A local regularity result for Neumann parabolic problems with nonsmooth data

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Abstract

In this work we analyze the relations between two different concepts of solution of the Neumann problem for a second order parabolic equation: the usual notions of weak solution and that of transposition solution, which allows well-posedness of problems with measure data. We give a regularity result for the transposition solution and we prove that, under smoothness assumptions for the principal part of the operator, the local regularity of the transposition solution is the same than that of the usual weak solution. As an interesting particular case, we present a rigorous proof of local continuity of the solution for a convection-diffusion problem with pointwise source term.

Keywords: Weak solution, Transposition solution, Convection-diffusion, Measure data, Neumann boundary condition

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1. Introduction

1.1. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary Γ. For $T > 0$, we denote $Q_T = \Omega \times (0,T)$, and $\Sigma_T = \Gamma \times (0,T)$.

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We consider the general second order parabolic problem with Neumann boundary condition:

$$\left.\begin{aligned} \frac{\partial y}{\partial t} + Ly = f \qquad & \text{in } Q_T, \\ \frac{\partial y}{\partial \nu_L} = g \qquad & \text{on } \Sigma_T, \\ y(x,0) = y_0 \qquad & \text{in } \Omega, \end{aligned}\right\} \tag{1}$$

where

$$Ly = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\{\sum_{j=1}^{n} a_{ij}(x,t)\frac{\partial y}{\partial x_j} + a_i(x,t)y\} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial y}{\partial x_i} + c(x,t)y,$$

$$\frac{\partial y}{\partial \nu_L} = \sum_{i=1}^{n}\{\sum_{j=1}^{n} a_{ij}(x,t)\frac{\partial y}{\partial x_j} + a_i(x,t)y\}\nu_i(x),$$

for $\vec{\nu}(x) = (\nu_1(x), \ldots, \nu_n(x))$ the outward unit normal vector to Γ in the point $x \in \Gamma$. We assume the following standard hypotheses on the coefficients of the operator L:

- $a_{ij} \in L^\infty(Q_T), \quad 1 \leq i, j \leq n,$
- $a_i, b_i \in L^\infty(Q_T) \quad 1 \leq i \leq n,$
- $c \in L^\infty(Q_T),$
- $\exists\ \alpha > 0 \ / \ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } (x,t) \in Q_T.$

Finally we introduce the associated family of bilinear forms

$$a(t;.,.) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$$

defined by:

$$\begin{aligned} a(t;w,v) = & \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x,t)\frac{\partial w}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx + \int_\Omega \sum_{i=1}^{n} a_i(x,t)w\frac{\partial v}{\partial x_i}\,dx \\ & + \int_\Omega \sum_{i=1}^{n} b_i(x,t)\frac{\partial w}{\partial x_i}v\,dx + \int_\Omega c(x,t)wv\,dx, \end{aligned} \tag{2}$$

and the family of operators $A(t) \in \mathcal{L}(H^1(\Omega), [H^1(\Omega)]')$ defined by:

$$\langle A(t)w, v\rangle = a(t; w, v), \quad \forall\, w, v \in H^1(\Omega), \quad \text{a.e. } t \in (0, T).$$

We shall also use the notation $a(t; w, v)$ with the same meaning as in equation (2) when $w \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ with $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

All along the mathematical literature several concepts of weak solution for second order parabolic equations have been given (see, for instance, Ladyzenskaja *et al.* [19], Dautray-Lions [14] or Showalter [28]). The analysis of the Dirichlet problem with measure data has been extensively studied for a long time (with entropy and renormalized solutions being the two main approaches, as can be seen, among others, in Boccardo *et al.* [6], Li-Zhao [21], Blanchard-Porretta [5], Amann-Quittner [2], Chasseigne-Vazquez [11], Meskine [23], Droniou-Prignet [16], Petitta [25] or Dall'Aglio *et al.* [12]. Further relevant results - concerning transposition/duality solutions, Neumann problems with non-smooth data, and local regularity - can be found in the interesting works of Stampacchia [29], Dall'Aglio-Orsina [13], Porzio-Vespri [26], Andreu *et al.* [3], Leoni-Pellacci [20], Petitta [24], Boccardo *et al.* [7], and the references therein). However, the study of the Neumann problem has been usually unaddressed. In the present work we analyze two types of solution for Neumann parabolic problems, allowing measure data: the weak solution and the transposition solution. The main aim of this work is related to proving a new regularity result for the transposition solution, and demonstrating that, under suitable hypotheses, the local regularity of the transposition solution agrees with that of the weak solution.

We begin our paper (Subsections 1.1 and 1.2) by presenting the more usual definitions of weak solution, depending on the smoothness of data, and their main properties and connections. In Section 2 we introduce the more general concept of transposition solution for the case of a parabolic problem with measure data, we recall the main existence and uniqueness results for this kind of solution and prove rigorously that, with the usual definitions of weak solution, any weak solution is a transposition solution. In Section 3 we use interpolation to obtain a regularity result for the transposition solution of (1) when the Neumann boundary condition is homogeneous and the source term $f \in L^q(Q_T)$, $1 < q < 2$. Section 4 contains the main result of this note: roughly speaking, we prove that, if the principal part of operator L satisfy $a_{ij} \in L^\infty(0, T; W^{1,\infty}(\Omega))$, $\ 1 \le i, j \le n$, the local regularity of the transposition solution is the same that the usual weak solution of a similar problem with second member in $L^{\tilde{r}}(0,T;L^{\tilde{q}}(\Omega))$, with $\tilde{r}\geq 2$ and $\tilde{q}\geq 2$. Then, we apply this result to a convection-diffusion problem in dimension $n=2$ with pointwise source terms and Neumann boundary conditions (motivated by a previous work of the authors [22]), attaining continuous solutions in closed sets outside the source points. Finally, in the Appendix, some auxiliary results are demonstrated.

1.2. Weak solution for "smooth" data

Let us assume that we are dealing with "smooth" data, i.e., $y_0\in L^2(\Omega)$, $f\in L^2(0,T;L^2(\Omega))$, and $g\in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$. We recall several well-known aspects about the solution of problem (1):

Definition 1. *We say that* y *is a* **usual weak solution** *of problem (1) if*

$$\left.\begin{array}{rl} y\in L^2(0,T;H^1(\Omega))\cap\mathcal{C}([0,T];L^2(\Omega)), & \\ \dfrac{d}{dt}(y(t),v)_{L^2(\Omega)}+a(t;y(t),v)=(f(t),v)_{L^2(\Omega)}+\langle g(t),v\rangle_\Gamma, & \\ \forall v\in H^1(\Omega), & \text{in } \mathcal{D}'(0,T),\\ y(0)=y_0 & \text{in } \Omega, \end{array}\right\} \tag{3}$$

where we have used the notation:

$$\langle g(t),v\rangle_\Gamma=\langle g(t),\gamma v\rangle_{H^{-\frac{1}{2}}(\Gamma),H^{\frac{1}{2}}(\Gamma)}$$

for $\gamma: H^1(\Omega)\to H^{\frac{1}{2}}(\Gamma)$ *the trace operator.*

Remark 1. *Previous definition is equivalent to looking for a solution* $y\in L^2(0,T;H^1(\Omega))\cap\mathcal{C}([0,T];L^2(\Omega))$ *satisfying the partial differential equation in the sense of distributions in* Q_T*, the initial condition in the sense of* $L^2(\Omega)$*, and the boundary condition in the sense of* $H^{-\frac{1}{2}}(\Sigma_T)$*. (See [9]).*

The following results are also well known (a detailed proof can be found, for instance, in the monograph of Casas [9]):

Result 1. *The problem (1) has a unique usual weak solution* y*. Moreover, the following estimate is verified:*

$$\begin{aligned}&\|y\|_{L^2(0,T;H^1(\Omega))}+\|y\|_{\mathcal{C}([0,T];L^2(\Omega))}\\ &\qquad\leq C\{\|y_0\|_{L^2(\Omega)}+\|f\|_{L^2(0,T;L^2(\Omega))}+\|g\|_{L^2(0,T;H^{-1/2}(\Gamma))}\}\end{aligned} \tag{4}$$

Let $F_{[f,g]} : (0,T) \longrightarrow [H^1(\Omega)]'$ be the function defined by:

$$\langle F_{[f,g]}(t), v\rangle = (f(t), v)_{L^2(\Omega)} + \langle g(t), v\rangle_\Gamma, \quad \forall v \in H^1(\Omega).$$

Result 2. *Function $F_{[f,g]} \in L^2(0,T;[H^1(\Omega)]')$. Moreover, y is a usual weak solution of (1) if and only if*

$$\left.\begin{array}{rl} y \in L^2(0,T;H^1(\Omega)) & \\ \dfrac{dy(t)}{dt} + A(t)y(t) = F_{[f,g]}(t), & \text{in} \quad L^2(0,T;[H^1(\Omega)]') \\ y(0) = y_0 & \text{in } \Omega. \end{array}\right\} \tag{5}$$

In particular, $y \in W(0,T;H^1(\Omega),[H^1(\Omega)]') \subset \mathcal{C}([0,T];L^2(\Omega))$.

Remark 2. *We must recall that:*

$$W(0,T;H^1(\Omega),[H^1(\Omega)]') = \{z \in L^2(0,T;H^1(\Omega)) \ : \ \frac{dz}{dt} \in L^2(0,T;[H^1(\Omega)]')\}$$

with the identification $H^1(\Omega) \subset L^2(\Omega) \equiv [L^2(\Omega)]' \subset [H^1(\Omega)]'$.

Result 3. *y is a usual weak solution of (1) if and only if*

$$\left.\begin{array}{l} y \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)), \\ -\displaystyle\int_{Q_T} y\frac{\partial \eta}{\partial t}\, dxdt + \int_0^T a(t;y(t),\eta(t))dt \\ = \displaystyle\int_0^T \langle F_{[f,g]}(t), \eta(t)\rangle\, dt + (y_0, \eta(0))_{L^2(\Omega)}, \\ \forall \eta \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \qquad \text{such that } \eta_{|t=T} = 0. \end{array}\right\} \tag{6}$$

Remark 3. *It is worthwhile mentioning here that space $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) = H^1(Q_T)$ and, consequently, formulation (6) agrees with formulation used in Chapter III of Ladyzenskaja et al. [19].*

Results 2 and 3 arise, for instance, from [28, Ch. III, Prop. 2.1], where a more general abstract case is considered.

