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The Peano–Sard theorem for fractional operators with Mittag-Leffler kernel and application in classical numerical approximation

Marc Jornet [a],[*], Juan J. Nieto [b]

[a] Departament de Matemàtiques, Universitat de València, 46100 Burjassot, Spain
[b] CITMAga, Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

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ABSTRACT

We investigate fractional Peano kernels for continuous linear functionals, in the context of differintegral operators with Mittag-Leffler kernel. New bounds for polynomial interpolation are obtained and numerical computations are shown, indicating improvements.

1. Introduction

1.1. Motivation

Fractional calculus extends classical mathematical analysis, with ordinary derivatives and integrals, to non-integer orders. For example, which is the $1/2$ derivative or integral of a function? There are different conceptions and theories of fractional calculus and related operators and equations [1–4].

In the nineties, Diethelm adapted and proved Peano kernel theorems in the fractional context: with the Riemann–Liouville derivative [5–7], and with the Caputo derivative in the particular case of quadrature integration [8]. Recently, Fernandez and Buranay found the Peano kernel with respect to Caputo operators, in a general setting, and illustrated the theoretical results in numerical analysis [9].

To move beyond Riemann–Liouville and Caputo derivatives, we aim at investigating these types of results on Peano kernels for fractional operators of non-singular kernels, for the first time, such as those based on the Mittag-Leffler function [10–14]. While these operators may have inconveniences when working with differintegral equations [15,16], we will see that they may become useful when representing, bounding and computing functionals, for example, the polynomial-interpolation error, due to the simpler structure of the integration kernel. The mathematical development and computations will show a number of improvements over previous publications.

Our article thus contributes to the theory of numerical methods related to fractional operators [17]. This is relevant, as fractional-based models and computational schemes are currently of use in different scientific and engineering fields [18,19].

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[*] Corresponding author.
E-mail addresses: marc.jornet@uv.es (M. Jornet), juanjose.nieto.roig@usc.es (J.J. Nieto).

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1.2. Preliminaries on fractional operators

The basic notations along the paper are: $[a,b]$, real interval in $\mathbb{R}$; $\mathrm{L}^p[a,b]$, Lebesgue integrable functions of order $1 \leq p \leq \infty$ with norm $\|\cdot\|_p$; $C[a,b]$, continuous functions; $\mathrm{AC}[a,b]$, absolutely continuous functions; $\circ$, composition of operators.

If $x \in \mathrm{L}^1[a,b]$, then the Riemann–Liouville fractional integral of order $\alpha \in (0,\infty)$ [2,20] is

$$ {}^{RL}I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\mathrm{d}s, \tag{1.1}$$

for $t \in [a,b]$, where Γ is the gamma function. The function ${}^{RL}I_a^\alpha x$ exists almost everywhere and belongs to $\mathrm{L}^1[a,b]$. If $x \in C[a,b]$, then ${}^{RL}I_a^\alpha x \in C[a,b]$. If $x \in \mathrm{AC}[a,b]$, then ${}^{RL}I_a^\alpha x \in \mathrm{AC}[a,b]$ (in general, the condition $x \in C[a,b]$ is not sufficient).

If $x \in \mathrm{AC}[a,b]$ and $\alpha \in (0,1)$, the fractional operator with non-singular kernel of Mittag-Leffler type in the sense of Caputo [10] is

$$ D_a^\alpha x(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t x'(s) E_\alpha\left(-\alpha \frac{(t-s)^\alpha}{1-\alpha}\right) \mathrm{d}s, \tag{1.2}$$

where $B(\alpha)$ is a normalization function satisfying $B(0) = B(1) = 1$, x' is the ordinary derivative, and E_α is the (one-parameter) Mittag-Leffler function. The associated fractional integral of (1.2) is

$$ I_a^\alpha x(t) = \frac{1-\alpha}{B(\alpha)} x(t) + \frac{\alpha}{B(\alpha)} \cdot {}^{RL}I_a^\alpha x(t), \tag{1.3}$$

where ${}^{RL}I_a^\alpha$ is (1.1) and $x \in \mathrm{L}^1[a,b]$. For $x \in \mathrm{AC}[a,b]$, it is known that

$$ I_a^\alpha \circ D_a^\alpha x(t) = x(t) - x(0), \tag{1.4}$$

but

$$ D_a^\alpha \circ I_a^\alpha x(t) = x(t) - E_\alpha\left(-\frac{\alpha}{1-\alpha} t^\alpha\right) x(0). \tag{1.5}$$

Eq. (1.4) may be seen as a Barrow's rule. Eq. (1.5) is not exactly a fundamental theorem of calculus, because $x(0) \neq 0$ in general. However, in contrast to fractional differential equations and their associated Volterra integral representations [15], we will not need (1.5) here.

For higher order [21], let $n < \alpha < n+1$, where $n \geq 1$ is an integer, and $\beta = \alpha - n \in (0,1)$. Then, for $x \in \mathrm{AC}^{n+1}[a,b]$ (i.e., $x, x', \dots, x^{(n)} \in \mathrm{AC}[a,b]$), one defines

$$ \begin{aligned} D_a^\alpha x(t) &= D_a^\beta [x^{(n)}](t) \\ &= \frac{B(\beta)}{1-\beta} \int_a^t x^{(n+1)}(s) E_\beta\left(-\beta \frac{(t-s)^\beta}{1-\beta}\right) \mathrm{d}s, \end{aligned} \tag{1.6}$$

where $x^{(j)}$ is the ordinary jth derivative of x and D_a^β is defined by (1.2). The fractional integral (1.3) is generalized to that $\alpha \in (0,\infty)$ as

$$ I_a^\alpha x(t) = {}^{RL}I_a^n \circ I_a^\beta x(t), \tag{1.7}$$

by using (1.1), where $x \in \mathrm{L}^1[a,b]$. If $x \in \mathrm{AC}^{n+1}[a,b]$, one has

$$ I_a^\alpha \circ D_a^\alpha x(t) = x(t) - \sum_{k=0}^n \frac{x^{(k)}(a)}{k!}(t-a)^k. \tag{1.8}$$

This property (1.8), which may be viewed as an alternative form of the residual of Taylor's series, extends (1.4). A generalization of (1.5) is not required in our context.

