma}$.

Let $a < \beta$ so $\deg_{\beta}(A) = \deg_{\bar{\beta}}(A) = 1$. Let $\gamma \neq \beta$. We construct a monomial B by replacing $\bar{\beta}$ by $\bar{\gamma}$; $\deg_{\bar{\beta}} B = 0$ and A is obtained from B by changing $\bar{\gamma} \rightarrow \bar{\beta}$. We apply Lemma 6.5 to find $A_1 \in \mathcal{B}(B)$ which appears in P with $A \neq A_1$. Since $\deg_{\bar{\beta}} B = 0$ and $\deg_{\bar{\beta}}(A_1) \neq 0$, A_1 is obtained from B by changing $\bar{\gamma} \rightarrow \bar{\beta}$ or, equivalently, A_1 arises from A by interchanging a $\bar{\beta}$ with a $\bar{\gamma}$ index. Thus, in particular, since $A_1 \neq A$, two anti-holomorphic indices of degree 1 in A can not touch in A .

Choose A of the form given in Equation (7.a) so the number of anti-holomorphic indices which touch themselves in A is maximal. Suppose $\deg_{\bar{\beta}}(A) = 1$ and $\bar{\beta}$ touches another anti-holomorphic index $\bar{\gamma}$ in A . Then there is a monomial A_1 of P different from A defined by interchanging $\bar{\beta}$ and $\bar{\gamma}$. This is not possible since $\bar{\gamma}$ would touch itself in A_1 which contradicts the maximality of A . Thus $\{\bar{V}_1, \dots, \bar{V}_a\}$ consists solely of anti-holomorphic indices of degree 2 in A and hence must comprise all the anti-holomorphic indices of degree 2 in A .

We say that a holomorphic index α touches an anti-holomorphic index $\bar{\beta}$ in A if A is divisible by $\omega_{(\alpha; \bar{\beta})}$, by $\bar{\omega}_{(\alpha; \bar{\beta})}$ or by $\Xi(\alpha; \bar{\beta})$. If A is as constructed above, then every holomorphic index α of degree 1 touches an anti-holomorphic index $\bar{\beta}$ of degree 1 in A . Among all the monomials A constructed above, choose A so the number of holomorphic indices α which touch $\bar{\alpha}$ in A is maximal. Let $a + b < \alpha$. If α touches $\bar{\beta}$ in A with $\alpha \neq \beta$, we can interchange $\bar{\beta}$ and $\bar{\alpha}$ to construct a monomial A_1 of P where there is one more holomorphic - anti-holomorphic touching which is impossible. Therefore A has the form given in the Lemma. \square

We use Lemma 7.1 to improve Equation (2.c). Let $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. Since $a_{m_i, 2n_i} = 0$ for $2n_i < m_i$, taking $2n = m$ in Equation (2.c) yields

$$a_{m,m}(\mathcal{E}_{\text{Dol}}(\mathcal{M}))(z_1, z_2) = a_{m_1, m_1}(\mathcal{E}_{\text{Dol}}(\mathcal{M}_1))(z_1) \cdot a_{m_2, m_2}(\mathcal{E}_{\text{Dol}}(\mathcal{M}_2))(z_2). \quad (7.b)$$

Let $\mathcal{N}_{\mathbf{k}}(0) = (N^{2\mathbf{k}}, J_N, g_N, E_N, h_N, 0)$ be a structure of complex dimension \mathbf{k} with trivial twisting $(1,0)$ -form $\omega_0 = 0$. Let

$$\mathbb{T}^2(\omega_i) = (S^1 \times S^1, dx^2 + dy^2, J_2, \mathbf{1}, h_0, \omega_i)$$

be the torus $S^1 \times S^1$ with the flat metric $dx^2 + dy^2$, usual complex structure $J_2 : \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial y}$ and $J_2 : \frac{\partial}{\partial y} \rightarrow -\frac{\partial}{\partial x}$, flat line bundle $\mathbf{1} = (S^1 \times S^1) \times \mathbb{C}$, trivial Hermitian metric h_0 , and (possibly) non-trivial twisting $(1,0)$ -form ω_i with $\partial\omega_i = 0$. Let $\vec{\omega} = (\omega_1, \dots, \omega_{\mathbf{m}-\mathbf{k}})$. Set

$$\mathcal{M}(\mathbf{k}; \vec{\omega}) := \mathcal{N}_{\mathbf{k}}(0) \times \mathbb{T}(\omega_1) \times \dots \times \mathbb{T}(\omega_{\mathbf{m}-\mathbf{k}}).$$