1.3. "Relaxation" of the hypothesis on f

Let us assume, for the sake of simplicity, that $y_0 \in L^2(\Omega)$. Let $f \in L^{r_1}(0,T;L^{q_1}(\Omega))$ and $g \in L^{r_2}(0,T;L^{q_2}(\Gamma))$ where r_1, q_1 and r_2, q_2 satisfy respectively the following conditions:

$$\left.\begin{array}{rr}\dfrac{1}{r_1}+\dfrac{n}{2q_1}=1+\dfrac{n}{4}, & \\ q_1\in\left[\dfrac{2n}{n+2},2\right],\quad r_1\in[1,2] & \text{for } n\geq 3,\\ q_1\in(1,2],\quad r_1\in[1,2) & \text{for } n=2,\\ q_1\in[1,2],\quad r_1\in\left[1,\dfrac{4}{3}\right] & \text{for } n=1.\end{array}\right\} \tag{7}$$

$$\left.\begin{array}{ll}\dfrac{1}{r_2}+\dfrac{n-1}{2q_2}=\dfrac{1}{2}+\dfrac{n}{4},\quad q_2\in\left[\dfrac{2(n-1)}{n},2\right],\quad r_2\in\left[\dfrac{4}{3},2\right] & \text{for } n>2,\\ \dfrac{1}{r_2}+\dfrac{1}{2q_2}=1,\quad q_2\in(1,2],\quad r_2\in\left[\dfrac{4}{3},2\right) & \text{for } n=2,\end{array}\right\} \tag{8}$$

(If $n=1$, we require $g\in L^{\frac{4}{3}}(\Sigma_T)$).

Remark 4. *We must note that condition (7) is in fact condition (1.6) introduced in Chapter III of Ladyzenskaja et al. [19]. We also recall that* $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\subset L^{r_1'}(0,T;L^{q_1'}(\Omega))$*. So, term* $\int_{Q_T} f\eta dxdt$ *makes sense for any* $\eta\in L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$*.*

Remark 5. *Our condition (8) is related to condition (III.5.3') of [19]. The correct range of admissible values of* r_2 *and* q_2 *is that of our condition (8). This range is more restricted that the one that appears in condition (III.5.3') of [19]. Determination of the range of admissible values of* r_2 *and* q_2 *requires the previous determination of the range of values* r *and* q *such that the trace operator* γ *verifies:*

$$\gamma\in\mathcal{L}(L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega));L^r(0,T;L^q(\Gamma))) \tag{9}$$

(cf. (II.3.11) and (II.3.10) of [19]). This in turn requires the previous determination of the range of values q *for which the inequality*

$$\|\gamma v\|_{L^q(\Gamma)}\leq C\|v\|^{\alpha}_{H^1(\Omega)}\|v\|^{1-\alpha}_{L^2(\Omega)}\quad \forall v\in H^1(\Omega),\quad \alpha=\frac{n}{2}-\frac{n-1}{q}, \tag{10}$$

holds. (It can be easily seen that this inequality is equivalent to inequality

(II.2.21) of [19].) It must be pointed out that inequality (10) is valid for $q \in [2, \frac{2(n-1)}{(n-2)}]$ *if* $n = 3$ *and for* $q \in [2, \infty)$ *if* $n = 2$ *and, as we shall see in next remark, it is not valid for* $q < 2$ *(contrary to what is said in the English edition of [19]).)*

Let us justify first that inequality (10) holds for such ranges of q*. We consider first the case* $n \geq 3$*. Inequality (10) for* $q = 2$ *arises from the last equation of page 41 of [18] and for* $q = \frac{2(n-1)}{(n-2)}$ *from Sobolev injection (see, for instance, [1], Th. 5.22). Then, the intermediate case* $2 < q < \frac{2(n-1)}{(n-2)}$ *is obtained through the interpolation inequality. In the case* $n = 2$*, inequality (10) results from the continuity of the trace operator* $\gamma : H^1(\Omega) \mapsto L^2(\Gamma)$*, the inequality*

$$\|v\|_{H^s(\Gamma)} \leq C_s \|v\|^{1-2s}_{L^2(\Gamma)} \|v\|^{2s}_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}(\Gamma), \quad 0 < s < \frac{1}{2}, \tag{11}$$

and the Sobolev imbedding $H^s(\Gamma) \subset L^q(\Gamma)$*, where* $0 < s < \frac{1}{2}$ *and* $\frac{1}{q} = \frac{1}{2} - s$*. (See [18], Theorem 1.4.4.1).*

Using (10), it is easy to see that the trace operator γ *satisfies (9) when* r *and* q *are such that:*

$$\left.\begin{array}{ll} \frac{1}{r} + \frac{n-1}{2q} = \frac{n}{4}, \quad q \in \left[2, \frac{2(n-1)}{n-2}\right], \quad r \in [2,4] & \text{for } n \geq 3, \\ \frac{1}{r} + \frac{1}{2q} = \frac{1}{2}, \quad q \in [2, \infty), \quad r \in (2,4] & \text{for } n = 2, \end{array}\right\} \tag{12}$$

For $n = 1$, $\gamma \in \mathcal{L}(L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)); L^4(\Sigma_T))$*. (Note that the range of admissible values of* r *and* q *is more restricted than that stated in (II.3.10) of [19]). The condition (12) leads to the condition (8) for* r_2 *and* q_2*.*

Remark 6. *The inequality (10) does not hold for* $q < 2$ *(that is,* $\alpha < \frac{1}{2}$*) as the following counterexample shows.*
Let $\Omega = Q_+ = \{x = (x', x_n); |x'| < 1, 0 < x_n < 1\}$*, where* $x' = (x_1, \ldots, x_{n-1})$ *and* $|x'|$ *stands for the Euclidean norm of* x'*. For any* $0 < \varepsilon < 1$*, let* $\eta_\varepsilon : [0,1] \mapsto \mathbb{R}$ *be the unique continuous piecewise linear function which takes the constant value* 1 *over* $[0, \varepsilon]$ *and vanishes over* $[2\varepsilon, 1]$*. Let* $u_\varepsilon : Q_+ \mapsto \mathbb{R}$ *be defined by* $u_\varepsilon(x) = \eta_\varepsilon(x_n)$*. Clearly* $u_\varepsilon \in H^1(Q_+)$ *and it is straightforward to*

check that

$$\|u_\varepsilon\|_{L^2(Q_+)} = O(\varepsilon^{1/2}), \tag{13}$$

$$\|u_\varepsilon\|_{H^1(Q_+)} = O(\varepsilon^{-1/2}). \tag{14}$$

If the inequality (10) were true with $q < 2$ *(that is,* $\alpha < \frac{1}{2}$*), it would hold that*

$$\|u_\varepsilon\|_{L^q(\partial Q_+)} \leq C\varepsilon^{(1-2\alpha)/2}, \tag{15}$$

hence $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^q(\partial Q_+)} = 0$*, which is impossible because*

$$\|u_\varepsilon\|^q_{L^q(\partial Q_+)} \geq \int_{|x'|<1} |u_\varepsilon(x',0)|^q dx_1 ... dx_{n-1} = \text{meas}_{n-1}(|x'| < 1) > 0 \ . \tag{16}$$

This counterexample can be extended to any bounded domain Ω *with* C^1 *boundary by means of partitions of the unity and local charts.*

In this case, following Ladyzenskaja *et al.* [19] we introduce the following definition, which is clearly an extension of formulation (6):

Definition 2. *We say that* y *is a* **usual weak solution** *of problem (1) if*

$$\left.\begin{aligned} & y \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)), \\ & -\int_{Q_T} y\frac{\partial \eta}{\partial t} dxdt + \int_0^T a(t;y(t),\eta(t))dt \\ = & \int_{Q_T} f\eta dxdt + \int_{\Sigma_T} g\eta d\sigma dt + (y_0,\eta(0))_{L^2(\Omega)}, \\ & \forall \eta \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \qquad \textit{such that } \eta_{|t=T} = 0. \end{aligned}\right\} \tag{17}$$

Theorem 1. *The problem (1) has a unique usual weak solution* y*. Moreover, following estimate is satisfied:*

$$\begin{aligned} & \|y\|_{L^2(0,T;H^1(\Omega))} + \|y\|_{\mathcal{C}([0,T];L^2(\Omega))} \\ & \qquad \leq C\{\|y_0\|_{L^2(\Omega)} + \|f\|_{L^{r_1}(0,T;L^{q_1}(\Omega))} + \|g\|_{L^{r_2}(0,T;L^{q_2}(\Gamma))}\}. \end{aligned} \tag{18}$$

Proof. See Chapter III of Ladyzenskaja *et al.* [19]: existence and uniqueness is derived from Theorem 5.1, and estimate (18) is obtained by adapting Theorem 2.1 for Neumann conditions. □

Remark 7. *Above result requires a lower time-space smoothness than Result 1. We will analyze below in detail the extreme cases in the hypothesis*

(7) on f*:*

- *(i) For any* n*:* $r_1 = 1,\ q_1 = 2$*, that is, it is enough that* $f \in L^1(0,T;L^2(\Omega))$ *in order to obtain existence and uniqueness of solution for problem (17), weakening the time regularity of* f*.*

- *(ii) For* $n \geq 3$*:* $r_1 = 2,\ q_1 = \dfrac{2n}{n+2} = (2^*)'$*, that is, in this case it is enough that* $f \in L^2(0,T;L^{(2^*)'}(\Omega))$*, where* 2^* *is the critical exponent for the Sobolev injection* $H^1(\Omega) \subset L^{2^*}(\Omega)$*, and* p' *denotes the conjugate exponent of* p*.*