Physically, $D_a^\alpha x$ is non-local and involves the whole history of x along $[0,t]$. It converges to $x^{(n+1)}$ when $\alpha \to (n+1)^-$, for $n \geq 0$, so results from ordinary calculus may be extended. We aim at applying these extensions to numerical analysis.

1.3. The Peano kernel in ordinary and fractional calculus

We denote by $\mathcal{L}(C[a,b])$ the set of continuous linear functionals, from $C[a,b]$ into $\mathbb{R}$. We denote by $\mathrm{NBV}[a,b]$ the set of normalized functions (i.e., $g(a) = 0$ and right continuous) with bounded variation.

Riesz theorem states that every functional of $\mathcal{L}(C[a,b])$ can be represented as a Riemann–Stieltjes integral [22,23]:

Theorem 1.1 (*[24, Chapter 3, Theorem 34]*).: *For every $L \in \mathcal{L}(C[a,b])$, there is a unique integrator $g_L \in \mathrm{NBV}[a,b]$ such that*

$$ Lx = \int_a^b x(t)\mathrm{d}g_L(t), \tag{1.9}$$

for all $x \in C[a,b]$.

The classical Peano–Sard theorem in ordinary calculus is the following:

Theorem 1.2 ([9, Theorem 1.2]). *Fix an integer $n \geq 1$. Let $L \in \mathcal{L}(C[a,b])$ such that $Lr = 0$ for every polynomial r of degree $\leq n-1$. Then, for $x \in \mathrm{AC}^n[a,b]$, it can be represented as*

$$Lx = \int_a^b K_n(t)x^{(n)}(t)\mathrm{d}t,$$

where

$$K_n(t) = L\left(\frac{(\cdot - t)_+^{n-1}}{(n-1)!}\right) \tag{1.10}$$

is the Peano kernel and $s_+ = \max\{s, 0\}$. In particular,

$$|Lx| \leq \|K_n\|_p \|x^{(n)}\|_q, \tag{1.11}$$

for any $1 \leq p, q \leq \infty$ that satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Since $x^{(n)}$ may be generalized to a fractional order, it is intuitive that a fractional version of the Peano–Sard theorem should exist. For Caputo fractional calculus, the recent theorem is the following:

Theorem 1.3 ([9, Theorem 2.1]). *Fix a non-integer number $\nu > 0$ and let $m-1 < \nu < m$, where m is an integer. Let $L \in \mathcal{L}(C[a,b])$ such that*

$$L((\cdot - a)^k) = 0 \text{ for } k = 0, \ldots, m-1.$$

(i) If $\nu > 1$, $x \in \mathrm{AC}^m[a,b]$, and

$$K_\nu(t) = L\left(\frac{(\cdot - t)_+^{\nu-1}}{\Gamma(\nu)}\right), \tag{1.12}$$

then

$$Lx = \int_a^b K_\nu(t) \cdot {}^C D_a^\nu x(t)\mathrm{d}t, \tag{1.13}$$

where ${}^C D_a^\nu$ is the Caputo fractional operator, defined by

$${}^C D_a^\nu x(t) = {}^{RL}I_a^{m-\nu} x^{(m)}(t) = \frac{1}{\Gamma(m-\nu)} \int_a^t \frac{x^{(m)}(\tau)}{(t-\tau)^{\nu+1-m}}\mathrm{d}\tau.$$

(ii) Suppose that $0 < \nu < 1$ and $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If

$$K_\nu(t) = \int_a^b \frac{(s-t)_+^{\nu-1}}{\Gamma(\nu)}\mathrm{d}g_L(s)$$

and it belongs to $\mathrm{L}^p[a,b]$, then L is given by (1.13) for every $x \in \mathrm{AC}^m[a,b]$ such that ${}^C D_a^\beta x \in \mathrm{L}^q[a,b]$.

In both cases,

$$|Lx| \leq \|K_\beta\|_p \|{}^C D_a^\beta x\|_q. \tag{1.14}$$

Remark 1.4.

Theorem 1.3 is also true in the setting of generalized proportional Caputo fractional calculus [25], with the operator ${}^C D_a^{\nu,\rho}$, $\rho \in [0,1]$. In this case, L must satisfy

$$L\left((\cdot - a)^k \mathrm{e}^{\frac{\rho-1}{\rho}(\cdot - a)}\right) = 0 \text{ for } k = 0, \ldots, m-1,$$

and the Peano kernel is

$$K_{\nu,\rho}(t) = \int_a^b \frac{(s-t)_+^{\nu-1}\mathrm{e}^{\frac{\rho-1}{\rho}(s-a)}}{\Gamma(\nu)}\mathrm{d}g_L(s).$$

The hypotheses of the theorem are the same. This remark has not been previously specified in the literature, but its use may be limited because polynomials do not belong to the null space of the functional.

2. Main result on representation of functionals

In the following lemma, we compute (1.7) explicitly, as a convolution. This representation is key to derive the Peano–Sard theorem for the fractional operator in Theorem 2.2, as it occurs with the Caputo fractional derivative. From the new formula for Lx, one can deduce bounds for $|Lx|$.

Lemma 2.1. *Let $n<\alpha<n+1$, where $n\geq 1$ is an integer, and $\beta=\alpha-n\in(0,1)$. The exact form of (1.7) is*

$$\begin{aligned} I_a^\alpha x(t)={}&\frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^t (t-s)^{n-1}x(s)\mathrm{d}s\\ &+\frac{\beta}{B(\beta)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\mathrm{d}s, \end{aligned} \tag{2.1}$$

where $x\in \mathrm{L}^1[a,b]$.