By Equation (7.b),

$$\begin{aligned} & a_{m,m}(\mathcal{E}_{\text{Dol}}(\mathcal{M}(\mathbf{k}; \vec{\omega}))) \\ &= a_{2\mathbf{k}, 2\mathbf{k}}(\mathcal{E}_{\text{Dol}}(\mathcal{N}_{\mathbf{k}})) a_{2,2}(\mathcal{E}_{\text{Dol}}(\mathbb{T}^2(\omega_1))) \dots a_{2,2}(\mathcal{E}_{\text{Dol}}(\mathbb{T}^2(\omega_{\mathbf{m}-\mathbf{k}}))). \end{aligned}$$

By Lemma 7.3, if $0 \neq P \in \ker(r) \cap \mathcal{P}_{\mathbf{m}, m}$, then $P(\mathcal{M}(\mathbf{k}; \vec{\omega})) \neq 0$ for some \mathbf{k} and some $\vec{\omega}$. On the other hand, we may use Equation (7.b), Theorem 1.3,

and Lemma 7.2 to see that

$$\{a_{m,m} - \{\text{Td} \wedge \text{ch} \wedge \Theta\}_m\}(\mathcal{M}(\mathbf{k}; \vec{\omega})) = 0 \text{ for all } \mathbf{k} \text{ and } \vec{\omega}.$$

Theorem 1.4 now follows. \square

7.1. The kernel of r

It seems useful to identify $\ker(r : \mathcal{P}_{m,2m} \rightarrow \mathcal{P}_{m-1,2m})$ in a bit more detail. Let ch_k be the k^{th} component of the Chern character (see [15]). We decompose the graded ring of characteristic forms into homogeneous components

$$\mathcal{C}_m := \mathbb{C}[\text{ch}_k(TM, J, g), \text{ch}_k(E, h)] = \bigoplus_k \mathcal{C}_m^{2k}.$$

We also consider the graded ring

$$\mathcal{D}_m := \mathbb{C}[\text{ch}_k(TM, J, g), \text{ch}_k(E, h), d\omega, d\bar{\omega}, \omega, \bar{\omega}] = \bigoplus_k \mathcal{D}_m^{2k}.$$

Let $\mathcal{P}_{m,2m}^{g,E}$ be the subspace of invariants which are independent of ω .

Lemma 7.4.

1. $\ker(r : \mathcal{P}_{m,m}^{g,E} \rightarrow \mathcal{P}_{m-1,m}^{g,E}) = \star \mathcal{C}_m^m$.
2. $\ker(r : \mathcal{P}_{m,m} \rightarrow \mathcal{P}_{m-1,m}) = \star \mathcal{D}_m^m$.

Proof. Assertion (1) follows from Theorem 1 of Gilkey [10]. Assertion (2) is a scholium to the arguments we have given above. We use Assertion (1) to control the metric terms. We express $P \in \ker(r)$ on $\mathcal{M}(\mathbf{k}; \vec{\omega})$ as a metric invariant on $\mathcal{M}_{\mathbf{k}}$ times invariants on the $\mathbb{T}^2(\omega_i)$. The metric invariant is itself in the kernel of r and hence is a characteristic class; the remaining invariants only involve Θ . Thus there is an element of \mathcal{D} which hits this invariant and hence by Lemma 7.3, P can be decomposed appropriately. \square

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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J. Álvarez López*

Department of Mathematics, Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain
e-mail: jesus.alvarez@usc.es

P. Gilkey

Mathematics Department, University of Oregon, Eugene OR 97403-1222, USA
e-mail: gilkey@uoregon.edu