- *(iii) For* $n = 2$*:* $q_1 = 1 + \epsilon$ *(with any* $\epsilon > 0$*),* $r_1 = \dfrac{2+2\epsilon}{1+3\epsilon} < 2$*, that is, in this case it is enough that* $f \in L^{\frac{2+2\epsilon}{1+3\epsilon}}(0,T;L^{1+\epsilon}(\Omega))$*.*

- *(iv) For* $n = 1$*:* $q_1 = 1,\ r_1 = \dfrac{4}{3}$*, that is, in this case it is enough that* $f \in L^{\frac{4}{3}}(0,T;L^1(\Omega))$*.*

As we can easily observe, in all of the cases, the required regularity for f *is always lower than the required for the "smooth" case* $f \in L^2(0,T;L^2(\Omega))$*.*

2. A parabolic problem with measure data

Let us deal again with problem (1), but now with "non-smooth" data, that is, we will keep the assumption on $y_0 \in L^2(\Omega)$, but we will consider here $f = \mu_{Q_T} \in \mathcal{M}(Q_T) = [\mathcal{C}_0(Q_T)]'$, and $g = \mu_{\Sigma_T} \in \mathcal{M}(\Sigma_T) = [\mathcal{C}_0(\Sigma_T)]'$. We recall the definition of weak solution given in Casas [10, Def. 6.1] for a related problem:

Definition 3. *Given* $r, p \in [1,2)$ *such that* $\dfrac{2}{r} + \dfrac{n}{p} > n+1$*, we say that a function* y *is a* **weak solution** *of problem (1) if:*

$$\left.\begin{aligned} y \in L^r(0,T;W^{1,p}(\Omega)),\\ -\int_{Q_T} y \frac{\partial \eta}{\partial t}\,dxdt + \int_0^T a(t;y(t),\eta(t))dt\\ = \int_{Q_T} \eta\, d\mu_{Q_T}(x,t) + \int_{\Sigma_T} \eta\, d\mu_{\Sigma_T}(x,t) + \int_\Omega y_0(x)\eta(x,0)\,dx,\\ \forall \eta \in \mathcal{C}^1(\overline{Q_T}) \text{ such that } \eta_{|t=T} = 0. \end{aligned}\right\} \tag{19}$$

In this case, we also introduce the concept of *transposition solution*. In order to do that, we consider the functional spaces:

$$
\begin{aligned}
&Y = L^2(0,T;H^1(\Omega)) \cap \mathcal{C}(\overline{Q_T}),\\
&Y_T = \{z \in Y \ : \ z(T) = 0\},\\
&Y_T^\infty = \{z \in Y_T \ : \ -\frac{\partial z}{\partial t} + L^* z \in L^\infty(Q_T),\ \frac{\partial z}{\partial \nu_{L^*}} \in L^\infty(\Sigma_T)\},
\end{aligned}
$$

where L^* is the formal adjoint of operator L given by:

$$L^* z = -\sum_{j=1}^n \frac{\partial}{\partial x_j}\{\sum_{i=1}^n a_{ij}(x,t)\frac{\partial z}{\partial x_i} + b_j(x,t) z\} + \sum_{i=1}^n a_i(x,t)\frac{\partial z}{\partial x_i} + c(x,t) z,$$

and

$$\frac{\partial z}{\partial \nu_{L^*}} = \sum_{j=1}^n \{\sum_{i=1}^n a_{ij}(x,t)\frac{\partial z}{\partial x_i} + b_j(x,t) z\}\nu_j(x).$$

Definition 4. *We say that a function y is a* **transposition solution** *of problem (1) if, for some r, $p \in [1,2)$ with $\frac{2}{r} + \frac{n}{p} > n+1$, it verifies*

$$\left.\begin{aligned}
&y \in L^r(0,T;W^{1,p}(\Omega)),\\
&\int_{Q_T} y(-\frac{\partial \eta}{\partial t} + L^*\eta)\,dx\,dt + \int_{\Sigma_T} y\frac{\partial \eta}{\partial \nu_{L^*}}\,d\sigma\,dt\\
= &\int_{Q_T} \eta\,d\mu_{Q_T}(x,t) + \int_{\Sigma_T} \eta\,d\mu_{\Sigma_T}(x,t) + \int_\Omega y_0(x)\eta(x,0)\,dx, \qquad \forall \eta \in Y_T^\infty.
\end{aligned}\right\} \quad (20)$$

Arguing as in [10, Th. 6.3] we can demonstrate the following result:

Result 4. *The problem (1) has a weak solution y, which is the unique transposition solution of (1). In fact, $y \in L^r(0,T;W^{1,p}(\Omega)), \forall r, p \in [1,2)$ such that $\frac{2}{r} + \frac{n}{p} > n+1$.*

Remark 8. *Uniqueness of the transposition solution is a consequence of the surjectivity of mapping:*

$$z \in Y_T^\infty \ \longrightarrow \ \big(-\frac{\partial z}{\partial t} + L^* z, \frac{\partial z}{\partial \nu_{L^*}}\big) \in L^\infty(Q_T) \times L^\infty(\Sigma_T),$$

that can be obtained by adapting results from Di Benedetto [15, Th. 4] and Ladyzenskaja et al. [19, Ch. III, Th. 7.1].

This implies that the zero function of $L^r(0,T;W^{1,p}(\Omega))$ is the only one satisfying:

$$\int_{Q_T} y(-\frac{\partial\eta}{\partial t} + L^*\eta)\,dx\,dt + \int_{\Sigma_T} y\frac{\partial\eta}{\partial\nu_{L^*}}\,d\sigma\,dt = 0, \quad \forall\eta\in Y_T^\infty,$$

which shows the uniqueness of y.

It is worthwhile mentioning here that, although each transposition solution is also a weak solution, a weak solution is not necessarily a transposition solution. For instance, let us consider the case corresponding to the choice $a_{ij}(x,t) = (\varepsilon^{-2}-1)x_ix_j(x_1^2+x_2^2)^{-1}+\delta_{ij}$ for $i,j=1,2$, $a_i=b_i=0$ for $i=1,2$, $c=0$ and the function $u(x,t)=x_1(x_1^2+x_2^2)^{-(1+\varepsilon)/2}$ with $0<\varepsilon<1$ defined in Ω the unit ball of $\mathbb{R}^2$ (Serrin [27]), that is a weak solution of the problem:

$$\left.\begin{array}{rl} \dfrac{\partial y}{\partial t} + Ly = 0 & \text{in } Q_T,\\ \dfrac{\partial y}{\partial\nu_L} = \dfrac{\partial u}{\partial\nu_L} & \text{on } \Sigma_T,\\ y(x,0) = u(x,0) & \text{in } \Omega. \end{array}\right\} \tag{21}$$

However, u is not a transposition solution of this problem, since it is not smooth. We must take into account that this problem admits a usual weak solution w, that will be smooth due to the fact that data are regular. This weak solution w is the unique transposition solution and it is evidently different from u. (Note that the spatial derivatives of u are in $L^p(\Omega)$ $\forall p<\frac{2}{1+\epsilon}$ but not in $L^2(\Omega)$).

We are going to analyze now the different types of solution depending on the smoothness of data.

Theorem 2. *Assume $y_0\in L^2(\Omega)$, $f\in L^2(0,T;L^2(\Omega))$ and $g\in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$. If y is the usual weak solution of problem (1), then y is a transposition solution and, consequently, a weak solution of (1).*

Proof. For a given $\eta\in Y_T^\infty$, we consider $\psi = -\dfrac{\partial\eta}{\partial t} + L^*\eta \in L^\infty(Q_T)$ and

$\phi = \dfrac{\partial \eta}{\partial \nu_{L^*}} \in L^\infty(\Sigma_T)$. Then, η is the unique usual weak solution of

$$\left.\begin{array}{rl} \eta \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)), & \\ -\dfrac{\partial \eta}{\partial t} + L^*\eta = \psi & \text{in } Q_T, \\ \dfrac{\partial \eta}{\partial \nu_{L^*}} = \phi & \text{on } \Sigma_T, \\ \eta(x,T) = 0 & \text{in } \Omega. \end{array}\right\} \tag{22}$$

Let $A^*(t) \in \mathcal{L}(H^1(\Omega), [H^1(\Omega)]')$ be given by:

$$\langle A^*(t)w, v\rangle = a(t;v,w) = \langle A(t)v, w\rangle, \quad \forall\, w, v \in H^1(\Omega), \qquad \text{a.e.} \quad t \in (0,T)\ .$$

Then, from Result 2, we have that y, $\eta \in W(0,T;H^1(\Omega),[H^1(\Omega)]')$ satisfy (5) and

$$-\frac{d\eta(t)}{dt} + A^*(t)\eta(t) = F_{[\psi,\phi]}(t) \qquad \text{in} \quad L^2(0,T;[H^1(\Omega)]'). \tag{23}$$

From this regularity, the following computations can be done:

$$\begin{aligned} &\int_{Q_T} y\Big(-\frac{\partial \eta}{\partial t} + L^*\eta\Big)\,dx\,dt + \int_{\Sigma_T} y\frac{\partial \eta}{\partial \nu_{L^*}}\,d\sigma\,dt \\ &\quad = \int_{Q_T} y\psi\,dx\,dt + \int_{\Sigma_T} y\phi\,d\sigma\,dt = \int_0^T \langle F_{[\psi,\phi]}(t), y(t)\rangle\,dt \\ &\quad = \int_0^T \langle -\frac{d\eta(t)}{dt} + A^*(t)\eta(t), y(t)\rangle\,dt = \int_0^T \langle \frac{dy(t)}{dt}, \eta(t)\rangle\,dt \\ &\quad - \int_\Omega \eta(x,T)y(x,T)dx + \int_\Omega \eta(x,0)y_0(x)dx + \int_0^T \langle A(t)y(t), \eta(t)\rangle\,dt \\ &\quad = \int_0^T \langle \frac{dy(t)}{dt} + A(t)y(t), \eta(t)\rangle\,dt + \int_\Omega \eta(x,0)y_0(x)dx \\ &\quad = \int_{Q_T} f\eta\,dx\,dt + \int_\Omega \eta(x,0)y_0(x)dx + \int_{\Sigma_T} g\eta\,d\sigma\,dt. \end{aligned}$$