Proof. We carry out the following computations, using (1.1), (1.3) and (1.7):

$$\begin{aligned} I_a^\alpha x(t) &= {}^{RL}I_a^n\circ I_a^\beta x(t)\\ &=\frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1}\cdot I_a^\beta x(s)\mathrm{d}s\\ &=\frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^t (t-s)^{n-1}x(s)\mathrm{d}s\\ &\quad+\frac{\beta}{B(\beta)\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{\Gamma(\beta)}\int_a^s (s-\tau)^{\beta-1}x(\tau)\mathrm{d}\tau\mathrm{d}s\\ &=\frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^t (t-s)^{n-1}x(s)\mathrm{d}s\\ &\quad+\frac{\beta}{B(\beta)\Gamma(n)\Gamma(\beta)}\int_a^t\left(\int_\tau^t (t-s)^{n-1}(s-\tau)^{\beta-1}\mathrm{d}s\right)x(\tau)\mathrm{d}\tau\\ &=\frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^t (t-s)^{n-1}x(s)\mathrm{d}s\\ &\quad+\frac{\beta}{B(\beta)\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)\mathrm{d}\tau. \end{aligned}$$

Fubini's theorem is justified here because $t^{\alpha-1}$ and $x(t)$ are in $\mathrm{L}^1[a,b]$, so their convolution is in $\mathrm{L}^1[a,b]$. ☐

Theorem 2.2. *Let $n<\alpha<n+1$, where $n\geq 1$ is an integer, and $\beta=\alpha-n\in(0,1)$. Let $L\in\mathcal{L}(C[a,b])$ such that*

$$L((\cdot-a)^k)=0 \text{ for } k=0,\dots,n,$$

that is, $Lr=0$ for every polynomial r of degree $\leq n$. Define

$$\begin{aligned} K_\alpha(t)&=L\left[\frac{1-\beta}{B(\beta)\Gamma(n)}(\cdot-t)_+^{n-1}+\frac{\beta}{B(\beta)\Gamma(\alpha)}(\cdot-t)_+^{\alpha-1}\right]\\ &=\frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^b (s-t)_+^{n-1}\mathrm{d}g_L(s)+\frac{\beta}{B(\beta)\Gamma(\alpha)}\int_a^b (s-t)_+^{\alpha-1}\mathrm{d}g_L(s)\\ &=\frac{1-\beta}{B(\beta)\Gamma(n)}\int_t^b (s-t)^{n-1}\mathrm{d}g_L(s)+\frac{\beta}{B(\beta)\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}\mathrm{d}g_L(s), \end{aligned} \tag{2.2}$$

for every $t\in[a,b]$, which belongs to $\mathrm{L}^\infty[a,b]$. If $x\in\mathrm{AC}^{n+1}[a,b]$, then

$$Lx=\int_a^b K_\alpha(t)\cdot D_a^\alpha x(t)\mathrm{d}t, \tag{2.3}$$

where D_a^α is given by (1.6). In particular,

$$|Lx|\leq \|K_\alpha\|_p\|D_a^\alpha x\|_q<\infty, \tag{2.4}$$

where $p,q\in[1,\infty]$ satisfy $1/p+1/q=1$. If K_α is of one sign on $[a,b]$, then

$$Lx=D_a^\alpha x(\xi)\int_a^b K_\alpha(t)\mathrm{d}t, \tag{2.5}$$

where $\xi\in[a,b]$.

Proof. By Jordan's decomposition theorem, $g_L=g_L^+-g_L^-$, where g_L^+ and g_L^- are right-continuous and increasing functions on $[a,b]$. Let $g_L^\sharp=g_L^++g_L^-$. For functions $h\geq 0$, we have the inequalities

$$\left|\int_a^b h(t)\mathrm{d}g_L(t)\right|\leq\int_a^b h(t)\mathrm{d}g_L^\sharp(t)\leq\|h\|_\infty\left(g_L^\sharp(b)-g_L^\sharp(a)\right). \tag{2.6}$$

This function $g_L^\sharp$ is used to bound Lebesgue–Stieltjes integrals.

As $t_+^{n-1},t_+^{\alpha-1}\in C[a,b]$, hence bounded, we can use (2.6) to obtain $K_\alpha\in\mathrm{L}^\infty[a,b]$.

Let $x \in \mathrm{AC}^{n+1}[a,b]$. Considering (1.6), which is a convolution of $x^{(n+1)} \in \mathrm{L}^1[a,b]$ and the Mittag-Leffler function $E_\beta(t^\beta) \in C[0,\infty)$, we have that ${}_a^{}D^\alpha x \in \mathrm{C}[a,b] \subseteq \mathrm{L}^1[a,b]$. Then the integral in (2.3) exists, by Hölder's inequality with indices ∞ and 1.

Let us prove (2.3). First,

$$\frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^b\int_a^b (s-t)_+^{n-1}\mathrm{d}g_L^\sharp(s)\left|{}_aD^\alpha x(t)\right|\mathrm{d}t + \frac{\beta}{B(\beta)\Gamma(\alpha)}\int_a^b\int_a^b (s-t)_+^{\alpha-1}\mathrm{d}g_L^\sharp(s)\left|{}_aD^\alpha x(t)\right|\mathrm{d}t < \infty,$$

by continuity of the integrands on $[a,b]^2$. In consequence, by Fubini's theorem, (2.1) in Lemma 2.1, Barrow's rule (1.8), and the annihilation of polynomials of degree $\le n$, we deduce the following equalities:

$$\begin{aligned}\int_a^b K_\alpha(t)\cdot {}_aD^\alpha x(t)\mathrm{d}t &= \frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^b\int_a^b (s-t)_+^{n-1}\mathrm{d}g_L(s)\,{}_aD^\alpha x(t)\mathrm{d}t \\ &\quad + \frac{\beta}{B(\beta)\Gamma(\alpha)}\int_a^b\int_a^b (s-t)_+^{\alpha-1}\mathrm{d}g_L(s)\,{}_aD^\alpha x(t)\mathrm{d}t \\ &= \frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^b\int_a^b (s-t)_+^{n-1}\cdot {}_aD^\alpha x(t)\mathrm{d}t\mathrm{d}g_L(s) \\ &\quad + \frac{\beta}{B(\beta)\Gamma(\alpha)}\int_a^b\int_a^b (s-t)_+^{\alpha-1}\cdot {}_aD^\alpha x(t)\mathrm{d}t\mathrm{d}g_L(s) \\ &= \frac{1-\beta}{B(\beta)\Gamma(n)}\int_a^b\int_a^s (s-t)^{n-1}\cdot {}_aD^\alpha x(t)\mathrm{d}t\mathrm{d}g_L(s) \\ &\quad + \frac{\beta}{B(\beta)\Gamma(\alpha)}\int_a^b\int_a^s (s-t)^{\alpha-1}\cdot {}_aD^\alpha x(t)\mathrm{d}t\mathrm{d}g_L(s) \\ &= \int_a^b {}_aI^\alpha \circ {}_aD^\alpha x(s)\mathrm{d}g_L(s) \\ &= \int_a^b \left(x(s) - \sum_{k=0}^n \frac{x^{(k)}(a)}{k!}(s-a)^k\right)\mathrm{d}g_L(s) \\ &= L\left[x - \sum_{k=0}^n \frac{x^{(k)}(a)}{k!}(\cdot - a)^k\right] \\ &= Lx.\end{aligned}$$

Finally, (2.4) follows from Hölder's inequality. Notice that the right-hand side is finite, because $K_\alpha \in \mathrm{L}^\infty[a,b] \subsetneq \mathrm{L}^p[a,b]$ and ${}_aD^\alpha x \in \mathrm{C}[a,b] \subseteq \mathrm{L}^q[a,b]$. Eq. (2.5) holds by the mean-value theorem for integrals. ☐

Remark 2.3.