Thus, y is a transposition solution of (1). $\square$

Theorem 3. *Assume* $y_0 \in L^2(\Omega)$, $f \in L^{r_1}(0,T;L^{q_1}(\Omega))$, $g \in L^{r_2}(0,T;L^{q_2}(\Gamma))$ *with* r_1, q_1 *and* r_2, q_2 *satisfying hypothesis (7) and (8). If* y *is the usual weak* *solution of problem (1), then y is a transposition solution and, consequently, a weak solution of (1).*

Proof. Since $\mathcal{C}(\overline{Q_T})$ is dense in $L^{r_1}(0,T;L^{q_1}(\Omega))$, and $\mathcal{C}(\overline{\Sigma_T})$ is dense in $L^{r_2}(0,T;L^{q_2}(\Gamma))$, we can consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(\overline{Q_T})$ convergent to f in $L^{r_1}(0,T;L^{q_1}(\Omega))$, and a sequence $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(\overline{\Sigma_T})$ convergent to g in $L^{r_2}(0,T;L^{q_2}(\Gamma))$. Let y_n be the solution of problem (17) with second member f_n, boundary condition g_n and initial condition $y_0 \in L^2(\Omega)$. Since $f_n \in \mathcal{C}(\overline{Q_T}) \subset L^2(0,T;L^2(\Omega))$ and $g_n \in \mathcal{C}(\overline{\Sigma_T}) \subset L^2(0,T;L^2(\Gamma))$, y_n is the solution of (6) and, consequently, the solution of (3) (with f_n and g_n instead of f and g). From theorem 2, y_n is the transposition solution, that is:

$$\begin{aligned}&\int_{Q_T} y_n\left(-\frac{\partial\eta}{\partial t} + L^*\eta\right) dx\,dt + \int_{\Sigma_T} y_n \frac{\partial\eta}{\partial\nu_{L^*}}\, d\sigma\, dt\\ &\qquad = \int_{Q_T} f_n\eta\, dx\, dt + \int_\Omega \eta(x,0)y_0(x)dx + \int_{\Sigma_T} g_n\eta\, d\sigma\, dt, \quad \forall\eta\in Y_T^\infty.\end{aligned}$$

From estimate (18), $\{y_n\}$ converges to y in $L^2(0,T;H^1(\Omega))\cap\mathcal{C}([0,T];L^2(\Omega))$. So

$$\begin{aligned}&\int_{Q_T} y\left(-\frac{\partial\eta}{\partial t} + L^*\eta\right) dx\,dt + \int_{\Sigma_T} y \frac{\partial\eta}{\partial\nu_{L^*}}\, d\sigma\, dt\\ &\qquad = \int_{Q_T} f\eta\, dx\, dt + \int_\Omega \eta(x,0)y_0(x)dx + \int_{\Sigma_T} g\eta\, d\sigma\, dt, \quad \forall\eta\in Y_T^\infty,\end{aligned}$$

i.e. y is the transposition solution of (1). □

Finally, for the case of measure data $f = \mu_{Q_T} \in \mathcal{M}(Q_T)$, $g = \mu_{\Sigma_T} \in \mathcal{M}(\Sigma_T)$ and $y_0 \in L^2(\Omega)$ (the two following results remain true even for the more general case $y_0 \in \mathcal{M}(\overline{\Omega}) = [\mathcal{C}_0(\overline{\Omega})]'$), we have the following relations:

(R1) If y is a weak solution of (1) in the sense of (19), then y is a solution of the partial differential equation $\dfrac{\partial y}{\partial t} + Ly = f$ in the sense of distributions in Q_T.

(R2) If y is the unique transposition solution of (1), then y is also a solution of $\dfrac{\partial y}{\partial t} + Ly = f$ in the sense of distributions in Q_T.

It is important to remark here that we cannot take $\eta \in \mathcal{D}(Q_T)$ as test function in the definition of the transposition solution, since $\mathcal{D}(Q_T) \not\subset Y_T^\infty$. This is due to the fact that we have only assumed the boundedness of a_{ij} and b_i, and this only assures that $-\dfrac{\partial \eta}{\partial t} + L^* \eta \in W^{-1,\infty}(Q_T)$, but not necessarily $-\dfrac{\partial \eta}{\partial t} + L^* \eta \in L^\infty(Q_T)$. Nevertheless, this relation (R2) holds true since, as commented above, the unique transposition solution is a weak solution in the sense of (19), and, consequently, also a distributional solution of the partial differential equation.

3. The solution operator

Let us assume for simplicity that $g = 0$. Then we can define the solution operator:

$$\begin{array}{rcccc} S & : & L^1(Q_T) \times L^2(\Omega) & \longrightarrow & L^r(0,T;W^{1,p}(\Omega)) \\ & & (f, y_0) & \longmapsto & y \end{array}$$

where y is the unique transposition solution (and also a weak solution in the sense of (19)) of problem (1) with $g = 0$, and r, $p \in [1,2)$ such that $\dfrac{2}{r} + \dfrac{n}{p} > n + 1$.

Since the usual weak solution (that is, the solution of (3) or (17)) is the transposition solution (and a weak solution in the sense of (19)) of (1), we have that:

$$S(L^2(Q_T) \times L^2(\Omega)) \subset L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)),$$

$$S(L^{r_1}(0,T;L^{q_1}(\Omega)) \times L^2(\Omega)) \subset L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)),$$

for any numbers r_1, q_1 satisfying condition (7).

It is also obvious that, due to the linearity and the uniqueness of solution,

$$S(f, y_0) = S(f, 0) + S(0, y_0), \quad \forall f \in L^1(Q_T),\ \forall y_0 \in L^2(\Omega).$$

Thus, we can consider the case $y_0 = 0$ and, in order to simplify the notation, we can rewrite $S_0(f) = S(f, 0)$.

Now, thanks to estimate (6.7) of Casas [10], the linear operator:

$$\begin{array}{rcccc} S_0 & : & L^1(Q_T) & \longrightarrow & L^r(0,T;W^{1,p}(\Omega)) \\ & & f & \longrightarrow & y \end{array}$$

is continuous and, due to estimates (4) and (18), the linear operator S_0 also verifies:

$$S_0 \in \mathcal{L}(L^2(Q_T), L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))),$$

$$S_0 \in \mathcal{L}(L^{r_1}(0,T;L^{q_1}(\Omega)), L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))),$$

for any numbers r_1, q_1 satisfying condition (7).

If we assume that r_1, q_1 satisfy stronger condition (7.2) of Ladyzenskaja *et al.* [19, Ch. III], then we have that:

$$S_0 \in \mathcal{L}(L^{r_1}(0,T;L^{q_1}(\Omega)), \mathcal{C}(\overline{Q_T})).$$

This stems from an easy adaptation of theorem 7.1 of [19, Ch. III] to the case of homogeneous Neumann boundary conditions, combined with [15, Th. 4].

3.1. Global regularity of the solution for a second member $f \in L^q(Q_T)$, $1 < q < 2$

We consider again problem (1) with $g = 0$. We are interested now in analyzing the regularity of the transposition solution in the "intermediate" case where $f \in L^q(Q_T)$, for $1 < q < 2$. Due to the linearity of the problem, it is enough to study the case $y_0 = 0$. For this case we know that the solution operator satisfies:

$$S_0 \in \mathcal{L}(L^1(Q_T), L^r(0,T;W^{1,p}(\Omega))),$$

$\forall\, r$, $p \in [1,2)$ such that $\dfrac{2}{r} + \dfrac{n}{p} > n+1$. Then, taking $p = r$, we have:

$$S_0 \in \mathcal{L}(L^1(Q_T), L^r(Q_T)), \quad \forall r \in [1, \frac{n+2}{n+1}),$$

and

$$\frac{\partial}{\partial x_i} \circ S_0 \in \mathcal{L}(L^1(Q_T), L^r(Q_T)), \quad \forall r \in [1, \frac{n+2}{n+1}).$$

On the other hand, since $S_0 \in \mathcal{L}(L^2(Q_T), L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)))$, we also have:

$$S_0 \in \mathcal{L}(L^2(Q_T), L^2(Q_T)),$$

and

$$\frac{\partial}{\partial x_i} \circ S_0 \in \mathcal{L}(L^2(Q_T), L^2(Q_T)).$$

From the Riesz-Thorin Theorem (see, for instance, Bergh-Löfström [4]), the operators S_0, $\dfrac{\partial}{\partial x_i} \circ S_0 \in \mathcal{L}(L^q(Q_T), L^s(Q_T))$ for q, s satisfying

$$\frac{1}{q} = \theta + (1-\theta)\frac{1}{2},$$

$$\frac{1}{s} = \theta\frac{1}{r} + (1-\theta)\frac{1}{2},$$

with $\theta \in (0,1)$, and $r \in [1, \dfrac{n+2}{n+1})$.

Considering the case $r = \dfrac{n+2}{n+1} - \epsilon$, $\epsilon > 0$, and after the elimination of θ, we arrive to:

$$s = s(\epsilon) = \frac{1}{\frac{1}{2} + \frac{2-q}{q}\frac{\frac{n}{n+1}+\epsilon}{\frac{2n+4}{n+1}-2\epsilon}} \longrightarrow s(0) = \frac{(n+2)q}{q+n}, \quad \text{as} \;\; \epsilon \to 0^+,$$

and $s(\epsilon) < s(0)$, for $\epsilon > 0$.

Consequently, the operators S_0, $\dfrac{\partial}{\partial x_i} \circ S_0$ lie in $\mathcal{L}(L^q(Q_T), L^{\frac{(n+2)q}{q+n}-\delta}(Q_T))$, $\forall q \in [1,2)$, and $\forall \delta > 0$ arbitrarily small.