The function (2.2) is a combination of the classical Peano kernel (1.10) and the Caputo-type Peano kernel (1.12). This is, essentially, a consequence of (1.3).

Remark 2.4.

Both $E_\alpha(-\alpha\frac{(t-s)^\alpha}{1-\alpha})$ and K_α in (2.2) are kernels, but they are distinct. The former refers to the fractional operator, which can be non-singular (bounded, as in this paper) and singular (when it blows up at some point). The latter is the integrator in the functional L, with (2.3).

3. Application to interpolation errors

We apply Theorem 2.2 in numerical analysis, for the polynomial interpolation of functions with some absolutely continuous derivative. New error formulae are derived.

Physically, an interpolating polynomial is a simple curve that approximates a given function, maybe complex, by passing through certain data nodes, and it important to measure the discrepancy. Given $[a,b]$, consider nodes $a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$, for an integer $n \ge 1$. Fix a point $\tau \in (a,b)$. The error of the Lagrange interpolation formula [26, Chapter 3] [27, Sections 3.4] is

$$L_{n,\tau}x = x(\tau) - \sum_{j=0}^n L_j(\tau)x(\xi_j), \tag{3.1}$$

where

$$L_j(\tau) = \prod_{i\neq j}\frac{\tau-\xi_i}{\xi_j-\xi_i}$$

for $j = 0,\dots,n$. It is known that, if $x \in C^{n+1}[a,b]$, then

$$L_{n,\tau}x = \frac{x^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^n(\tau-\xi_j), \tag{3.2}$$

where $\xi \in [\xi_0, \xi_n]$. The interpolation is exact for polynomials of degree $\leq n$, therefore we can apply Theorem 2.2 with the functional $L_{n,\tau} \in \mathcal{L}(C[a,b])$.

Let $\alpha \in (n, n+1)$ and $x \in \mathrm{AC}^{n+1}[a,b]$. By (2.2) and (3.1),

$$\begin{aligned} K_\alpha(t) = {} & \frac{1-\beta}{B(\beta)\Gamma(n)}(\tau - t)_+^{n-1} + \frac{\beta}{B(\beta)\Gamma(\alpha)}(\tau - t)_+^{\alpha-1} \\ & - \sum_{j=0}^{n} L_j(\tau)\left(\frac{1-\beta}{B(\beta)\Gamma(n)}(\xi_j - t)_+^{n-1} + \frac{\beta}{B(\beta)\Gamma(\alpha)}(\xi_j - t)_+^{\alpha-1}\right), \end{aligned} \tag{3.3}$$

for $t \in [a,b]$. This is a new Peano kernel. With (3.3), the exact form (2.3) and the bound (2.4) hold.

If K_α, given by (3.3), does not change sign on $[a,b]$, then

$$\begin{aligned} \int_a^b K_\alpha(t)\mathrm{d}t = {} & \frac{1-\beta}{B(\beta)n!}(\tau - a)^n + \frac{\beta}{B(\beta)\Gamma(\alpha+1)}(\tau - a)^\alpha \\ & - \sum_{j=0}^{n} L_j(\tau)\left(\frac{1-\beta}{B(\beta)n!}(\xi_j - a)^n + \frac{\beta}{B(\beta)\Gamma(\alpha+1)}(\xi_j - a)^\alpha\right) \\ = {} & \frac{\beta}{B(\beta)\Gamma(\alpha+1)}\left((\tau - a)^\alpha - \sum_{j=0}^{n} L_j(\tau)(\xi_j - a)^\alpha\right). \end{aligned} \tag{3.4}$$

Notice that, in the previous expression (3.4), $(\tau - a)^n - \sum_{j=0}^n L_j(\tau)(\xi_j - a)^n = 0$, because the degree of $(\tau - a)^n$ is n. With (3.4), equality (2.5) is satisfied. If $\alpha = n+1$, $\beta = \alpha - n = 1$, $B(\beta) = 1$, and $x \in C^{n+1}[a,b]$, then (3.4) becomes

$$\frac{1}{\Gamma(n+2)}\left((\tau - a)^{n+1} - \sum_{j=0}^{n} L_j(\tau)(\xi_j - a)^{n+1}\right) = \frac{1}{(n+1)!}\prod_{j=0}^{n}(\tau - \xi_j)$$

and (2.5) is the classical formula (3.2). Thus, our methodology extends the known theory on Lagrange interpolation error, to functions $x \in \mathrm{AC}^{n+1}[a,b]$ and fractional values $\alpha \in (n, n+1)$.

Let

$$E(t,s) = \frac{B(\beta)}{1-\beta}E_\beta\left(-\beta\frac{(t-s)^\beta}{1-\beta}\right).$$

When

$$D_a^\alpha x = \int_a^t x^{(n+1)}(s)E(t,s)\mathrm{d}s$$

(see (1.6)) is unknown in explicit form for (2.4) and (2.5), one may use the bounds

$$\|D_a^\alpha x\|_\infty \leq \|x^{(n+1)}\|_1 \cdot \left(\max_{t\in[a,b],\, s\in[a,t]} E(t,s)\right) = \|x^{(n+1)}\|_1 \frac{B(\beta)}{1-\beta} \tag{3.5}$$

and, for $1 \leq p < \infty$ and by Hölder's inequality,

$$\begin{aligned} \|D_a^\alpha x\|_p & \leq \left(\int_a^b \left|\int_a^t x^{(n+1)}(s)E(t,s)\mathrm{d}s\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ & \leq \left(\int_a^b \left(\int_a^t |x^{(n+1)}(s)|^{\frac{1}{p}}|x^{(n+1)}(s)|^{\frac{1}{q}}E(t,s)\mathrm{d}s\right)^p \mathrm{d}t\right)^{\frac{1}{p}} \\ & \leq \left(\int_a^b \left(\int_a^t |x^{(n+1)}(s)|E(t,s)^p\mathrm{d}s\right)\left(\int_a^t |x^{(n+1)}(s)|\mathrm{d}s\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}} \\ & \leq \|x^{(n+1)}\|_1^{\frac{1}{q}}\left(\int_a^b\int_s^b E(t,s)^p\mathrm{d}t|x^{(n+1)}(s)|\mathrm{d}s\right)^{\frac{1}{p}} \\ & \leq \|x^{(n+1)}\|_1 \cdot \left(\max_{s\in[a,b]} \|E(\cdot,s)\mathcal{X}_{[s,b]}(\cdot)\|_p\right) \\ & = \|x^{(n+1)}\|_1 \cdot \frac{B(\beta)}{1-\beta}\left(\int_0^b E_\beta\left(-\beta\frac{u^\beta}{1-\beta}\right)^p \mathrm{d}u\right)^{\frac{1}{p}}. \end{aligned} \tag{3.6}$$

The previous norms are considered on $[a,b]$.

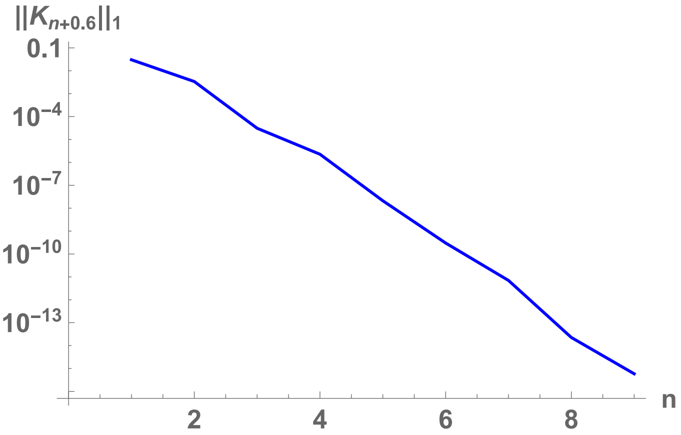

Fig. 1. Decay of $\|K_\alpha\|_1$, in Example 3.1.

In applications, the described method is specially useful when $x \in \mathrm{AC}^{n+1}[a,b]$ but $x \notin C^{n+1}[a,b]$, so that standard techniques are not of use (see (3.2)). The simplest bound for (3.1) is (2.4), with $p=1$, $q=\infty$, $B(\beta)=1$, (3.3) (its $\|\cdot\|_1$-norm), and (3.5) (generally, K_α is not of one sign):

$$|L_{n,\tau}x| \le \mathcal{B}^*_{\alpha,n} := \|K_\alpha\|_1 \left\|D_a^\alpha x\right\|_\infty \tag{3.7}$$

$$\le \mathcal{B}_{\alpha,n} := \|K_\alpha\|_1 \|x^{(n+1)}\|_1 \frac{1}{1-\beta}, \tag{3.8}$$

for $x \in \mathrm{AC}^{n+1}[a,b]$. In general, bounds (3.7) and (3.8), with $p=1$ and $q=\infty$, are not possible in ordinary calculus or with fractional operators of singular kernels, because $\|x^{(n+1)}\|_\infty = \infty$ (i.e., $x^{(n+1)}$ is not Lipschitz continuous on $[a,b]$). The use of a non-singular kernel such as in D_a^α permits having $\|D_a^\alpha x\|_\infty \lesssim \|x^{(n+1)}\|_1 < \infty$, and this may be an important advantage in this context of numerical-error estimation.

We show some examples. The software Mathematica® [28] is used, with the built-in functions *NMaximize* for the norm $\|\cdot\|_\infty$, *NIntegrate* for $\|\cdot\|_1$, and *MittagLefflerE* for E_β.

Example 3.1. Fix $[a,b]=[0,0.5]$ and $\tau = 1/\pi$. We consider partitions of $[0,0.5]$ with n equally separated nodes and fractional orders $\alpha = n+\beta = n+0.6$, for $n=1,2,\ldots,9$. With $B(\beta)=1$ and (3.7), the Lagrange remainder evolves as follows:

$$|L_{1,\tau}x| \le 0.0302207 \left\|D_0^{1.6}x\right\|_\infty,$$

$$|L_{2,\tau}x| \le 0.00338978 \left\|D_0^{2.6}x\right\|_\infty,$$

$$|L_{3,\tau}x| \le 0.0000308980 \left\|D_0^{3.6}x\right\|_\infty,$$

$$|L_{4,\tau}x| \le 2.26346\times 10^{-6} \left\|D_0^{4.6}x\right\|_\infty,$$

$$|L_{5,\tau}x| \le 2.10973\times 10^{-8} \left\|D_0^{5.6}x\right\|_\infty,$$

$$|L_{6,\tau}x| \le 3.00826\times 10^{-10} \left\|D_0^{6.6}x\right\|_\infty,$$

$$|L_{7,\tau}x| \le 6.98079\times 10^{-12} \left\|D_0^{7.6}x\right\|_\infty,$$

$$|L_{8,\tau}x| \le 2.25586\times 10^{-14} \left\|D_0^{8.6}x\right\|_\infty,$$

$$|L_{9,\tau}x| \le 5.86410\times 10^{-16} \left\|D_0^{9.6}x\right\|_\infty.$$

Notice that $\|K_\alpha\|_1 \to 0$ as $\alpha\to\infty$. The decay is exponential with n, as the log-scale of Fig. 1 shows.

We illustrate the function K_α in the four panels of Fig. 2, for $\alpha \in \{2.6, 3.6, 8.6, 9.6\}$. It is near 0 and of two signs, and it tends to be smoother as α grows.