Thus, we have demonstrated the following global regularity result:

Theorem 4. *If $f \in L^q(Q_T)$, with $q \in [1,2)$, then the transposition solution of the problem (1) lies in $L^{\frac{(n+2)q}{q+n}-\delta}(0,T; W^{1,\frac{(n+2)q}{q+n}-\delta}(\Omega))$, $\forall \delta > 0$.*

Remark 9. *An analogous result for the case of nonlinear equations with homogeneous Dirichlet conditions has been proved by Boccardo et al. [7, Th. 2.9 and Th. 2.11]. Other related results for the case of Dirichlet boundary conditions can be found in the works of Dall'Aglio-Orsina [13] (for measure data), Porzio-Vespri [26] (for Hölder regularity results), and Leoni-Pellacci [20] (for unbounded domains).*

4. Local regularity for a parabolic equation with non-smooth data

Let us consider now the problem

$$\left.\begin{aligned} \frac{\partial y}{\partial t} + Ly = f \qquad & \text{in } Q_T, \\ \frac{\partial y}{\partial \nu_L} = 0 \qquad & \text{on } \Sigma_T, \\ y(x,0) = y_0 \qquad & \text{in } \Omega, \end{aligned}\right\} \tag{24}$$

with measure data $f = \mu_{Q_T} \in \mathcal{M}(Q_T)$, and $y_0 \in L^2(\Omega)$.

In the sequel we shall assume also that

$$a_{ij} \in L^\infty(0,T; W^{1,\infty}(\Omega)), \quad 1 \le i,j \le n.$$

Let as assume, as supplementary hypothesis, that, for a given open subset $\omega \subset\subset \Omega$, the restriction of f to $\omega_T := \omega \times (0,T)$ lies in $L^{\tilde{r}}(0,T; L^{\tilde{q}}(\omega))$, with $\tilde{r} \ge 2$ and $\tilde{q} \ge 2$. The goal of this Section is to prove that, for any open subset $\omega' \subset\subset \omega$, the restriction of the transposition solution y of (24) to $\omega' \times (0,T)$ has the same regularity that the usual weak solution of a similar problem with second member in $L^{\tilde{r}}(0,T; L^{\tilde{q}}(\Omega))$.

We know that, if $f \in \mathcal{M}(Q_T)$ and $y_0 \in L^2(\Omega)$, then (see Casas [10, Th. 6.3]) there exists a unique $y \in L^r(0,T; W^{1,p}(\Omega))$, $\forall\, r,\, p \in [1,2)$ satisfying $\frac{2}{r} + \frac{n}{p} > n+1$, that is the transposition solution of (24). Moreover, y is a weak solution of (24) in the sense of (19).

We choose $\zeta \in \mathcal{C}_0^\infty(\Omega)$ satisfying:

$$\left.\begin{aligned} &\zeta = 1 \text{ in } \overline{\omega'}, \\ &\operatorname{supp}(\zeta) \subset \omega, \end{aligned}\right\} \tag{25}$$

so that the restriction of ζy to ω' agrees with the restriction of y to ω'. It is clear that $\zeta y \in L^r(0,T; W^{1,p}(\omega'))$, $\forall\, r,\, p \in [1,2)$ such that $\frac{2}{r} + \frac{n}{p} > n+1$.

We shall first see that ζy is the distributional solution of a partial differential equation like that appearing in (24) but with an appropriate right hand side $\tilde{f}$.

Multiplying now equation (24) by ζ (which makes sense, since $\zeta \in \mathcal{C}^\infty(\bar{\Omega})$

and the equation is understood in $\mathcal{D}'(Q_T)$):

$$\zeta\frac{\partial y}{\partial t}+\zeta Ly=f\zeta.$$

But we know that:

$$\frac{\partial(\zeta y)}{\partial t}=\zeta\frac{\partial y}{\partial t}$$

and

$$\begin{aligned}\zeta Ly &= -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\{\sum_{j=1}^{n}a_{ij}(x,t)\frac{\partial(\zeta y)}{\partial x_j}+a_i(x,t)\,\zeta y\}+\sum_{i=1}^{n}b_i(x,t)\frac{\partial(\zeta y)}{\partial x_i}\\ &\quad +c(x,t)\,\zeta y+\sum_{i=1}^{n}\{\frac{\partial\zeta}{\partial x_i}\sum_{j=1}^{n}a_{ij}(x,t)\frac{\partial y}{\partial x_j}+\sum_{j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x,t)y)\frac{\partial\zeta}{\partial x_j}\\ &\quad +\sum_{j=1}^{n}a_{ij}(x,t)y\frac{\partial^2\zeta}{\partial x_i\partial x_j}+a_i(x,t)y\frac{\partial\zeta}{\partial x_i}\}-\sum_{i=1}^{n}b_i(x,t)y\frac{\partial\zeta}{\partial x_i}\\ &= L(\zeta y)+\sum_{i=1}^{n}\{\frac{\partial\zeta}{\partial x_i}\sum_{j=1}^{n}a_{ij}(x,t)\frac{\partial y}{\partial x_j}+\sum_{j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x,t)y)\frac{\partial\zeta}{\partial x_j}\\ &\quad +\sum_{j=1}^{n}a_{ij}(x,t)y\frac{\partial^2\zeta}{\partial x_i\partial x_j}+a_i(x,t)y\frac{\partial\zeta}{\partial x_i}\}-\sum_{i=1}^{n}b_i(x,t)y\frac{\partial\zeta}{\partial x_i}.\end{aligned}$$

Thus,

$$\frac{\partial(\zeta y)}{\partial t}+L(\zeta y)=\tilde{f}$$

in $\mathcal{D}'(Q_T)$, where

$$\begin{aligned}\tilde{f} &= \zeta f-\sum_{i=1}^{n}\{\frac{\partial\zeta}{\partial x_i}\sum_{j=1}^{n}a_{ij}(x,t)\frac{\partial y}{\partial x_j}+\sum_{j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x,t)y)\frac{\partial\zeta}{\partial x_j}\\ &\quad +\sum_{j=1}^{n}a_{ij}(x,t)y\frac{\partial^2\zeta}{\partial x_i\partial x_j}+a_i(x,t)y\frac{\partial\zeta}{\partial x_i}\}+\sum_{i=1}^{n}b_i(x,t)y\frac{\partial\zeta}{\partial x_i}\end{aligned} \qquad (26)$$

lies in $L^r(0,T;L^p(\Omega))$, $\forall\, r,\, p\in[1,2)$ such that $\frac{2}{r}+\frac{n}{p}>n+1$ if $\tilde{r}\geq 2$ and $\tilde{q}\geq 2$. Thus, here we find the necessity of the analysis of the regularity of the transposition solution of problem (24) for the "intermediate case" $f \in L^q(Q_T)$, $1 < q < 2$, which we have still developed in Subsection 3.1.

Our second aim in this Section consists of proving that ζy is in fact the transposition solution of a similar problem to (24) with second member $\tilde{f}$. A first attempt to do this, based on the fact that y is the transposition solution of problem (24), could begin from taking as test functions $\zeta\eta$, with $\eta \in Y_T^\infty$. But this cannot be done, since $\eta \in Y_T^\infty$ does not imply that $\zeta\eta \in Y_T^\infty$.

In fact, if we consider, for instance, $Ly = -\beta\Delta y + \vec{b}\cdot\nabla y$, where $\vec{b} \in (L^\infty(Q_T))^n$ and $\beta > 0$ is a constant, then

$$Y_T^\infty = \{z \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}(\overline{Q_T}) \ : \ -\frac{\partial z}{\partial t} + L^* z$$
$$= -\frac{\partial z}{\partial t} - \nabla\cdot(\vec{b}z) - \beta\Delta z \in L^\infty(Q_T),\ \frac{\partial z}{\partial \nu_{L^*}} = \beta\frac{\partial z}{\partial \nu} + z\vec{b}\cdot\vec{\nu} \in L^\infty(\Sigma_T)\}$$

Hence, if $\eta \in Y_T^\infty$ we have

$$-\frac{\partial(\zeta\eta)}{\partial t} + L^*(\zeta\eta) = -\zeta\frac{\partial \eta}{\partial t} - \nabla\cdot(\vec{b}\zeta\eta) - \beta\Delta(\zeta\eta)$$
$$= -\zeta\frac{\partial \eta}{\partial t} - \beta\zeta\Delta\eta - \beta\eta\Delta\zeta - 2\beta\nabla\zeta\cdot\nabla\eta - \zeta\nabla\cdot(\vec{b}\eta) - \eta\vec{b}\cdot\nabla\zeta$$
$$= \zeta(-\frac{\partial \eta}{\partial t} + L^*\eta) - 2\beta\nabla\zeta\cdot\nabla\eta - \beta\eta\Delta\zeta - \eta\vec{b}\cdot\nabla\zeta$$

Since, in general, $\nabla\zeta\cdot\nabla\eta \notin L^\infty(Q_T)$, then $\zeta\eta \notin Y_T^\infty$.

Thus, in order to demonstrate that that ζy is the transposition solution of a similar problem to (24) with second member $\tilde{f}$, we can argue as in Casas [10, Th. 6.3] and prove the following:

Theorem 5. *Let $f = \mu_{Q_T} \in \mathcal{M}(Q_T)$ and $y_0 \in L^2(\Omega)$. Assume that for a given open subset $\omega \subset\subset \Omega$, the restriction of f to $\omega_T := \omega \times (0,T)$ lies in $L^{\tilde{r}}(0,T;L^{\tilde{q}}(\omega))$, with $\tilde{r} \geq 2$ and $\tilde{q} \geq 2$. Let y be the transposition solution (and also a weak solution in the sense of (19)) of problem (24). Then, ζy is the transposition solution (and also a weak solution in the sense of (19)) of* *problem:*

$$\left.\begin{aligned} \frac{\partial\phi}{\partial t} + L\phi = \tilde{f} \qquad & \text{in } Q_T, \\ \frac{\partial\phi}{\partial\nu_L} = 0 \qquad & \text{on } \Sigma_T, \\ \phi(x,0) = \zeta y_0 \qquad & \text{in } \Omega, \end{aligned}\right\} \tag{27}$$

where $\tilde{f}$ is given by (26).