If we work on $[a,b]=[0,2]$ and $\tau = 1+1/\pi$, with n nodes and $\alpha = n+0.6$, then

$$|L_{1,\tau}x| \le 0.268295 \left\|D_0^{1.6}x\right\|_\infty,$$

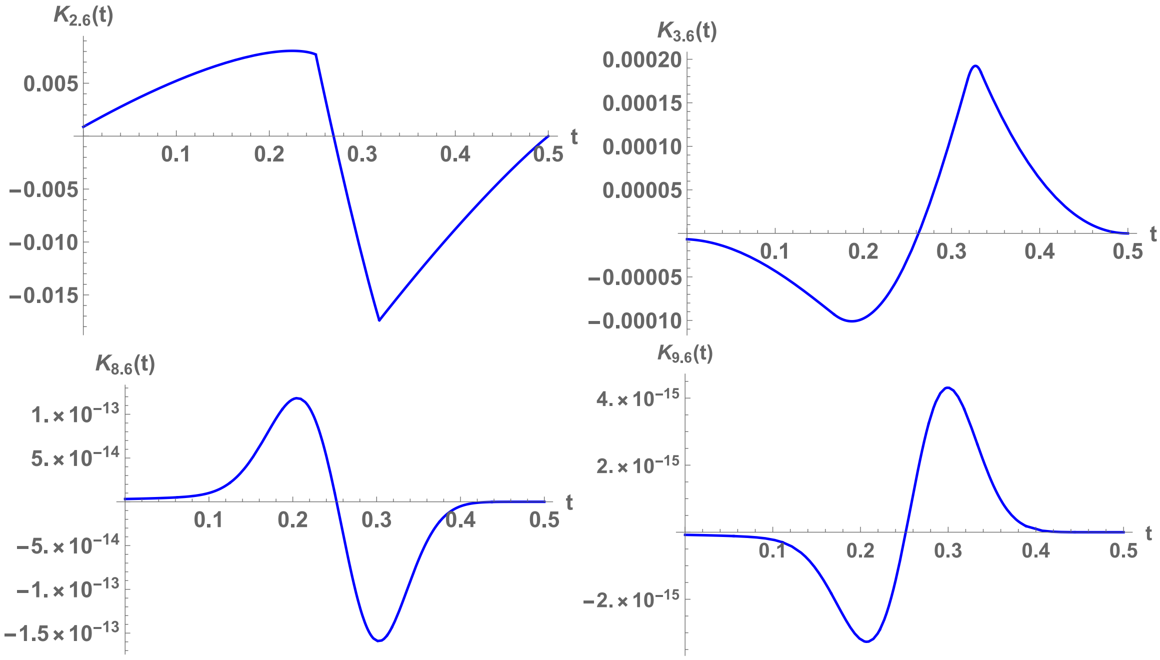

Fig. 2. Some graphs of K_α, in Example 3.1.

$$|L_{2,\tau}x| \le 0.0764129 \left\| D_0^{2.6} x \right\|_\infty ,$$

$$|L_{3,\tau}x| \le 0.000648979 \left\| D_0^{3.6} x \right\|_\infty ,$$

$$|L_{4,\tau}x| \le 0.000695504 \left\| D_0^{4.6} x \right\|_\infty ,$$

$$|L_{5,\tau}x| \le 0.0000419925 \left\| D_0^{5.6} x \right\|_\infty ,$$

$$|L_{6,\tau}x| \le 4.31853 \times 10^{-7} \left\| D_0^{6.6} x \right\|_\infty ,$$

$$|L_{7,\tau}x| \le 1.45534 \times 10^{-7} \left\| D_0^{7.6} x \right\|_\infty ,$$

$$|L_{8,\tau}x| \le 5.28462 \times 10^{-9} \left\| D_0^{8.6} x \right\|_\infty ,$$

$$|L_{9,\tau}x| \le 5.92940 \times 10^{-11} \left\| D_0^{9.6} x \right\|_\infty ,$$

and we also have $\|K_\alpha\|_1 \to 0$ as $\alpha \to \infty$, not necessarily in a monotonically decreasing way, but still at exponential convergence rate. The shape of K_α is similar to that presented in Fig. 2 for the other case.

Example 3.2. Consider nodes $\xi_0 = 0$, $\xi_1 = 0.2$ and $\xi_2 = 0.5$. Let $\tau = 0.3$ and $x(t) = t^{2.99}\mathrm{e}^t$. We have $n = 2$, $B(\beta) = 1$, $a = 0$, $b = 0.5$, $x \in \mathrm{AC}^3[0, 0.5]$ and $x \notin C^3[0, 0.5]$. The norms of x''' are

$$\|x'''\|_1 = 7.62865$$

and

$$\|x'''\|_\infty = x'''(0^+) = \infty.$$

Table 1 reports results for different values of the fractional order $\alpha \in (2, 3)$, up to six significant digits: $\|K_\alpha\|_1$ (see (3.3)), $\mathcal{B}_{\alpha,2}$ (see (3.8)), $\|D_0^\alpha x\|_\infty = D_0^\alpha x(0.5)$ and $\mathcal{B}^*_{\alpha,2}$ (see (3.7)). With $\mathcal{B}_{\alpha,2}$, the best bound for the Lagrange error is obtained when $\alpha \approx 2$, which is ≈ 0.07629. With $\mathcal{B}^*_{\alpha,2}$, it is ≈ 0.02782, when $\alpha \approx 3$.

Some plots of K_α are displayed in Fig. 3. We observe that it changes sign and there are some points of non-differentiability. At the limit $\alpha = 3$, it is of one sign and smooth. On the other hand, the fractional derivative $D_0^\alpha x(t)$ is shown in Fig. 4. Notice

Table 1
Values of $\|K_\alpha\|_1$ (see (3.3)), $\mathcal{B}_{\alpha,2}$ (see (3.8)), $\|D_0^\alpha x\|_\infty$ and $\mathcal{B}^*_{\alpha,2}$ (see (3.7)), for different fractional orders $\alpha \in (2,3)$, in Example 3.2.

α	$\|K_\alpha\|_1$	$\mathcal{B}_{\alpha,2}$	$\|D_0^\alpha x\|_\infty$	$\mathcal{B}^*_{\alpha,2}$
2.0001	0.01	0.0762941	7.62865	0.0762865
2.01	0.00999764	0.0770389	7.62979	0.0762799
2.1	0.00978061	0.0829032	7.73739	0.0756764
2.2	0.00919486	0.0876804	8.05172	0.0740344
2.3	0.00834436	0.0909374	8.57565	0.0715583
2.8	0.00257632	0.0982691	17.2282	0.0443852
2.9	0.00146726	0.111933	21.9199	0.0321623
2.99	0.0010018	0.764235	27.7660	0.0278158

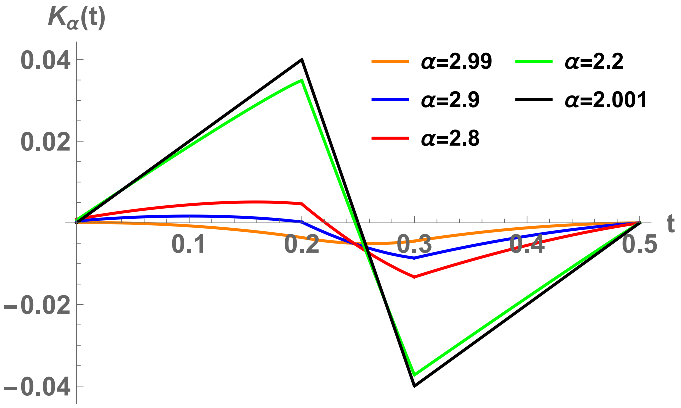

Fig. 3. Some graphs of K_α, in Example 3.2.

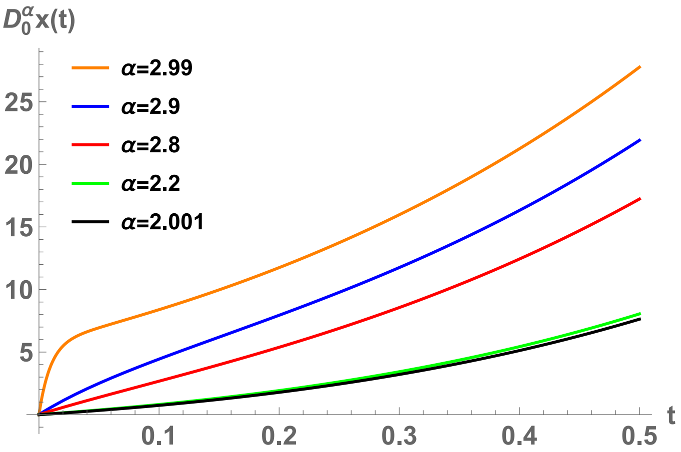

Fig. 4. Some graphs of $D_0^\alpha x$, in Example 3.2.

that $D_0^\alpha x(0) = 0$, as a consequence of the non-singularity of the Mittag-Leffler function. The fractional derivative is observed to be increasing on the interval. Its slope at $t = 0$ seems to tend to ∞ as $\alpha \to 3^-$. In Fig. 5, we compare $D_0^{2.99} x$ and x''', emphasizing the fact that fractional operators of non-singular nature regularize the derivative.

With the classical formula (3.2), since $x \in C^2[0, 0.5]$, we can only use $\xi_0 = 0$ and $\xi_2 = 0.5$, with larger error bound:

$$\frac{\|x''\|_\infty}{2} |\tau - \xi_0 \,\|\, \tau - x_2| = 0.22886.$$

The exact Lagrange error (3.1) is

$$\left| x(\tau) - \sum_{j=0}^{2} L_j(\tau) x(\xi_j) \right| = 0.0145468.$$

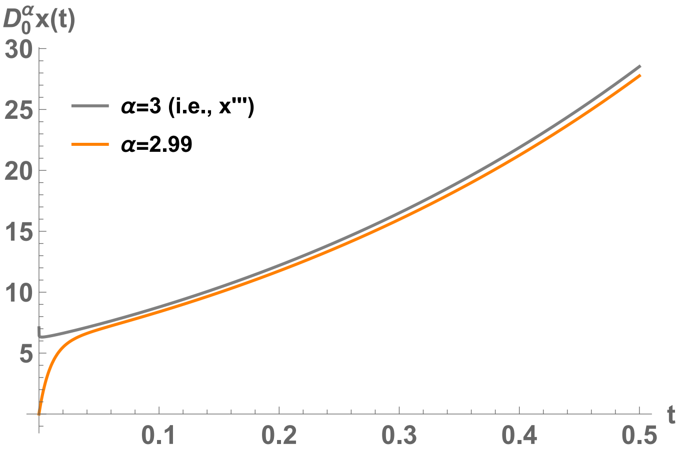

Fig. 5. A comparison of $D_0^\alpha x$ and x''', in Example 3.2.

Table 2
Values of $\|K_\alpha\|_1$ (see (3.3)), $\mathcal{B}_{\alpha,2}$ (see (3.8)), $\|D_0^\alpha x\|_\infty$ and $\mathcal{B}^*_{\alpha,2}$ (see (3.7)), for different fractional orders $\alpha \in (2,3)$, in Example 3.3.

α	$\|K_\alpha\|_1$	$\mathcal{B}_{\alpha,2}$	$\|D_0^\alpha x\|_\infty$	$\mathcal{B}^*_{\alpha,2}$
2.0001	0.01	0.0515106	5.01062	0.0501062
2.01	0.00999764	0.0520135	5.01130	0.0501011
2.1	0.00978061	0.0559727	5.07570	0.0496434
2.2	0.00919486	0.0591982	5.26522	0.0484129
2.3	0.00834436	0.0613972	5.58199	0.0465781
2.8	0.00257632	0.0663472	10.5438	0.0271641
2.9	0.00146726	0.0755722	12.8214	0.0188123
2.99	0.00146726	0.51598	15.0081	0.0150351

Example 3.3. In the context of the previous example, let $x(t) = t^{2.0001}\mathrm{e}^t$. We have

$$\|x'''\|_1 = 5.15055$$

and

$$\|x'''\|_{1.0001} = \infty. \tag{3.9}$$

Due to (3.9), computations are more prone to numerical instabilities, compared to Example 3.2. We include, among the options for *NIntegrate*, the Global Adaptive method with Gauss–Kronrod quadrature rule and 100 recursive bisections, to achieve convergence for integrals. Table 2 tabulates results for different values of the fractional order $\alpha \in (2,3)$, up to six significant digits: $\|K_\alpha\|_1$ (see (3.3)), $\mathcal{B}_{\alpha,2}$ (see (3.8)), $\|D_0^\alpha x\|_\infty = D_0^\alpha x(0.5)$ and $\mathcal{B}^*_{\alpha,2}$ (see (3.7)). With $\mathcal{B}_{\alpha,2}$, the best bound for the Lagrange error is obtained when $\alpha \approx 2$, which is ≈ 0.05151. With $\mathcal{B}^*_{\alpha,2}$, it is ≈ 0.01504, when $\alpha \approx 3$.