Proof. Let $\tilde{\mu} \in \mathcal{M}(\overline{Q_T})$ be the regular real Borel measure on $\overline{Q_T}$ whose restriction to Q_T is f and whose restriction to ∂Q_T is the zero measure (see Section 4.2). We consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(\overline{Q_T})$ weak* convergent to $\tilde{\mu}$ in $\mathcal{M}(\overline{Q_T})$ and satisfying $\|f_n\|_{L^1(Q_T)} \le \|f\|_{M(Q_T)}$, $\forall n \in \mathbb{N}$. Let y_n be the usual weak solution of approximate problem:

$$\left.\begin{aligned} y_n \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)), & \\ \frac{\partial y_n}{\partial t} + Ly_n = f_n \qquad & \text{in } Q_T, \\ \frac{\partial y_n}{\partial\nu_L} = 0 \qquad & \text{on } \Sigma_T, \\ y_n(x,0) = y_0 \qquad & \text{in } \Omega. \end{aligned}\right\} \tag{28}$$

Then, (cf. Casas [10, Th. 6.3]) $\{y_n\}$ is bounded in $L^r(0,T;W^{1,p}(\Omega))$, $\forall\, r,\, p \in [1,2)$ such that $\dfrac{2}{r} + \dfrac{n}{p} > n+1$. Thus, we can extract a subsequence, still denoted in the same way $\{y_n\}$, weakly convergent to y in $L^r(0,T;W^{1,p}(\Omega))$.

We know that y_n satisfies:

$$-\int_{Q_T} y_n \frac{\partial\eta}{\partial t}\,dx\,dt + \int_0^T a(t,y_n(t),\eta(t))\,dt = \int_{Q_T} f_n\eta\,dx\,dt + \int_\Omega y_0(x)\eta(x,0)\,dx,$$
$$\forall\eta \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \;\text{ such that }\; \eta_{|t=T} = 0.$$

Due to the properties of ζ, $\zeta\eta \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, and $(\zeta\eta)_{|t=T} = 0$; thus, we can use $\eta\zeta$ as a test function:

$$-\int_{Q_T} y_n\zeta \frac{\partial\eta}{\partial t}\,dx\,dt + \int_0^T a(t,y_n(t),\zeta\eta(t))\,dt = \int_{Q_T} f_n\zeta\eta\,dx\,dt + \int_\Omega y_0(x)\zeta(x)\eta(x,0)\,dx,$$
$$\forall\eta \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \;\text{ such that }\; \eta_{|t=T} = 0. \tag{29}$$

In particular, this equation remains valid $\forall \eta \in \mathcal{C}^1(\overline{Q_T})$ such that $\eta_{|t=T} = 0$. Using the weak* convergence of the sequence f_n to $\tilde{\mu}$ in $\mathcal{M}(\overline{Q_T})$, the definition of $\tilde{\mu}$ and the facts that $\mathrm{supp}(\zeta) \subset \omega$ and $\zeta f \in L^{\tilde{r}}(0,T;L^{\tilde{q}}(\omega))$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_{Q_T} f_n \zeta \eta \, dx \, dt &= \int_{\overline{Q_T}} \zeta\eta \, d\tilde{\mu} = \int_{Q_T} \zeta\eta \, d\mu_{Q_T}(x,t) \\ &= \int_{\omega_T} \zeta\eta \, d\mu_{Q_T}(x,t) = \int_{\omega_T} \zeta f \eta \, dx dt. \qquad (30)\end{aligned}$$

Taking this into account and passing to the limit as $n \to \infty$ in (29), we deduce:

$$-\int_{Q_T} y\zeta \frac{\partial \eta}{\partial t} + \int_0^T a(t, y(t), \zeta\eta(t)) \, dt = \int_{\omega_T} \zeta f \eta \, dx \, dt + \int_\Omega y_0(x)\zeta(x)\eta(x,0) \, dx,$$
$$\forall \eta \in \mathcal{C}^1(\overline{Q_T}) \text{ such that } \eta_{|t=T} = 0,$$

We also know that:

$$\begin{aligned}a(t;\zeta y(t), \eta(t)) &= a(t; y(t), \zeta\eta(t)) + \int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x,t) y(x,t) \frac{\partial \eta}{\partial x_i}(x,t) \frac{\partial \zeta}{\partial x_j}(x) \, dx \\ &- \int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x,t)\eta(x,t) \frac{\partial y}{\partial x_j}(x,t) \frac{\partial \zeta}{\partial x_i}(x) \, dx - \int_\Omega \sum_{i=1}^n a_i(x,t)\eta(x,t)y(x,t) \frac{\partial \zeta}{\partial x_i}(x) \, dx \\ &+ \int_\Omega \sum_{i=1}^n b_i(x,t) \frac{\partial \zeta}{\partial x_i}(x) \eta(x,t) y(x,t) \, dx.\end{aligned}$$

So,

$$\begin{aligned}-\int_{Q_T} y\zeta \frac{\partial \eta}{\partial t} &+ \int_0^T a(t, \zeta y(t), \eta(t)) \, dt = \int_{\omega_T} \zeta f \eta \, dx \, dt + \int_\Omega y_0(x)\zeta(x)\eta(x,0) \, dx \\ &+ \int_{Q_T} \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x,t) y \frac{\partial \eta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \, dx \, dt - \int_{Q_T} \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x,t) \eta \frac{\partial y}{\partial x_j} \frac{\partial \zeta}{\partial x_i} \, dx \, dt \\ &- \int_{Q_T} \sum_{i=1}^n a_i(x,t) \eta y \frac{\partial \zeta}{\partial x_i} \, dx \, dt + \int_{Q_T} \sum_{i=1}^n b_i(x,t) \frac{\partial \zeta}{\partial x_i} \eta y \, dx \, dt \\ &= \int_\Omega y_0(x)\zeta(x)\eta(x,0) \, dx + \int_{Q_T} \eta \tilde{f} \, dx \, dt, \quad \forall \eta \in \mathcal{C}^1(\overline{Q_T}) \text{ such that } \eta_{|t=T} = 0,\end{aligned}$$

Thus, ζy is a weak solution of (27) in the sense of (19). We can also demonstrate that ζy is an transposition solution of (27), that is, that:

$$\begin{aligned}\int_{Q_T} y\zeta(-\frac{\partial \eta}{\partial t} + L^*\eta)\,dx\,dt + \int_{\Sigma_T} y\zeta \frac{\partial \eta}{\partial \nu_{L^*}}\,d\sigma\,dt \\ = \int_{Q_T} \tilde{f}\eta\,dx\,dt + \int_\Omega y_0(x)\zeta(x)\eta(x,0)\,dx, \quad \forall \eta \in Y_T^\infty.\end{aligned} \tag{31}$$

To do this, given $\eta \in Y_T^\infty$, we consider $\psi = -\dfrac{\partial \eta}{\partial t} + L^*\eta \in L^\infty(0,T;L^\infty(\Omega))$ and $g = \dfrac{\partial \eta}{\partial \nu_{L^*}} \in L^\infty(\Sigma_T)$. Then, η is the unique solution of problem:

$$\left.\begin{aligned} \eta \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)), & \\ -\frac{\partial \eta}{\partial t} + L^*\eta = \psi \qquad & \text{in } Q_T, \\ \frac{\partial \eta}{\partial \nu_{L^*}} = g \qquad & \text{on } \Sigma_T, \\ \eta(x,T) = 0 \qquad & \text{in } \Omega. \end{aligned}\right\} \tag{32}$$

Let $F_{[\psi,g]} : (0,T) \to [H^1(\Omega)]'$ be the function defined by:

$$\langle F_{[\psi,g]}(t), v\rangle = \int_\Omega \psi(t) v dx + \langle g(t), v\rangle_\Gamma, \qquad \forall v \in H^1(\Omega).$$

Thus, η is solution of:

$$\left.\begin{aligned} & \eta \in W(0,T;H^1(\Omega),[H^1(\Omega)]'), \\ & -\frac{d\eta(t)}{dt} + A^*(t)\eta(t) = F_{[\psi,g]}(t) \qquad \text{in } \ L^2(0,T;[H^1(\Omega)]'). \end{aligned}\right\} \tag{33}$$

Because of $y_n\zeta \in W(0,T;H^1(\Omega),[H^1(\Omega)]')$, the following computations can be done:

$$
\begin{aligned}
&\int_{Q_T} y_n\zeta(-\frac{\partial\eta}{\partial t} + L^*\eta)\,dx\,dt + \int_{\Sigma_T} y_n\zeta\frac{\partial\eta}{\partial\nu_{L^*}}\,d\sigma\,dt\\
&\qquad = \int_{Q_T} y_n\zeta\psi\,dx\,dt + \int_{\Sigma_T} y_n\zeta g\,d\sigma\,dt\\
&\qquad = \int_0^T \langle F_{[\psi,g]}(t), \zeta y_n(t)\rangle\,dt = \int_0^T \langle -\frac{d\eta(t)}{dt} + A^*(t)\eta(t), \zeta y_n(t)\rangle\,dt\\
&\qquad = \int_0^T \langle\frac{d(\zeta y_n(t))}{dt}, \eta(t)\rangle\,dt - \int_\Omega \eta(x,T)\zeta(x)y_n(x,T)dx\\
&\qquad + \int_\Omega \eta(x,0)\zeta(x)y_0(x)dx + \int_0^T \langle A(t)(\zeta y_n(t)), \eta(t)\rangle\,dt.
\end{aligned}
$$