The plots of K_α are the same as in Fig. 3, because they are independent of the input function x. The fractional derivative $D_0^\alpha x(t)$ is illustrated in Fig. 6. It vanishes at $t = 0$ and its slope appears to grow towards ∞ as $\alpha \to 3^-$, like in the above example. The regularization of x''' is exemplified in Fig. 7. The two figures show the distinctive properties of differintegral operators with bounded kernels.

With the classical formula (3.2) for C^2 functions, the bound is larger:

$$\frac{\|x''\|_\infty}{2}|\tau-\xi_0||\tau-x_2| = 0.210217.$$

On the other hand, with (1.11), we essentially need $p = \infty$ and $q = 1$, by (3.9), which gives

$$|Lx| \le \|K_3\|_\infty \|x'''\|_1 = 0.005 \times 5.15055 = 0.0257527. \tag{3.10}$$

This number is greater than $\mathcal{B}^*_{2.9,2}$. Note that the complexity of $\|x'''\|_1$ and $\|D_0^\alpha x\|_\infty = D_0^\alpha x(0.5)$ is similar, by the non-singularity of the kernel. If the Caputo operator is used with (1.14), $p = \infty$ and $q = 1$, then

$$|Lx| \le \|K_{2.9}\|_\infty \|{}^C D_0^{2.9} x\|_1 = 0.00606319 \times 4.34169 = 0.0263245,$$

which is larger than $\mathcal{B}^*_{2.9,2}$ and (3.10). Furthermore, $\|{}^C D_0^{2.9} x\|_1$ is more complex than $\|x'''\|_1$ and $D_0^\alpha x(0.5)$.

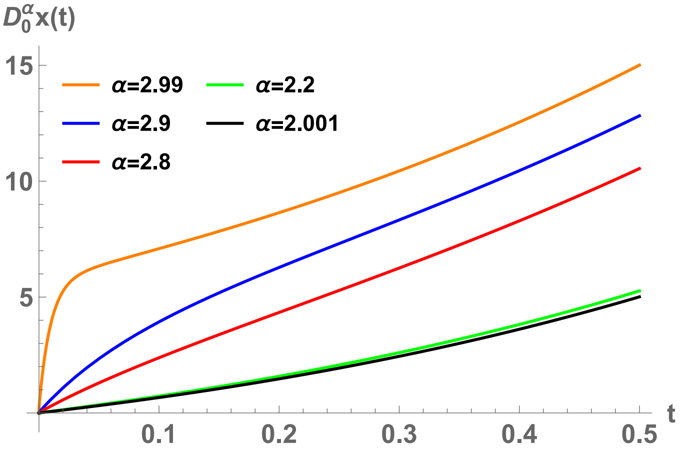

Fig. 6. Some graphs of $D_0^\alpha x$, in Example 3.3.

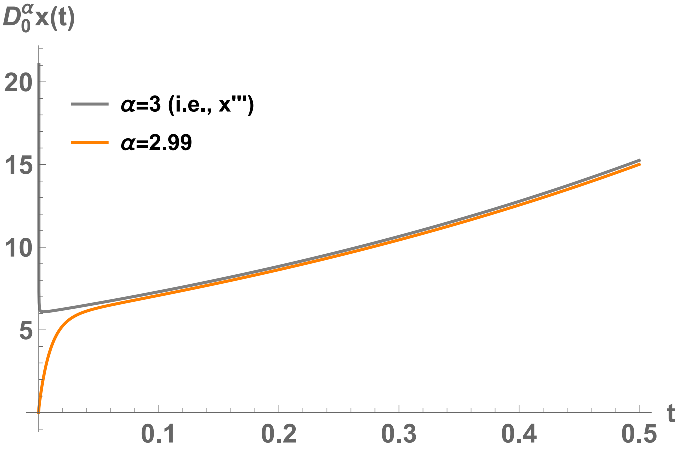

Fig. 7. A comparison of $D_0^\alpha x$ and x''', in Example 3.3.

The exact Lagrange remainder (3.1) is

$$\left| x(\tau) - \sum_{j=0}^{2} L_j(\tau) x(\xi_j) \right| = 0.00980593.$$

4. Conclusion

We consider the fractional calculus associated to operators of Mittag-Leffler kernel. Based on the extended Barrow's rule (1.8) and the explicit form for the integral operator (2.1), we can derive a Peano–Sard theorem for the representation of continuous linear functionals that annihilate some polynomials, Theorem 2.2. The input function needs to be in a certain space, related to absolute continuity. Our result complements the recent contribution [9]. Hölder's inequality (2.4) bounds the functional, and hence one may seek applications in numerical analysis. The non-singularity of the Mittag-Leffler function is relevant in the inequalities and computations. For interpolation, new error formulae can be obtained, such as (3.7) and (3.8). The numerical examples of the paper illustrate the theory and show that classical error bounds may be improved. Note that $\|K_\alpha\|_1$ and $\|D_a^\alpha x\|_\infty$ need to be computed.

In the future, other types of fractional operators, of singular and non-singular kernels, shall be investigated. It seems that, as opposed to fractional differential equations, the use of singular kernels does not give any advantage in this context. It is intuitive that, if the goal is bounding functionals (such as numerical-approximation remainders), bounded components are preferable.

Interpolating polynomials are very important in different scientific fields; for instance, in curve fitting and data analysis, image processing, modeling and forecasting, design of numerical methods for engineering simulations, etc. Hence the understanding of discrepancies in the approximations is essential. For next publications, other kinds of problems in classical numerical analysis could be studied via fractional operators. This is a new view on fractional calculus, which has often been concerned with mathematical modeling of time series based on memory effects.

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Data availability

No data was used for the research described in the article.

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