Since $\dfrac{d(\zeta y_n(t))}{dt} = \zeta\dfrac{dy_n(t)}{dt}$, $\eta_{|t=T} = 0$, and

$$
\begin{aligned}
&\langle A(t)(\zeta y_n(t)), \eta(t)\rangle = a(t;\zeta y_n(t), \eta(t)) = a(t; y_n(t), \zeta\eta(t))\\
&\quad + \int_\Omega \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)y_n(x,t)\frac{\partial\eta}{\partial x_i}(x,t)\frac{\partial\zeta}{\partial x_j}(x)\,dx\\
&\quad - \int_\Omega \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)\eta(x,t)\frac{\partial y_n}{\partial x_j}(x,t)\frac{\partial\zeta}{\partial x_i}(x)\,dx\\
&\quad - \int_\Omega \sum_{i=1}^n a_i(x,t)\eta(x,t)y_n(x,t)\frac{\partial\zeta}{\partial x_i}(x)\,dx + \int_\Omega \sum_{i=1}^n b_i(x,t)\frac{\partial\zeta}{\partial x_i}(x)\eta(x,t)y_n(x,t)\,dx\\
&\quad = \langle A(t)y_n(t), \zeta\eta(t)\rangle + \int_\Omega \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)y_n(x,t)\frac{\partial\eta}{\partial x_i}(x,t)\frac{\partial\zeta}{\partial x_j}(x)\,dx\\
&\quad - \int_\Omega \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)\eta(x,t)\frac{\partial y_n}{\partial x_j}(x,t)\frac{\partial\zeta}{\partial x_i}(x)\,dx\\
&\quad - \int_\Omega \sum_{i=1}^n a_i(x,t)\eta(x,t)y_n(x,t)\frac{\partial\zeta}{\partial x_i}(x)\,dx + \int_\Omega \sum_{i=1}^n b_i(x,t)\frac{\partial\zeta}{\partial x_i}(x)\eta(x,t)y_n(x,t)\,dx,
\end{aligned}
$$

we have that:

$$\begin{aligned}
&\int_{Q_T} y_n\zeta(-\frac{\partial\eta}{\partial t}+L^*\eta)\,dx\,dt+\int_{\Sigma_T} y_n\zeta\frac{\partial\eta}{\partial\nu_{L^*}}\,d\sigma\,dt\\
&\quad=\int_0^T\langle\zeta\frac{dy_n(t)}{dt},\eta(t)\rangle\,dt+\int_\Omega\zeta(x)y_0(x)\eta(x,0)dx+\int_0^T\langle A(t)y_n(t),\zeta\eta(t)\rangle\,dt\\
&\quad+\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)y_n\frac{\partial\eta}{\partial x_i}\frac{\partial\zeta}{\partial x_j}\,dx\,dt\,-\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)\eta\frac{\partial y_n}{\partial x_j}\frac{\partial\zeta}{\partial x_i}\,dx\,dt\\
&\quad-\int_{Q_T}\sum_{i=1}^n a_i(x,t)\eta y_n\frac{\partial\zeta}{\partial x_i}\,dx\,dt\,+\int_{Q_T}\sum_{i=1}^n b_i(x,t)\frac{\partial\zeta}{\partial x_i}\eta y_n\,dxdt\\
&\quad=\int_{Q_T} f_n\eta\zeta\,dx\,dt\,+\int_\Omega\zeta(x)y_0(x)\eta(x,0)dx\\
&\quad-\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n\frac{\partial}{\partial x_i}(a_{ij}(x,t)y_n)\frac{\partial\zeta}{\partial x_j}\eta\,dx\,dt\\
&\quad-\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)y_n\frac{\partial^2\zeta}{\partial x_i\partial x_j}\eta\,dx\,dt-\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)\eta\frac{\partial y_n}{\partial x_j}\frac{\partial\zeta}{\partial x_i}\,dx\,dt\\
&\quad-\int_{Q_T}\sum_{i=1}^n a_i(x,t)\eta y_n\frac{\partial\zeta}{\partial x_i}\,dx\,dt\,+\int_{Q_T}\sum_{i=1}^n b_i(x,t)\frac{\partial\zeta}{\partial x_i}\eta y_n\,dx\,dt.
\end{aligned}$$

Letting now $n\to\infty$ and using (30) we obtain:

$$\begin{aligned}
&\int_{Q_T} y\zeta(-\frac{\partial\eta}{\partial t}+L^*\eta)\,dx\,dt+\int_{\Sigma_T} y\zeta\frac{\partial\eta}{\partial\nu_{L^*}}\,d\sigma\,dt\\
&\quad=\langle f,\zeta\eta\rangle+\int_\Omega y_0(x)\zeta(x)\eta(x,0)\,dx-\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n\frac{\partial}{\partial x_i}(a_{ij}(x,t)y)\frac{\partial\zeta}{\partial x_j}\eta\,dx\,dt\\
&\quad-\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)y\frac{\partial^2\zeta}{\partial x_i\partial x_j}\eta\,dx\,dt-\int_{Q_T}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t)\eta\frac{\partial y}{\partial x_j}\frac{\partial\zeta}{\partial x_i}\,dx\,dt\\
&\quad-\int_{Q_T}\sum_{i=1}^n a_i(x,t)\eta y\frac{\partial\zeta}{\partial x_i}\,dx\,dt\,+\int_{Q_T}\sum_{i=1}^n b_i(x,t)\frac{\partial\zeta}{\partial x_i}\eta y\,dx\,dt \qquad (34)\\
&\quad=\int_\Omega y_0(x)\zeta(x)\eta(x,0)\,dx+\int_{Q_T}\tilde f\eta\,dx\,dt.
\end{aligned}$$

Finally, since (34) remains valid for all $\eta \in Y_T^\infty$, we have that ζy is a transposition solution of (27). $\square$

Remark 10. *In above computations we have made an implicit use of the following equality:*

$$\text{For any given } l \in [H^1(\Omega)]', \quad \langle \zeta l, v\rangle = \langle l, \zeta v\rangle, \ \forall v \in H^1(\Omega).$$

Thus, it is easy to prove that, if $z \in W(0,T;H^1(\Omega),[H^1(\Omega)]')$*, then:*

$$\zeta z \in W(0,T;H^1(\Omega),[H^1(\Omega)]'),$$

with $\dfrac{d(\zeta z)}{dt} = \zeta \dfrac{dz}{dt}$.

4.1. Local regularity of the solution for a second member $f = \mu_{Q_T} \in \mathcal{M}(Q_T)$, whose restriction to ω is in $L^{\tilde r}(0,T;L^{\tilde q}(\omega))$ with $\tilde r \geq 2$ and $\tilde q \geq 2$.

Finally, we return to the problem (24) in order to obtain a local regularity result for its transposition solution y.

Let us recall that, due to Result 4, the right hand side of equation (27) $\tilde f \in L^r(0,T;L^p(\Omega))$, $\forall\, r, p \in [1,2)$ such that $\dfrac{2}{r} + \dfrac{n}{p} > n+1$. In particular (by taking $r = p$), $\tilde f \in L^r(Q_T)$, $\forall\, r \in [1, \dfrac{n+2}{n+1})$.

From Theorems 4 and 5, since ζy is the transposition solution of (27), we have that $\zeta y \in L^{s_1}(0,T;W^{1,s_1}(\Omega))$, $\forall s_1 < \overline{s}_1 = \dfrac{n^2+2n+4}{n^2+2n+2}$.

Thus, y, $\dfrac{\partial y}{\partial x_i} \in L^{s_1}(\omega_1 \times (0,T))$, $\forall s_1 < \overline{s}_1$, and for all domain $\omega_1 \subset\subset \omega$. For the sake of homogeneity of the notation, we note $\zeta_1 = \zeta$ and $\tilde f_1 = \tilde f$.

Repeating the process in a subdomain $\omega_2 \subset\subset \omega_1$, the corresponding function $\zeta_2 y$ is the transposition solution of (24) with a second member $\tilde f_2 \in L^{s_1}(Q_T)$ $\forall s_1 < \overline{s}_1$, hence $\zeta_2 y \in L^{s_2}(0,T;W^{1,s_2}(\Omega))$ $\forall s_2 < \overline{s}_2 = \dfrac{\overline{s}_1(n+2)}{n+\overline{s}_1}$. Thus, y, $\dfrac{\partial y}{\partial x_i} \in L^{s_2}(\omega_2 \times (0,T))$, $\forall s_2 < \overline{s}_2$, and for all domain $\omega_2 \subset\subset \omega_1$. The iteration of this process leads to y, $\dfrac{\partial y}{\partial x_i} \in L^{s_k}(\omega_k \times (0,T))$, $\forall s_k < \overline{s}_k$ with $\overline{s}_k = \dfrac{\overline{s}_{k-1}(n+2)}{n+\overline{s}_{k-1}}$ for all domain $\omega_k \subset\subset \omega_{k-1}$. It is easy to see that

$1 < \overline{s}_{k-1} < \overline{s}_k < 2$ and $\lim_{k\to\infty} \overline{s}_k = 2$. Hence, given any open subset $\omega^{'} \subset\subset \omega$, we can choose an integer l big enough and a finite sequence of open sets

$$\omega^{'} \subset\subset \omega_l \subset\subset \omega_{l-1} \subset\subset \ldots \subset\subset \omega_2 \subset\subset \omega_1 \subset\subset \omega$$

in such a way that the function $\zeta_l y$ (with $\mathrm{supp}(\zeta_l) \subset \omega_l$) be the transposition solution of (27) for a right hand side $\tilde{f}_l$ which satisfies condition (7). Therefore we have that $\zeta_l y$ is the usual weak solution of this last problem and then the standard results of Ladyzenskaja *et al.* [10, Ch. III] about regularity of solution can be applied.

4.2. Local regularity for a convection-diffusion problem with pointwise source term in dimension $n = 2$

In this Subsection we are particularly interested in the study of the case with a pointwise source term $f = m(t)\delta(x - P_1)$, with $m \in L^\infty(0,T)$, and $P_1 \in \Omega$. Our interest is started from a previous work of the authors [22] where the design and management of a wastewater treatment system is analyzed by controlling a system of the type (24) with pointwise source terms, representing the pollution concentration in a shallow water area. In that work a result about the local regularity of the solution far from the point P_1 is used, but only a rough proof of it is presented. Here we deal with a detailed demonstration of this result, that is, that solution y is smooth in any closed set of Ω with a strictly positive distance from P_1.

We obtain, as in above Subsection, that y is an usual weak solution of (24), with null second member, for all domain $\Omega_1 \subset\subset \Omega\backslash\{P_1\}$. So, the results of Ladyzenskaja *et al.* [10, Ch. III] about Hölder continuity of solution can be applied. Consequently, we obtain the following continuity result for y in the compact subsets of $(\Omega\backslash\{P_1\}) \times [0,T]$:

Theorem 6. *If $f = m(t)\delta(x - P_1)$, with $m \in L^\infty(0,T)$ and $P_1 \in \Omega$, then the transposition solution of the problem (24) is continuous in $\bar{A} \times [0,T]$, for any subdomain $A \subset\subset \Omega\backslash\{P_1\}$.*

Appendix: A result about approximation of measures.

Let Ω be an open bounded subset of $\mathbb{R}^n$. We denote by $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}(\overline{\Omega})$) the set of the real regular Borel measures over Ω (resp. $\overline{\Omega}$). We recall that $\mathcal{M}(\Omega) = [C_0(\Omega)]^{'}$ and $\mathcal{M}(\overline{\Omega}) = [C(\overline{\Omega})]^{'}$.

Although the approximation of measures by regular functions seems to be a well known property that has been sometimes used in the literature (see, for instance, Temam [31] or Giusti [17]), after an almost exhaustive search through the mathematical literature, we have not been able to find a proof of this particular approximation result. Thus, for the sake of completeness, we address here a rigorous demonstration of the approximation of a compactly supported measure by a sequence of smooth functions, *via* the use of the convolution with a mollifier sequence.

Lemma 1. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *and let* $\lambda \in \mathcal{M}(\overline{\Omega})$, $\lambda \geq 0$, *such that* λ *is concentrated in an open set* $\omega \subset\subset \Omega$*. Let* $\tilde{\lambda}$ *be the measure defined over* $\mathbb{R}^n$ *by*

$$\tilde{\lambda}(E) = \lambda(E \cap \omega) \tag{35}$$

for all Borel set $E \subset \mathbb{R}^n$*. (* $\tilde{\lambda} \in \mathcal{M}(\mathbb{R}^n)$ *and* $\tilde{\lambda} \geq 0$*). Let* $\{\rho_m\}_{m\in\mathbb{N}}$ *be a mollifier sequence, that is,*

$$\rho_m \in C_0^\infty(\mathbb{R}^n), \quad supp(\rho_m) \subset B(0, \frac{1}{m}), \quad \rho_m \geq 0, \quad \int_{\mathbb{R}^n} \rho_m(x)dx = 1\,,$$

and assume further that each ρ_m *is an even function. Let*

$$(\rho_m * d\tilde{\lambda})(x) = \int_{\mathbb{R}^n} \rho_m(x-y)d\tilde{\lambda}(y) \tag{36}$$

Then, we have:

$$\|\rho_m * d\tilde{\lambda}\|_{L^1(\mathbb{R}^n)} \leq \|\tilde{\lambda}\|_{\mathcal{M}(\mathbb{R}^n)} = \tilde{\lambda}(\mathbb{R}^n) = \lambda(\overline{\Omega})\,, \tag{37}$$

$\rho_m * d\tilde{\lambda} \in C_0^\infty(\mathbb{R}^n)$*, the support* $supp(\rho_m * d\tilde{\lambda}) \subset \overline{\omega} + \overline{B(0, \frac{1}{m})}$ *(so,* $\rho_m * d\tilde{\lambda} \in C_0^\infty(\Omega)$*) for* m *big enough, and*

$$\rho_m * d\tilde{\lambda} \rightharpoonup \tilde{\lambda} \text{ weakly} * \text{ in } \mathcal{M}(\mathbb{R}^n), \tag{38}$$

$$(\rho_m * d\tilde{\lambda})|_{\overline{\Omega}} \rightharpoonup \lambda \text{ weakly} * \text{ in } \mathcal{M}(\overline{\Omega}). \tag{39}$$

Proof. Inequality (37) arises from [30, Ch. I, Th. 1.3]. We center our efforts to obtain the convergence (39) from the convergence (38), since the rest of the assertions are straightforward to check. Let $\psi \in C(\overline{\Omega})$, we must prove that

$$\lim_{m\to\infty} \int_{\overline{\Omega}} (\rho_m * d\tilde{\lambda})(x)\psi(x)dx = \int_{\overline{\Omega}} \psi(x)d\lambda(x) \tag{40}$$

Since $\omega \subset\subset \Omega$, we can choose a compact set $K \subset \Omega$ and an integer m_0 such that $supp(\rho_m * d\tilde{\lambda}) \subset \overline{\omega} + \overline{B(0, \frac{1}{m})} \subset K$ for all integers $m \geq m_0$. Let $g \in C_0(\Omega)$ such that $0 \leq g \leq 1$ and g takes the value 1 over K. Then

$$\int_{\overline{\Omega}} (\rho_m * d\tilde{\lambda})(x)\psi(x)dx = \int_K (\rho_m * d\tilde{\lambda})(x)\psi(x)dx$$

$$= \int_K (\rho_m * d\tilde{\lambda})(x)g(x)\psi(x)dx = \int_{\mathbb{R}^n} (\rho_m * d\tilde{\lambda})(x)\,\overbrace{g\psi}\,(x)dx$$

where $\overbrace{g\psi} \in C_0(\mathbb{R}^n)$ is the extension of $g\psi$ by zero outside of Ω. Owing to convergence (38), this last integral converges to

$$\int_{\mathbb{R}^n} \overbrace{g\psi}\,(x)d\tilde{\lambda}(x) = \int_\omega g(x)\psi(x)d\tilde{\lambda}(x) = \int_\omega \psi(x)d\lambda(x) = \int_{\overline{\Omega}} \psi(x)d\lambda(x)$$

as required. $\square$

From this lemma we obtain the following result:

Lemma 2. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $\mu \in \mathcal{M}(\Omega)$. Let $\tilde{\mu}$ be the measure on $\overline{\Omega}$ which restriction to Ω is μ and which restriction to $\partial\Omega$ is the zero measure. Then $\tilde{\mu} \in \mathcal{M}(\overline{\Omega})$ and there is a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that*

$$f_n \rightharpoonup \tilde{\mu} \quad \text{weakly} * \ \text{ in } \ \mathcal{M}(\overline{\Omega})$$

Proof. It is easy to check that $\tilde{\mu}$ is in fact a real regular Borel measure over $\overline{\Omega}$. For proving the approximation result, we can assume without loss of generality that μ is positive. (Otherwise we use the decomposition of μ into its positive and negative variation.) The first step of the proof involves the approximation of $\tilde{\mu}$ by a sequence of measures with compact support contained in Ω. Let $\Omega_n = \{x \in \Omega \in ; dist(x, \partial\Omega) < \frac{1}{n}\}$. Since $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$, we have

$$\lim_{n\to\infty} \mu(\Omega_n) = \mu(\Omega)\,. \tag{41}$$

Let $\tilde{\mu}_n$ be the measure over $\overline{\Omega}$ defined by $\tilde{\mu}_n(E) = \mu(E \bigcap \Omega_n)$ for any Borel set $E \subset \overline{\Omega}$. We have that $\tilde{\mu}_n \in \mathcal{M}(\overline{\Omega})$, and $0 \leq \tilde{\mu}_n(E) \leq \tilde{\mu}(E)$ for any Borel set $E \subset \overline{\Omega}$. By construction $\tilde{\mu}_n$ is concentrated in Ω_n and, since $\tilde{\mu}_n$ is positive, we have

$$\|\tilde{\mu}_n\|_{\mathcal{M}(\overline{\Omega})} = \tilde{\mu}_n(\overline{\Omega}) = \mu(\Omega_n) \leq \mu(\Omega) \tag{42}$$

Since $\tilde{\mu} - \tilde{\mu}_n$ is positive, we have

$$\|\tilde{\mu} - \tilde{\mu}_n\|_{\mathcal{M}(\overline{\Omega})} = (\tilde{\mu} - \tilde{\mu}_n)(\overline{\Omega}) = \mu(\Omega) - \mu(\Omega_n) \tag{43}$$

This, together with (41), gives

$$\tilde{\mu}_n \to \tilde{\mu} \ \text{ strongly in } \ \mathcal{M}(\overline{\Omega}) \tag{44}$$

and a fortiori

$$\tilde{\mu}_n \rightharpoonup \tilde{\mu} \ \text{ weakly} * \text{ in } \ \mathcal{M}(\overline{\Omega}) \tag{45}$$

In the second step of the proof we apply Lemma 1 with $\lambda = \tilde{\mu}_n$ in order to approximate each measure $\tilde{\mu}_n$ by the sequence of smooth functions $\rho_m * \tilde{\mu}_n$. We combine the two approximation steps as follows. Since $\overline{\Omega}$ is a compact set of $\mathbb{R}^n$, the space $C(\overline{\Omega})$ is separable, hence the closed ball $B = \{\nu \in \mathcal{M}(\overline{\Omega}); \|\nu\|_{\mathcal{M}(\overline{\Omega})} \leq \mu(\Omega)\}$ is metrizable for the weak* topology in $\mathcal{M}(\overline{\Omega})$ (see [8, Th. III.25]). Let d be a distance over B which induces this topology. Thanks to (42) and (37), $\tilde{\mu}_n \in B$ and $\rho_m * \tilde{\mu}_n \in B$. So, convergence (45) and (39) are respectively equivalent to $\lim_{n\to\infty} d(\tilde{\mu}_n, \tilde{\mu}) = 0$ and $\lim_{m\to\infty} d(\rho_m * \tilde{\mu}_n, \tilde{\mu}_n) = 0$. Hence we can choose an appropriate sequence $f_n = \rho_{m(n)} * \tilde{\mu}_n$ such that $lim_{n\to\infty} d(f_n, \tilde{\mu}) = 0$, which concludes the proof. ☐

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